Physica 29D (1987) 191–201
North-Holland, Amsterdam

SYMBOLIC DYNAMICS OF ORDER-PRESERVING ORBITS

Peter VEERMAN*
Department of Theoretical and Applied Mechanics, Cornell University, Ithaca, NY 14853, USA

Received 1 May 1986
Revised manuscript received 22 June 1987

The maps we consider are roughly those that can be obtained by truncating non-invertible maps to weakly monotonic maps (they have a flat piece). The binary sequences that correspond to order-preserving orbits are shown to satisfy a minimax principle (which was already known for order-preserving orbits with rational rotation number). The converse is also proven: all minimax orbits are order-preserving with respect to some rotation number.

For certain families of such circle maps one can solve exactly for the parameter values for which the map has a specified rotation number rho. For rho rational we obtain the endpoints of the resonance intervals recursively. These parameter values can be organized in a natural way as the nodes of a Farey tree. We give some applications of the ideas discussed.

1. Introduction

Symbolic dynamics is often used to describe aspects of complicated behaviour (Guckenheimer and Holmes [1]). In this paper, we study the full shift on $\{0,1\}^{\mathbf{N}}$ and applications thereof to a class of circle maps. In particular, we are interested in order-preserving orbits of circle maps.

Order-preserving orbits are models for similarly behaving orbits in two-dimensional area-preserving twist maps. These orbits, in turn, describe physical phenomena such as the minimal energy states of a chain of atoms submitted to a periodic potential (Aubry [2], Aubry and LeDaeron [3]). Non-periodic order-preserving orbits of twist maps are also important in the study of disappearance of invariant KAM tori (Mather [4], Katok [5], Mackay and Stark [6]) and in the study of the flux from one resonant region to another across cantori (Meiss et al. [7]).

In the latter case, the cantori are related to orbits of monotone circle maps with irrational rotation number and which are not dense on the circle. Circle maps which admit such orbits are sometimes called Denjoy counter examples (Nitecki [8], Herman [9], chapter 10).

In this paper we are interested in describing such orbits as they occur in certain families of circle maps. In particular, for a one-parameter family of continuous, piecewise linear "flat spot" circle maps φ_λ (see solid graph in fig. 1), we give an exact representation of the orbits for rational, as well as for irrational, rotation numbers. We also find the parameter value λ for which φ_λ admits orbits of prescribed rotation number ρ, thereby explicitly solving for the graph $R(\lambda)$ in some simple cases (figs. 7a, b and c). The properties of $R(\lambda)$ in a more general context have been investigated by many authors.

It was argued (numerical methods) in Jensen et al. [10, 11] that for a one-parameter family of circle maps with cubic inflection point, the set A of parameters for which $R(\lambda)$ is irrational, has fractal dimension 0.87.... This would imply that the set has measure zero. So far, proofs of these statements have not been given.

To our knowledge, the examples presented here are the only ones in the literature for which the graph of $R(\lambda)$ can be described analytically. The

*Present address: Physics department, Rockefeller University, New York, NY 10021, USA.

fact that A has measure zero, for instance, can be easily proven. Unfortunately, our maps are not cubically critical. Nevertheless, they may provide exemplary insight.

In section 2, we limit ourselves to the study of binary sequences in $\{0,1\}^{\mathbf{N}}$. A minimal set under f is a closed, f-invariant set that has no f-invariant closed subsets. The main theorem states that for each number $\rho \in [0,1]$ there is a unique minimal set $\overline{\Sigma}_\rho$ in $\{0,1\}^{\mathbf{N}}$ under the shift with the following properties: 1) Each sequence in $\overline{\Sigma}_\rho$ has an average of ρ ones. 2) The shift on $\overline{\Sigma}_\rho$ is semi-conjugate to rotation by ρ (order preservation). 3) Among all minimal sets with rotation number ρ, the greatest element of $\overline{\Sigma}_\rho$ is smaller than the greatest element of any other set. 4) The corresponding statement for the smallest element of $\overline{\Sigma}_\rho$. The theorem is a generalization of that given in Gambaudo et al. [12], who proved it for rational ρ (after some partial results had appeared in Bernhardt [13]).

Various authors have argued that the organization in a Farey tree of the midpoints of the resonance intervals of families of circle maps leads to some understanding of $R(\lambda)$ (Feigenbaum [14], Ostlund and Kim [15], Cvitanovic et al. [16]). The main result of section 3 asserts that for the families of circle maps we consider, the organization of the endpoints of the resonance intervals in a Farey tree gives complete understanding of $R(\lambda)$. The idea is not to order the rotation numbers themselves in the tree, but, rather, to order the binary sequences that describe the symbolic dynamics pertaining to monotone maps with those rotation numbers (see fig. 4).

In section 5 we mention some applications of the ideas developed in sections 2 and 3. Among other things, we note that the A, defined as before, has zero Lebesgue measure and Hausdorff dimension zero.

Finally we give some of the conventions we use. The shift σ is defined on $\{0,1\}^{\mathbf{N}}$ as follows:

$$\sigma(i_1 i_2 i_3 \cdots) = i_2 i_3 i_4 \cdots .$$

Throughout, we will use the distance $|\cdots|$ on $\{0,1\}^{\mathbf{N}}$, $(s, s' \in \{0,1\}^{\mathbf{N}})$

$$x(s) = \sum_{n=1}^{\infty} \frac{i_n}{\tau^n}, \quad |s - s'| = |x(s) - x(s')|, \quad (1.1)$$

where usually $\tau = 2$.

Identifying s and s' whenever $x(s) = x(s')$, we effectively give $\{0,1\}^{\mathbf{N}}$ the topology of the real interval [0, 1]. This is a connected space.

The symbol $\rho_n(s)$ is used to denote the average number of ones in the first n digits of a sequence s_0 (or: $\rho_n = (1/n)\Sigma_{j=1}^{n} i_j$). The limit $\lim_n \rho_n(s)$ (if it exists) is called the rotation number.

2. Order-preserving orbits

The main result of this section will be the extension of a theorem of Gambaudo et al. [12].

Consider the shift σ acting on $\{0,1\}^{\mathbf{N}}$. We call an orbit orb (s) or $\sigma^k(s)$, $s \in \{0,1\}^{\mathbf{N}}$, $k > 0$ minimax if it satisfies the following requirements: s has a rotation number ρ, and for any other sequence s' with the properties

$$\sup_k x(\sigma^k(s)) \leq \sup_k x(\sigma^k(s')),$$

where $x(s)$ is the real number associated with the sequence s. There is a similar definition for maximin orbits $\sigma^k(s)$, with the inequality

$$\inf_k x(\sigma^k(s)) \geq \inf_k x(\sigma^k(s')).$$

We will briefly recall some notions from Veerman [17] for later use. Define $\varphi = 2x \bmod 1$ and φ_β as the truncation of φ as depicted in fig. 1 (Kadanoff [18] and Boyland [19] used this construction to study non-invertible map). Each φ_β is a (weakly) monotone degree-one continuous circle map. Let Φ_β be the lift of φ_β. The rotation number

$$R(\varphi_\beta) = \lim_{n \to \infty} \frac{\Phi_\beta^n(x) - x}{n} = \rho$$

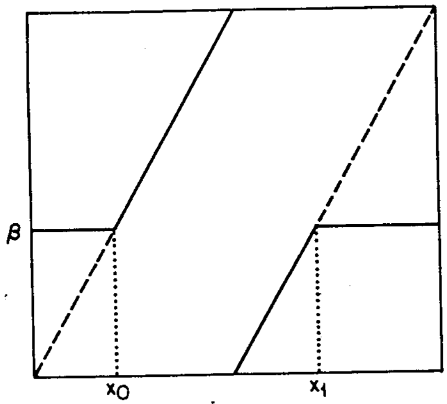

Fig. 1. Construction of φ_β from $x \to 2x \bmod 1$.

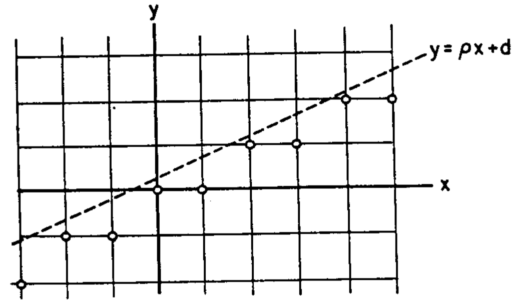

Fig. 2. Construction of sequences of $\bar{\Sigma}_\rho$.

is well-defined and independent of x (see Nitecki [8], Herman [9]). In fact R is a continuous function of β. For each rational value $R(\beta) = p/q$, in smallest terms, the graph of R has a plateau of length $(2^q - 1)^{-1}$ (see Veerman [20] and related results in Nagumo et al. [21] and Boyd [22]). For later use we define the resonance interval $I_{p/q}$ as the interval of parameter values where $R(\beta) = p/q$. The complement of the union of the inverse iterates of (x_1, x_0) (the flat piece of the map) contains the unique nonwandering set, $\Lambda_\rho \subset [x_0, x_1]$ which is also a minimal set (Nitecki [8], Boyland [19], Herman [9]). The orbits in Λ_ρ preserve the cyclic order of rotation by ρ (order-preservation), that is: for β such that $R(\beta) = \rho$ and for $x \in \Lambda_\rho$ (ρ irrational),

$$\Phi_\beta^{k+n}(x) < \Phi_\beta^k(x) + (p_n + 1) \Leftrightarrow n\rho < p_n + 1 \tag{2.1}$$

(see also Nitecki [8]). Note further that Λ_ρ might vary with β when $\rho =$ constant. That this is not the case, is one of the direct consequences of theorem 2.1.3 and 5, and the fact that Λ_ρ is unique (Veerman [17]).

To study the non-wandering set, note that it is contained in an interval of length $\frac{1}{2}$ and that φ_β restricted to Λ_ρ is equal to $2x \bmod 1$. Writing the elements of Λ_ρ as binary sequences, it becomes clear that Λ_ρ is a minimal set under the shift σ operating on $\{0,1\}^{\mathbb{N}}$ as studied for instance, in Hedlund [23]. As a minimal set under the shift, we will refer to this set as $\bar{\Sigma}_\rho$, as a minimal set under φ_β, we will call it Λ_ρ.

Let p_n be the (unique) sequence of integers that satisfy (fixed d, see fig. 2)

$$-d \le n\rho - p_n < 1 - d, \quad d \in [0,1),\ n \in \mathbb{N}. \tag{2.2}$$

Then, define sequences with digits i_n as follows:

$$i_n = p_n - p_{n-1}, \quad n \in \mathbb{N}. \tag{2.3}$$

We will refer to d as the index. For each index $d \in [0,1)$ we obtain in a different binary sequence. We call this set of sequences Σ_ρ. The Σ_ρ^* is defined similarly but with the equality sign in (2.2) under the second inequality and $d \in (0,1]$ and is equal to Σ_ρ for ρ rational. The extension to doubly infinite binary sequences is straightforward: just take $n \in \mathbb{Z}$ in (2.2) and (2.3). (By projecting this construction onto the vertical axis one obtains Σ_ρ and Σ_ρ^* by pure rotation as in Hedlund [23].)

The closure of either Σ_ρ or Σ_ρ^* turns out to be the union of these two sets.

We can now state the main result of this section. For rational rotation numbers this theorem was proven by Gambaudo et al. [12], some of whose results, in turn, were anticipated by Bernhardt [13].

Theorem 2.1. For a fixed rotation number ρ the following statements are equivalent for s in $\{0,1\}^{\mathbb{N}}$

and an element of a minimal set:
1) $\sigma^k(s)$ is minimax,
2) $\sigma^k(s)$ is maximin,
3) $\sup_k(x(\sigma^k(s))) - \inf_k(x(\sigma^k(s))) < \frac{1}{2}$ (ρ rational),
$\sup_k(x(\sigma^k(s))) - \inf_k(x(\sigma^k(s))) = \frac{1}{2}$ (ρ irrational),
4) $\sigma|_{o(s)}$ is order-preserving, where $o(s) = \{\sigma^n(s) | n \in \mathbb{N}\}$,
5) $s \in \bar{\Sigma}_\rho$.

In the following proof we suppose that ρ is irrational. If ρ is rational, say p/q, the proof goes through with only the modification that sequence in $\bar{\Sigma}_{p/q}$ have exactly p ones among every q digits.

To prove the equivalence of statements 1 to 5 of theorem 2.1, consider first:

Lemma 2.2. 5) $\Leftrightarrow$ 4).

Proof. It was shown in Veerman [17] that for s to be a member of $\bar{\Sigma}_\rho$ is equivalent to the following: For every k, the first n digits of $\sigma^k(s)$ contain p_n or p_n+1 ones where p_n and n satisfy (ρ irrational)

$$p_n < n\rho < p_n + 1. \tag{2.4}$$

Now, φ_β restricted to Λ_ρ is equal to the shift on $x \in \Lambda_\rho$ written as a binary sequence s. Every time the integer value of $\Phi_\beta^{k+i}(x)$ increases by one, the first digit of $\Phi_\beta^{k+i-1}(x)$ must have been a one. So (2.4) is equivalent to

$$\forall k, n \in \mathbb{N}:$$
$$\Phi_\beta^k(x) + p_n < \Phi_\beta^{k+n}(x) < \Phi_\beta^{k+n}(x) + (p_n + 1)$$
$$\Leftrightarrow p_n < n\rho < p_n + 1,$$

which in turn is equivalent to (2.1). □

Lemma 2.3. i) 5) $\Leftrightarrow$ 1); ii) 5) $\Leftrightarrow$ 2).

Proof. According to the previous lemma we may replace 5 by 4 in lemma 2.3.

i) Fix ρ and let p_j satisfy eq. (2.5). Recall that we only consider sequences that are members of a minimal set.

Suppose that s is a sequence that satisfies 4) while s' does not. Then s' must have a subsequence of length n with either 1) more than $p_n + 1$ ones or 2) less than p_n ones.

In the first case, shift s' so that its first n digits are t_n. All sequence in $\bar{\Sigma}_\rho$ have no more than $p_n + 1$ ones among their n digits. It is easy to see that there is an $N \in \{0, 1, \ldots, n-2\}$ such that $\sigma^N(s')$ is greater than any element of $\bar{\Sigma}_\rho$. (Namely, choose r such that t_r' of length r is the shortest subsequence of t_n having $p_r + 2$ ones. Then t_r' has a one as its first and last digit. The sequences in $\bar{\Sigma}_\rho$ can (at most) have t_{r-1}' as initial digits, followed by a zero.)

The second case reduces to the first one by making the following observations. There are $M_i < M$, $i \in \mathbb{N}$, such that, whenever t_n appears for the ith time it will appear for the $i+1$st time exactly $n + M_i$ digits later. (This assertion is proven by noting that if s' contains arbitrarily long subsequences without t_n, then there is a sequence z in the closure of $\bigcup_k \sigma^k(s')$ that does not contain t_n. The closure of the iterates of z do not contain t_n, and that would constitute a proper invariant subset of the original minimal set.) If none of the subsequences of length M_i contain more than $p_{M_i} + 1$ ones, the following contradiction arises: Since in $n + M_i$ digits there are at most $p_n - 1 + p_{M_i} + 1$ ones, we have

$$\rho < \sup_i \left(\frac{p_n - 1 + p_{M_i} + 1}{n + M_i} \right) = \sup_i \left(\frac{p_n + p_{M_i}}{n + M_i} \right) < \rho.$$

The latter inequality is derived from (2.5) and the boundedness of M_i.

It follows that $\sup x(\sigma^k(s))$ is smaller than $\sup x(\sigma^k(s'))$ for any choice of s and s'.

Note that the reasoning goes either way, so that we have proven equivalence.

ii) The proof is entirely analogous to i). □

Lemma 2.4. 5) $\Leftrightarrow$ 3).

Proof. $\Leftarrow$: Suppose there is a sequence s' that satisfies 3). Then a β can be chosen such that the

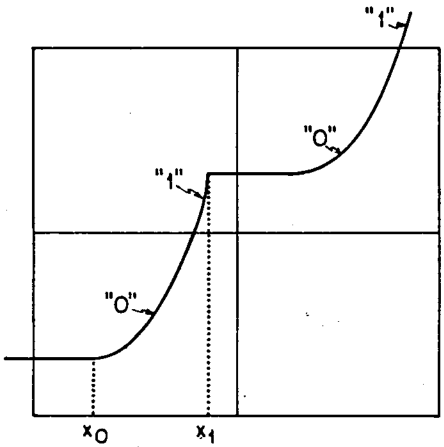

Fig. 3. $F|_{\Lambda_\rho}$ is monotone for ρ irrational.

orbit of $x(s')$ under φ_β never lands in the flat spot. By hypothesis, s' is member of a minimal set. The only minimal set φ_β allowed is Λ_ρ, as remarked above, or $\bar{\Sigma}_\rho$, under the shift.

⇒ : As proved by Veerman [17] □

Remark. The theorem partially generalizes with respect to the base τ in $x(s)=\sum_{n=1}^{\infty} i_n/n$. It is straightforward to check that if one replaces $\frac{1}{2}$ by $1/\tau$ in 3) the theorem holds for any $\tau \geq 2$. Statements 3), 4), and 5) are equivalent for any base $\tau > 1$.

The formulate a corollary, let f be a continuous, degree-one circle map, such that its lift F is non-decreasing. Suppose further, that f has a rotation number ρ and a non-wandering set Λ_ρ. Define the map s that assigns binary "addresses" to each point $x \in \Lambda_\rho$: the nth digit is a zero when int $F^n(x)$ and int $F^{n-1}(x)$ have the same value, a one if not (see fig. 3).

Corollary 2.5. s is a non-decreasing map $\Lambda_\rho \to \bar{\Sigma}_\rho$ ($=\Sigma_\rho$ if ρ is rational).

Proof. Suppose $x_i \in \Lambda_\rho$ and $x_2 > x_1$. Then $F^n(x_2) \geq F^n(x_1)$ for all n, and so

$$s(x_2) \geq s(x_1).$$

Into $\bar{\Sigma}_\rho$ follows from the fact that the order has to be preserved (F monotone) and that all order-preserving orbits are elements of $\bar{\Sigma}_\rho$ according to theorem 2.1. □

Remark. In Hockett and Holmes [24] these orbits, along with others, are constructed with the aid of the somewhat more indirect methods outlined in Katok [5]. These methods use Hausdorff limits and apply in general to Aubry–Mather sets of 2-dimensional twist maps (Mather [4], Katok [5]).

3. Resonance intervals and the Farey tree

The purpose of this section is to give a simple recursive scheme that generates $R(\beta)$ (see figs. 7a, b and c) for a piecewise linear "flat spot" map. The only information one needs to generate $R(\beta)$ is the slope τ of the slant part of φ_β.

Recall the definition of the family φ_β and the resonance interval $I_{p/q}$ from section 2. It was shown in Veerman [20] that the coordinate of the right-most end-point of $I_{p/q}$ (see fig. 7a) is equal to $x(\alpha(p/q))=\sum_1^\infty i_n/2^n$ where $\alpha(p/q)=\{i_n\}_1^\infty$ is the element of $\Sigma_{p/q}$ constructed with index $d=p/q$ (see 2.2 and 2.3). This sequence is periodic with period q and has p ones in each period. Periodic sequences will be denoted by: $\langle \cdots \rangle$. Note that a sequence is interpreted in two ways. The average number of ones defines a rotation number p/q and the value on the base two defines the endpoint $x(\alpha(p/q))$ of the resonance interval $I_{p/q}$. As indicated before, the length of $I_{p/q}$ is $(2^q-1)^{-1}$.

Define Farey addition ⊕ on the periodic sequences (where it is non-commutative) and on the rationals (where it is commutative). as follows (t_1 are finite sequences):

$$\langle t_1 \rangle \oplus \langle t_2 \rangle = \langle t_1 t_2 \rangle = .t_1 t_2 t_1 t_2 t_1 t_2 \cdots,$$

$$\frac{p_1}{q_1} \oplus \frac{p_2}{q_2} = \frac{p_1+p_2}{q_1+q_2}.$$

It is clear that if $\langle t_1 \rangle$ has rotation number p_1/q_1 and $\langle t_2 \rangle$ has rotation number p_2/q_2, then $\langle t_1 \rangle \oplus$

$\langle t_2 \rangle$ has rotation number $p_1/q_1 \oplus p_2/q_2$. Suppose that $p_1/q_1 < p_2/q_2$. We define these two rationals to be Farey-adjacent whenever $q_1 p_2 - q_2 p_1 = 1$ (Hardy and Wright [25]). We now have:

Theorem 3.1. Let p_1/q_1 and p_2/q_2 be Farey-adjacent, then

i) $\alpha\left(\frac{p_1}{q_1} \oplus \frac{p_2}{q_2}\right) = \alpha\left(\frac{p_1}{q_2}\right) \oplus \alpha\left(\frac{p_2}{q_2}\right)$
except when $\frac{p_2}{q_2} = \frac{1}{1}$.

ii) $\alpha\left(\frac{p_1}{q_1} \oplus \frac{p_2}{q_2}\right) = \alpha\left(\frac{1}{1}\right) \oplus \alpha\left(\frac{p_1}{q_1}\right)$.

Proof. i) The proof is straightforward and elementary, but somewhat long-winded. It relies on two basic ingredients, the first one being the definition of $\alpha(p/q)$.

$$\begin{aligned} \alpha\left(\frac{p_1}{q_1}\right)&: \quad -\frac{p_1}{q_1} < \frac{p_1}{q_1}k - h_k < 1 - \frac{p_1}{q_1}, \\ \alpha\left(\frac{p_2}{q_2}\right)&: \quad -\frac{p_2}{q_2} < \frac{p_2}{q_2}k - l_k < 1 - \frac{p_2}{q_2}, \end{aligned} \tag{3.1}$$

where h_k, resp. l_k, counts the number of ones in the first k digits of $\alpha(p_1/q_1)$, resp. $\alpha(p_2/q_2)$ (see 2.2 and 2.3, we have left out the equality sign because we assume $k > 0$).

We define $\alpha[(p_1+p_2)/(q_1+q_2)]$ (abbreviated as α^*):

$$\alpha^*: \quad -\frac{p_1+p_2}{q_1+q_2} < \frac{p_1+p_2}{q_1+q_2}k - g_k < 1 - \frac{p_1+p_2}{q_1+q_2}. \tag{3.2}$$

In a) it will be proven that one can replace g_k by h_k for $k \in \{1 \cdots q_1\}$. In b) it will be proven that g_{k+q_1} can be replaced by l_k for $k \in \{1 \cdots q_2\}$. The composition of the rest of the sequence follows from its q_1+q_2 periodicity. The second ingredient is that p_1/q_1 be Farey-adjacent to p_2/q_2

$$q_1 p_2 - q_2 p_1 = 1 \Leftrightarrow \frac{p_1 q_2}{q_1} = p_2 - \frac{1}{q_1}. \tag{3.3}$$

a) Let $k \in \{1 \cdots q_1\}$. Since α^* has a greater rotation number and greater index than $\alpha(p_1/q_1)$, it is clear from fig. 2 that $h_k \le q_k$. So the first inequality in (3.2) with g_k replaced by h_k follows immediately.

To prove the second inequality, note that from (3.1) it follows that

$$-h_k < 1 - \frac{p_1}{q_1} - \frac{p_1}{q_1}k.$$

Together with (3.3), this implies

$$\begin{aligned} &(p_1+p_2)k - h_k(q_1+q_2) \\ &\quad < q_1 + q_2 - p_1 - p_2 - \frac{k+1}{q_1}. \end{aligned} \tag{3.4}$$

For $k < q_1 - 1$ we can drop the only fractional term:

$$(p_1+p_2)k - h_k(q_1+q_2) \le q_1 + q_2 - p_1 + p_2.$$

Adding $p_1 + p_2$ to both sides and dividing by $q_1 + q_2$ we see that the left-hand side is fractional whereas the right-hand side is not. So we can drop the equality in the last equation. For $k < q_1 - 1$ the second inequality of (3.3) is proven.

To deal with the last two digits, we note (Veerman [20]) that the q_1st digit of the p_1/q_1 scaling sequence is a zero. Then

$$h_{q_1} - 1 = p_1 \quad \text{and} \quad h_{q_1} = p_1.$$

For $k = q_1 - 1$, (3.4) becomes (using (3.3))

$$1 - p_1 - p_2 < q_1 + q_2 - p_1 - p_2 + \frac{h+1}{q_1}.$$

This inequality is satisfied also if $(k+1)/q_1$ is dropped, because, by assumption, $q_1 + q_2 > 1$. For $k = q_1$, using (3.4)

$$1 < q_1 + q_2 - p_1 - p_2 + \frac{k+1}{q_1}.$$

It is straightforward to derive that the term $(k + 1)/q_1$ can be dropped iff $p_2/q_2 \ne 1/1$.

b) Let $k \in \{1 \cdots q_2\}$. To prove that g_{k+q_1} can be replaced by $p_1 + l_k$, note first that from (3.1) it follows that

$$-\frac{p_2}{q_2} - \frac{p_2}{q_2}k < l_k < 1 - \frac{p_2}{q_2} - \frac{p_2}{q_2}k.$$

From this follows

$$-p_1 - p_2 - \frac{k+1}{q_2} < (p_1+p_2)k - l_k(q_1+q_2)$$
$$< q_1 + q_2 - p_1 - p_2 - \frac{k+1}{q_2}.$$

Then add

$$\frac{q_2}{q_2} = (p_1+p_2)q_1 - p_1(q_1+q_2)$$

to each of the terms:

$$-p_1 - p_2 + \frac{q_2-k-1}{q_2}$$
$$< (p_1+p_2)(q_1+k) - (p_1+l_k)(q_1+q_2)$$
$$< q_1 + q_2 - p_1 - p_2 - \frac{q_2-k-1}{q_2}.$$

One can then proceed as in a) to prove that the fractional can be dropped.

ii) The proof is entirely analogous to that in i). □

Remarks. Note that every relative prime rational smaller than one in absolute value appears in the Farey tree (Hardy and Wright [25]). The theorem then implies that the right-most endpoint can be found by Farey addition of $\langle 0\rangle$ and $\langle 1\rangle$ (see fig. 4). According to the theorem by Farey concatenating the smallest sequence first, except when $\langle 1\rangle$ is one of the sequences (in that case concatenate $\langle 1\rangle$ first). The graph of $R(\beta)$ for the family φ_β can then be drawn as follows (see fig. 7). Draw the Farey tree up to a desired level, as in fig. 4. The right-most endpoint of the plateau $I_{2/3}$, for instance, is given by $\rho = 2/3$,

$$\beta = \frac{1}{2} + \frac{1}{2^2} + \frac{0}{2^3} + \frac{1}{2^4} + \frac{1}{2^5}\cdots$$
$$= \frac{6}{8}\left(1 + \frac{1}{8} + \frac{1}{8^2}\cdots\right) = \frac{6}{7}.$$

The left-most endpoint is given by $\rho = 2/3$,

$$\beta = \frac{6}{7} - \frac{1}{2^3-1} = \frac{5}{7}.$$

The results are, clearly, also valid for the families $\varphi_{\tau,\beta}$ ($\tau > 1$, fixed), defined as follows (see Veerman [20]): $\varphi_{\tau,\beta}$ is piecewise linear, the slope is τ and the flat spot has length $(\tau-1)/\tau$. The only difference being that now the sequences are interpreted on the base τ rather than on the base 2. In fig. 7b, the rotation number as a function of β (varying in $[0, 1/(\tau-1)]$ is plotted. In this case $\tau = 1.2$. Note that the resonance intervals, pertaining to the low rationals, tend to be smaller than

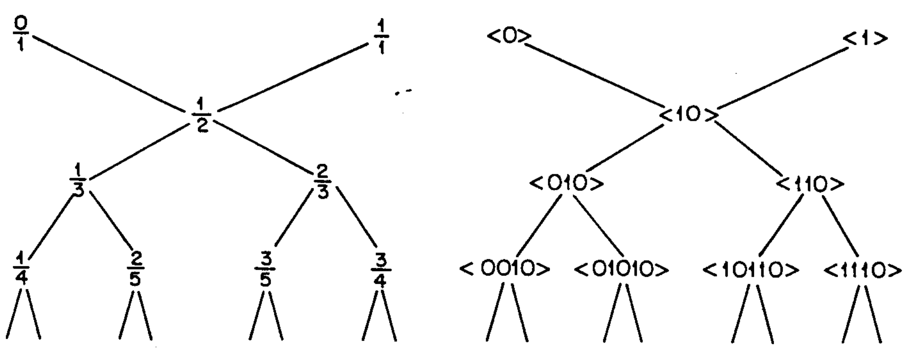

Fig. 4. The Farey tree for the rationals and the Farey tree for binary sequences.

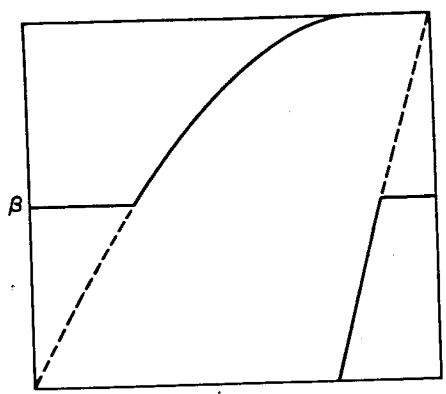

Fig. 5. $\varphi' = h \cdot \varphi \cdot h^{-1}$. Note that φ' is not differentiable.

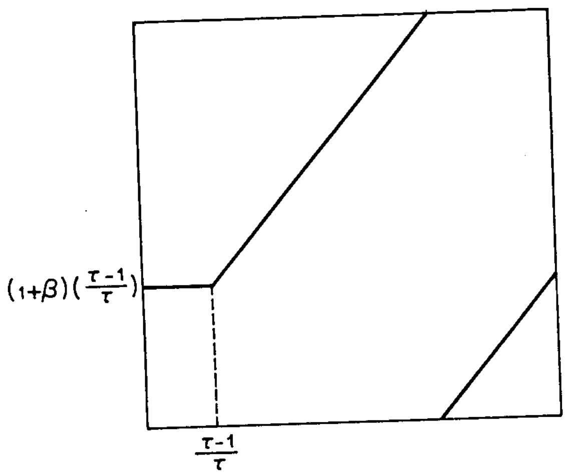

Fig. 6. Picture of $f_{\tau,\beta}$ for $\tau = 1.2$ and $\beta = 1$.

the corresponding intervals for greater τ (as in fig. 7a). However, because of the slower decay, more intervals are visible than in fig. 7a, where $\tau = 2$.

Instead of truncating $\varphi = 2x \bmod 1$, one could define a new family φ' by truncating $\varphi' = h \cdot \varphi \cdot h^{-1}$ and h is a homeomorphism (see fig. 5). It is easy to see that if we truncate φ at β and φ' at $\beta' + h(\beta)$, then $\varphi'_{\beta'}$ and φ_β are topologically conjugate (with the same conjugation h). It follows (Herman [9], II.2.10) that both have the same rotation number. The rotation number of φ_β as a function of β was given by $R(\beta)$. So,

$$R'(\beta') = R'(h(\beta)) = R(\beta)$$

or

$$R'(\beta') = R \cdot h^{-1}(\beta')$$

gives the rotation number of $\varphi'_{\beta'}$ as a function of the truncation height. An example of a conjugated family is given in fig. 5.

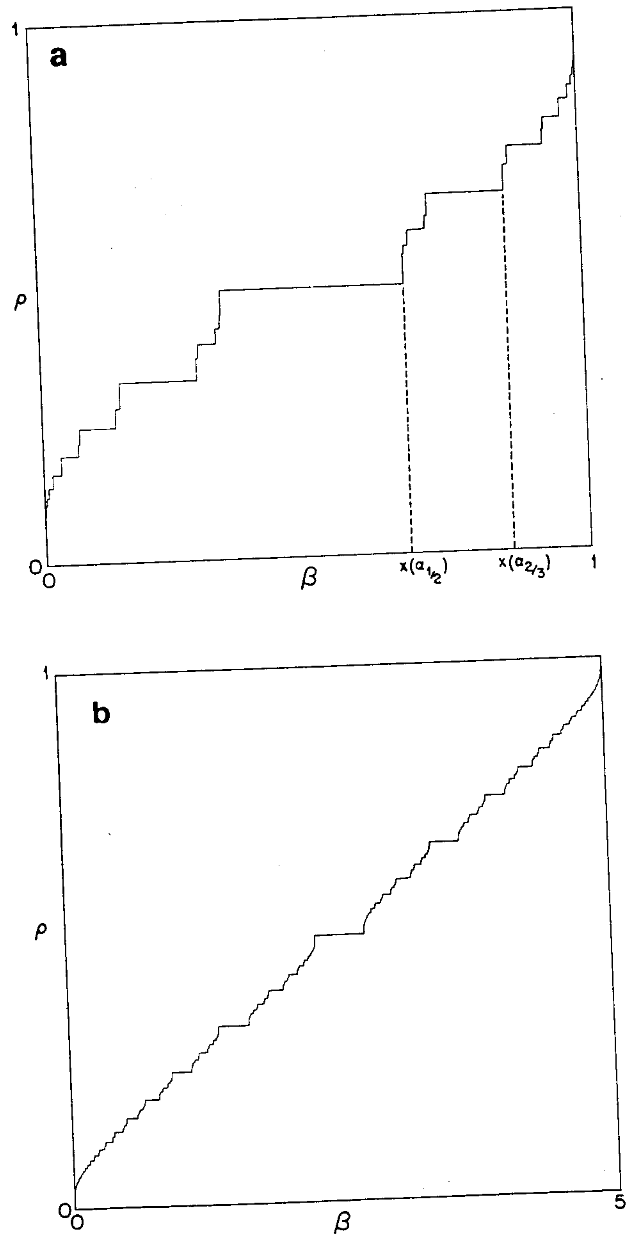

Fig. 7. (a) Rotation number versus parameter of φ_β. (b) Rotation number versus parameter of $\varphi_{1.2,\beta}$. (c) Rotation number versus parameter $\varphi'_{\beta'}$, where $\varphi' = h \cdot \varphi \cdot h^{-1}$ and $h^{-1} = 2y - 2y^2$.

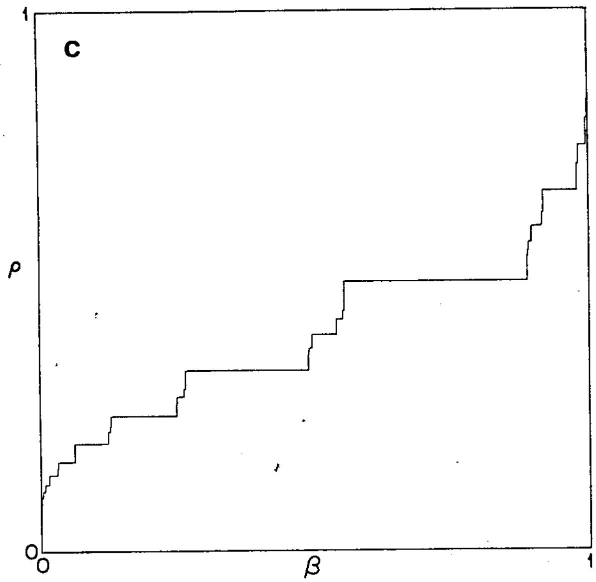

Fig. 7. continued

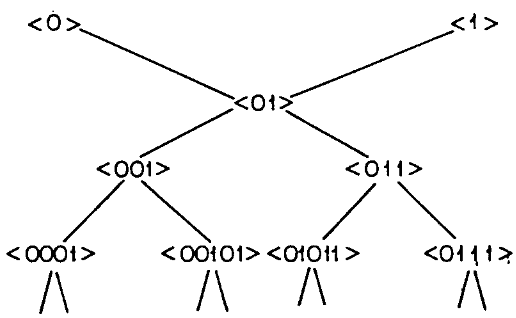

Fig. 8. Smallest elements of $\Sigma_{p/q}$.

In fig. 7c, we have plotted $R \cdot h^{-1}$, where $h^{-1} = 2y - 2y^2$. (Note that h is not differentiable. Also note that if φ and φ' are expanding, then they are conjugate according to a theorem of Shub's (Shub [26], also in Smale [27]). However, the function h is not necessarily known, nor easy to find.)

Similarly, $f_{\tau,\beta}$ and $\varphi_{\tau,\beta}$ have identical rotation numbers: $f_{\tau,\beta} = r_\delta \cdot \varphi_{\tau,\beta} \cdot r_{-\delta}$, where r_δ is a rotation by δ and $\varphi_{\tau,\beta}$ as defined before. Choosing $\delta = (\tau - 1 - \beta)/\tau$, one obtains a family, $f_{\tau,\beta}$, which is piecewise linear with slope τ, but now the flat spot occupies the interval $[0, (\tau-1)/\tau]$ for each β (see fig. 6). The new truncation height is $r_\delta(\beta) = (1+\beta)[(\tau-1)/\tau]$. We are interested in the parameter range of β such that there are no fixed points. It follows that we take $\beta \in [0, 1/(\tau - 1)]$.

As a final remark in this section, we mention that the Farey tree, as constructed in fig. 4, not only gives information about $R(\beta)$, it also gives the location of one of the periodic points (remember that the sequences are periodic) of $\varphi_{\tau,\beta}$. Applying the shift (or the inverse shift) to each sequence in the tree yields the image (or inverse image) of those points. In principle we can construct another of those points. In principle we can construct another of those points. In principle we can construct another Farey tree by applying σ^n, $n \in \mathbb{Z}$ to the current one. It can easily be proven (see Veerman [17]) that the new Farey tree can also be constructed recursively, by slightly varying the recipe given in theorem 3.1. In fig. 8 we have drawn the tree when σ^{-1} is applied to the tree in fig. 4.

4. Concluding remarks

From the considerations in the previous sections, it is clear that certain (simple) families of critical circle maps can be completely understood. In particular, the rotation number as a function of the parameter, and, for each parameter value, the structure of the orbits can be predicted. This appears to be useful in applications where circle maps that are very close to the described ones play a role. One example is that of Cherry flows on a torus (see Boyd [22]). Considering the Poincaré maps of such a flow results in a circle map with a flat spot. Another example is a measurement of internal and external rotation numbers in Birkhoff attractors of twist maps on the cylinder (Casdagli [28]). The measurement of each of these rotation numbers gives rise to a circle map with a discontinuity. The inverse of this map is a map with a gap, or, without changing the rotation number, a flat spot map.

It is tempting to try to find scaling results, as defined in Shenker [30], for the family $\varphi_{\tau,\beta}$. Following Shenker, define β_i as the parameter value at the midpoint of the q_i/Q_i resonance interval. Here q_i/Q_i are the continued fractions approaching an irrational number $\rho = [\alpha_1, \dots]$ (where α_i are the continued fraction coefficients of ρ [31]. Define

$$\delta = \lim_{i\to\infty} \left| \frac{\beta_{i-1} - \beta_i}{\beta_i - \beta_{i+1}} \right|. \tag{4.1}$$

The following exponent is also calculated in Wilbrink [29].

Proposition 4.1. $\delta = \infty$.

Proof. $\beta_{i-1} - \beta_i$ is equal to the half-width of the q_{i-1}/Q_{i-1} interval plus the half-width of the q_i/Q_i interval plus other resonance widths in between, the biggest of which has denominator $Q_{i-1} + Q_i$. Using the expression $(\tau - 1)/(\tau^q - 1)$ for the p/q resonance width, the quotient in (4.1) becomes

$$\delta = \lim_{n\to\infty} \frac{\frac{\tau-1}{2}(\tau^{Q_{i-1}} - 1)^{-1} + (\tau^{Q_i} - 1)^{-1} + S_i}{\frac{\tau-1}{2}(\tau^{Q_i} - 1)^{-1} + (\tau^{Q_{i+1}} - 1)^{-1} + S_{i+1}},$$

where the rest term S_i can be estimated as follows (i being large enough):

$$\begin{aligned} S_i &< \sum_{j=Q_{i-1}+Q_i}^{\infty} \frac{(\tau-1)\varphi(j)}{\tau^j - 1} < \sum_{j=Q_{i-1}+Q_i}^{\infty} \frac{(\tau-1)(j-1)}{\tau^j - 1} \\ &< (\tau-1)\sum_j (j+1)\tau^{-j} = (\tau-1)\partial_x\left(\frac{x^{Q_i+Q_i+1}}{1-x}\right) \\ &< \frac{(\tau-1)}{2} C(Q_i + Q_{i-1})\tau^{-(Q_i+Q_{i-1})}, \end{aligned}$$

where we have taken $x = \tau^{-1}$, C some constant depending on τ and φ is Euler's phi function, counting the relative primes p_0 to q, as defined in section 5.4 of Hardy and Wright [25]. Now, using that $Q_{i+1} = \alpha_{i+1}Q_i + Q_{i-1}$, expanding $(\tau^k - 1)^{-1} = (1/\tau^k)(1 - \tau^{-k})^{-1}$, and multiplying by $\tau^{Q_{i-1}} \cdot 2/(\tau - 1)$:

$$\delta = \lim_{n\to\infty} \frac{(1 + \tau^{-Q_{i-1}} \cdots) + \tau^{Q_{i-1}-Q_{i+1}}(1 + \cdots) + \tau^{Q_{i-1}}S_i}{\tau^{Q_{i-1}-Q_i}(1 + \cdots) + \tau^{Q_{i-1}-Q_{i+1}}(1 + \cdots) + \tau^{Q_{i-1}}S_{i+1}}.$$

All terms in this expansion, except 1 in the numerator approach zero. Therefore, $\delta = \infty$. □

Note that δ is the same for every irrational rotation number. The fact that $\delta = \infty$ agrees with the following intuition. Shenker's result [30] for the golden mean rotation number ($\delta = 2.83\dots$) is valid for maps with a cubical critical point. As the critical point becomes more degenerate (i.e., of higher order), the flat part of the map becomes more prominent, making it easier for low iterates of the map to have a fixed point. Large δ simply means that low order locking intervals are preferred with respect to higher order ones.

Define, for a one-parameter set of maps $\varphi_{\tau,\beta}$ (τ fixed), the set A as follows:

$$A = \{\beta \mid \text{the rotation number of } \varphi_{\tau,\beta} \text{ is irrational}\}.$$

Proposition 4.2. $\mu(A) = 0$ (μ is Lebesgue measure).

Proof. The idea is to add up all the lengths of the resonance intervals, and show that their sum is equal to the length of the entire parameter interval (see also Veerman [20]).

Again, φ is Euler's phi function. The function satisfies $\sum_{d|m}\varphi(d) = m$, where the sum runs over all divisors of m (including 1 and m), as in section 5.4 of Hardy and Wright [25].

$$\sum_{q\geq 2} I_{p/q} = \sum_{q\geq 2} \frac{\varphi(q)(\tau-1)}{\tau^q - 1} = (\tau-1)\sum_{q\geq 2}\varphi(q)\sum_{n\geq 2}\tau^{-nq} = (\tau-1)\sum_{k\geq 2}(k-1)\tau^{-k}$$
$$= (\tau-1)\sum_{k\geq 0}(k+1)\tau^{-k-2} = (\tau-1)\frac{\tau^{-2}}{(1-\tau^{-1})^2} = \frac{1}{\tau-1}. \qquad \square$$

Remark. A completely different proof of this fact (for a more general class of maps) was given in Boyd [22]. In fact, in his article, Boyd also proves that the Hausdorff dimension of A equals zero.

Acknowledgements

The author would like to thank Martin Casdagli, Stellan Ostlund, Philip Holmes, John Guckenheimer and Mitchell Feigenbaum for their valuable ideas and discussions on the subject matter. This work is partially supported by NSF grant MEA 84-02069.

References

[1] J. Guckenheimer and P.J. Holmes, Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields, (Springer, Berlin, 1983).
[2] S. Aubry, Physica 7D (1983) 240–258.
[3] S. Aubry and P.Y. LeDaeron, Physica 8D (1983) 381–422.
[4] J.N. Mather, Topology 21 (1982) 457–467.
[5] A. Katok, Erg. Th. Dyn. Syst. 2 (1982) 185–194.
[6] R.S. Mackay and J. Stark, Orbits of minimal action for area-preserving maps, preprint, Univ. of Warwick (1985).
[7] J.D. Meiss, R.S. Mackay and I.C. Percival, Phys. Rev. Lett. 52 (1984) 697–700.
[8] Z. Nitecki, Differentiable Dynamics (MIT Press, Cambridge, MA, 1971).
[9] M.R. Herman, Publ. Math. IHES no. 49 (1979) 5–234.
[10] M.H. Jensen, P. Bak and T. Bohr, Phys. Rev. Lett. 50, No. 21 (1983) 1637–1639.
[11] M.H. Jensen, P. Bak and T. Bohr, Phys. Rev. A 30 (1984) 1960–1969.
[12] J.M. Gambaudo, O. Lanford and C. Tresser, C.R. Acad. Sci. Paris 299, serie 1 (1984) 823–826.
[13] C. Berhardt, Proc. London Math. Soc. (3) 45 (1982) 258–280.
[14] M.J. Feigenbaum, Renormalization of the Farey map, preprint, Cornell Univ. (1984).
[15] S. Ostlund and S. Kim, Phys. Scr. T9 (1985) 193–198.
[16] P. Cvitanovic, B. Shraiman and B. Soderberg, Phys. Scr. 32 (1985) 263–270.
[17] J.J.P. Veerman, On resonance widths in dynamical systems, Thesis, Cornell Univ. (1986).
[18] L.P. Kadanoff, J. Stat. Phys. 31 (1983) 1–27.
[19] P.L. Boyland, Bifurcations of circle maps, Arnol'd Tongues, bistability and rotation intervals, preprint, Boston Univ. (1983).
[20] J.J.P. Veerman, Physica 134 A (1986) 543–576.
[21] J. Nagumo and S. Sato, Kybernetik, 10 Bd. Heft3 (1972) 155–164.
[22] C. Boyd, Erg. Th. Dyn. Syst. 5 (1985) 27–46.
[23] G.A. Hedlund, Am. J. Math. 66 (1944) 605–620.
[24] K. Hockett and P.J. Holmes, Bifurcation to rotating Cantor sets in maps of the circle, preprint, Cornell Univ. (1986).
[25] G.M. Hardy and E.M. Wright, An Introduction to the Theory of Numbers (Oxford Univ. Press, Oxford, 1960).
[26] M. Shub, in: Global Analysis, S. Chern and S. Smale, eds., Am. Math. Soc. (1970).
[27] S. Smale, Bull. Am. Math. Soc. 73 (1967) 747–817.
[28] M. Casdagli, Measuring chaos using rotation numbers: summary, preprint, La Jolla Institute Center for Nonl. Dyn. (1986).
[29] G. Wilbrink, Analytic results for the scaling behaviour of a piecewise-linear map of the circle, preprint, Univ. of California at Santa Cruz (1986).
[30] S.J. Shenker, Physica 5D (1982) 405–411.
[31] A.Ya. Khinchin, Continued Fractions (Univ. of Chicago Press, Chicago, 1964).