Some Current Research Interests

References are to my cv. Cross-references to my website at the bottom of this document. An updated version is maintained on my website: `http://web.pdx.edu/~veerman/`

Dynamics on Networks

General:

A dynamical system is essentially a rule that specifies how a system changes at any given moment. Examples of this are differential equations or difference equations. The central problem of dynamical systems is determining how the system evolves over a long time. Much of my research centers around dynamical systems and their applications in various areas of mathematics, science, and engineering. Recently, I have had a strong interest in large networks of coupled low-dimensional dynamical systems. If one of these systems influences another, we can draw an directed arrow between them. Thus we obtain a directed graph (or directed network) of similar systems. Methods to study such systems are a mix of dynamical systems and graph theory. These systems have many applications across all the sciences. With my students (and other collaborators), examples we have studied are social networks, movement of flocks (in particular traffic), a simplified model for biological evolution, and chemical reaction networks.

More Details:

Among the most common dynamical processes on directed graphs are Laplacian processes. Roughly speaking, one can divide these into two classes. One type is what we call "consensus". Every agent takes its input from "parents" in the graph, and averages it to define its position, velocity, opinion, etc. The other type is what we call "diffusion". It it, every agent sends its information, split in some way, to its "children". Graph theoretic Laplacians are defined to be row-stochastic with non-negative diagonal elements and non-positive off-diagonal elements. It turns out that one can consider these two processes as dual to one another in the following sense. Suppose we are given a Laplacian L. Let x be column vector in $\mathbb{R}^v$ where v is the number of vertices of the graph. A consensus process can be described by $\dot{x} = -Lx$, whereas a diffusion process by $\dot{x}^T = -x^T L$. A Laplacian is a matrix whose row-sum is zero; its diagonal elements are non-negative, while its non-diagonal elements are non-positive.

Over the years, we have formulated a standardized theory for such processes in arbitrary directed graphs (1.38, 4.9, 1.57). While some (though not all) of these results were known in diverse areas of applied science, no complete treatment existed in the mathematics literature. Among the main results are that for any Laplacian L, all eigenvalues are non-negative and all zero eigenvalues are simple. Furthermore, a precise characterization of the left and right kernel of L is given. We have proceeded to use these results to understand other applied problems. For example, in a recent paper (5.1), we drastically simplified the basic mathematical theory describing chemical reaction networks. Chemical networks consist of a (possibly very large) set of polynomial equations. The surprise is that, in spite of the highly nonlinear character of these equations, one can often analytically prove the existence of asymptotically stable fixed points, just by looking at the underlying network formed by the reactions and the properties of an associated Laplacian.

Another effort in this area is aimed at understanding the behavior of (linear) flocks. Here the idea is that each agent in a flock determines its acceleration based on its position relative to some of its neighbors and its velocity relative to its neighbors. This gives a problem similar to consensus, but now the equations are of second order as opposed to first order. Note that this is essentially a more general version of the study of coupled oscillators that underlies much of physics (where usually certain symmetries are present that simplify the problem). Using our earlier theory of Laplacians, we formulated a now very widely cited theory for general flocks whose neighbor structure is that of a general directed graph (1.35, 1.36, 1.38). More recently, we have studied 1-dimensional identical oscillators on the line, modeling certain traffic situations (1.49, 1.50, 1.51). In these papers, we generalized the usual 'periodic boundary conditions' approach. This enabled us to derive quantitative predictions for the dynamics of these models, such as criteria for stability and, if stable, the amplitude of the stop-and-go oscillation typical in very dense traffic. An even further generalization led to analytic work (5.4) on traffic situations where for example one knows only the distribution of the masses of the cars, but not the actual individual masses.

Dynamics on directed graphs is very active area of research, that has many applications in nearly all areas of science. Even in the social sciences, dynamical processes on graphs form an important tool. An example is the remarkable result that one very prevalent (and nonlinear) model for opinion forming in a group predicts the splitting up of that group into two antagonistic factions (1.54).

To the best of our knowledge there appear to be no good textbooks suitable for a graduate mathematics course on the theory of directed graphs. Part of our effort in this direction is to provide such a book based on lecture notes already available on my website (`http://web.pdx.edu/~veerman/`) and some of the papers mentioned here.

Mediatrices

General:
This research is closer to what is considered 'pure' mathematics. An important effort is dedicated to trying to figure out what equidistant sets in a metric space look like. Say we have a metric space X two disjoint (compact) sets A and B in it. A set $L \subset X$ is a mediatrix if each point $x \in L$ has the same distance to A as to B. It is as though A and B are both very jealous, and you can only live at locations that equally close (or far) from A and B. What does that set L look like? Our interest in this topic is mainly in the topological properties (such as how connected are they?) and geometric properties (how fractal can they be?) of these sets. This notion has many applications throughout the sciences and mathematics, and even in real life. For instance in methods of shape matching, after metrizing the space of shapes, one needs to determine whether a given shape is closer to one or the other of two shapes. In real life, the international court of justice in The Hague has sought to settle international territorial disputes by referring to "equidistant sets".

More Details:
In slightly more technical language, if X is a metric space, and A and B are disjoint sets, then L

would be given by:

$$\{x \in X : d(x, A) = d(x, B)\},$$

where d means distance to the closest point. Our initial interest was more restricted, namely A and B are distinct points and X a 2-dimensional closed manifold (a surface). The original motivation was to study focusing of geodesics on, for example, a torus (1.24). If we send out geodesics from a point a on a torus, then where (and when) do two or more geodesics meet? To find out, look at the universal cover of the torus, choose two representatives A and B of the point a. Geodesics out of a will meet at the projection (back to the torus) of the mediatrix relative to A and B.

After some time, we realized that surface mediatrices can be characterized topologically, namely they are finite embedded graphs that are *minimal separating sets*. This led to an effort to classify surface mediatrices topologically, initially using homology (1.31, 1.34). It turns out that there is one possibility in the sphere (the Jordan curve theorem), namely the topological circle. There are 5 possibilities in the torus (1.34). Further classification is difficult, because the combinatorial possibilities grow so fast. However, in an unpublished paper (`http://web.pdx.edu/~veerman/minseps.pdf`), we proved (now using methods of combinatorial topology) that in genus 2 surfaces, there are (up to topology) 26 types of mediatrices, while in genus 3, there are 217 types. Because of the combinatorial explosion, part of the proof is a numerical counting procedure. We are currently working on getting a user-friendly version of that program up and running, so that we can publish these results.

Another line of research aims to determine geometric properties of mediatrices. Surface mediatrices with respect to points A and B are embedded minimal separating graphs as observed above, but they are also Lipschitz structures. Near a vertex, all angles between incoming edges are (strictly) positive. Something stronger is true. At each vertex x of a mediatrix, we can draw shortest geodesics to the points A and B. These must make a positive angle with one another. One can define the bisector: the geodesic that divides this angle in 2. We proved that the mediatrix must be tangent to this bisector. This leads to a theorem where in some cases the total angle of all the singularities in the derivative of the mediatrix is bounded by the total curvature of the manifold (1.52).

Future work is aimed at generalizing to mediatrices in Alexandrov spaces (a generalization of smooth compact manifolds) with respect to disjoint compact *sets* (as opposed to points) A and B. The most important step in this direction has already been completed. This is, very loosely speaking, the statement that if you travel along a geodesic, the distance to a given set A has a one-sided derivative (1.61). This should enable us to answer open questions such as (Mario Ponce, Patricio Santibez (2014) On Equidistant Sets and Generalized Conics: The Old and the New, The American Mathematical Monthly, 121:1, 18-32): Characterize mediatrices in $\mathbb{R}^2$ with respect to disjoint closed connected sets.

Statistical Properties of Dynamical Systems

General:

Other research concerns the statistical properties of dynamical systems. Here the central question is how a system evolves ‘on average’. Ultimately, this line of thought came from the realization in the beginning of the 20th century, that for many physical systems, we do not really care much for the individual orbit, but for the aggregate behavior of the system. The classical example is air in a room. We are interested in its temperature (average kinetic energy of the molecules) and not in the details of the motion of every molecule. This gave rise to ergodic theory which studies statistical properties of dynamical systems.

It has been know for a long time that in very chaotic systems, the typical distribution of physical variables, such as the velocity mentioned earlier, is Gaussian. In a sense, these systems are extreme. Many common physical systems exhibit some form of long range interaction. In the last thirty or so years, Constantino Tsallis and others have formulated a more general distribution, sometimes called Tsallis distribution, of which the Gaussian distribution is a special case. They found evidence that variables of many physical systems with long range interaction indeed satisfy that more general distribution (but not the more specialized Gaussian one). In a recent publication (1.59), we proved a statistical result for a very simple simple system, that is *consistent* with Tsallis statistics. And even though weak evidence, it appears to be the first *analytic* result in this direction. We are currently working on further analytic results along these lines.

Another line of research in this direction is the study of the so-called Bak-Sneppen model. This is a highly controversial model of evolution of species formulated in the 90's of the last century. Given N vertices on a circle, we assign each of these a "fitness" in $[0, 1]$. The dynamics takes the following form: at each time-step, locate the vertex with the lowest fitness (or "rank") and replace it *and its immediate neighbors* with new fitnesses chosen from a uniform random distribution. The motivation is that the least fit species of course disappears, but also the species in close contact with it. The result is that eventually all species have a fitness in an interval $[c, 1]$, where c is very close to 1/3. There is still no analytic proof of this fact.

More Details:

The system in (1.59) that we found to be consistent with Tsallis statistics is the so-called standard map on the torus with $k = 0$:

$$f(x, y) = \begin{cases} x & \to & x + y \\ y & \to & y \end{cases}$$

Since this is a system laminated by pure rotations, this seems a simple problem. However, we had to rely on a difficult ergodic theorem by Kesten in 1960 to solve this. The upshot is that if you compute the average over randomly chosen initial conditions (x_0, y_0) of $\sum_{i=0}^{n-1} (x_i - 1/2)$, the distribution is Cauchy. (Here, x_i is the first coordinate of $f^i(x_0, y_0)$.) As it turns out, this is a special case of the Tsallis distribution but is not normal. To appreciate that this is not trivial, we also noted the following: for *most* individual rotations (i.e. fix y_0), the statement is actually *false*. We are currently looking into other models for which we can determine the statistical analytically (5.2).

As far as the Bak-Sneppen model is concerned, the source of the mathematical problems is the coupling between "rank" and space (the neighbors of the "worst" vertex are also replaced). We showed (1.45, 1.47) that under a certain unproved hypothesis the dynamics reduces to a simple dynamical system on the space of cumulative distribution functions. In turn, that dynamical system can be completely analyzed using the theory of 1-dimensional discrete dynamical systems on the interval. The hypothesis is (very roughly) that one of the neighbors of the lowest ranked species on average has a very low rank, and the other neighbor has an arbitrary rank (on average). We are currently looking into other versions of this model. For example, one can require that the species with the worst *relative* fitness (its fitness minus an average of the fitnesses of its neighbors) is replaced (and possibly its immediate neighbors).

Other

General:
Because dynamical systems intersects so many other areas of science and mathematics, many other topics occasionally attract my attention. In the interest of brevity, I make only brief mention of some of these here. These are in no particular order.

I have always had an interest in number theory, though I have rarely been able to follow up on it, except early on, in the context of low-dimensional dynamical systems (1.4, 7.5). As luck would have it, I have now taught undergraduate number theory for some years, and found the books I used wanting quality. I have been working on a general beginning graduate course in number theory, covering all aspects. Just the way someone driven by curiosity but without a lot of prior knowledge would approach it. The first part of (I hope) a future three part course, can be found on my website.

A conversation with John Milnor a few years ago, after I gave a talk at Stony Brook University, led to a new algorithm to monitoring the distance to a convex set in $\mathbb{R}^3$ while flying past it (1.58). The algorithm uses the curvature tensor of the surface forming the boundary of the convex set. The algorithm works if the boundary of the convex set is C^2. In the paper, we also give an example of a "strange convex set" in $\mathbb{R}^2$ whose boundary is $C^{(1,1)}$ and where the algorithm is guaranteed fail.

Tridiagonal matrices are important tools. The most familiar applications are discretizations some of the most common partial differential equations, and in one dimensional arrays of coupled ordinary differential equations. They are crucial in the study of crystal vibrations in solid physics and in studies of flocks on the line (see above). In (1.56) we derive explicit expressions for all eigenvalues of certain (large) class of tridiagonal matrices. These expressions are accurate to order $\mathcal{O}(n^{-2})$ where n is the dimension of the matrix. This leads to estimates of the associated eigenvectors accurate to order $\mathcal{O}(n^{-1})$ for each component. We also investigated the dependence of these quantities on boundary conditions, both in theory and in two practical applications (a system of coupled ODE and a PDE) and investigates the validity of the "periodic boundary" approach used in many applications without rigorous justification.

Cross-references to documents on my website

1.4 :`http://web.pdx.edu/~veerman/symbolic2.pdf`
1.24: `http://web.pdx.edu/~veerman/brillouin.pdf`
1.31: `http://web.pdx.edu/~veerman/mediatrix2.pdf`
1.34: `http://web.pdx.edu/~veerman/mediatrix3.pdf`
1.35: `http://web.pdx.edu/~veerman/flocks1.pdf`
1.36: `http://web.pdx.edu/~veerman/flocks2.pdf`
1.38: `http://web.pdx.edu/~veerman/laplacian.pdf`
1.45: `http://web.pdx.edu/~veerman/RankDriven.pdf`
1.47: `http://web.pdx.edu/~veerman/Erratumv2.pdf`

1.49: http://web.pdx.edu/~veerman/ThirdOrderSystem.pdf
1.50: http://web.pdx.edu/~veerman/signalvelocity.pdf
1.51: http://web.pdx.edu/~veerman/Transients.pdf
1.52: http://web.pdx.edu/~veerman/mediatrix-regularity.pdf
1.56: http://web.pdx.edu/~veerman/tridiagonal3.pdf
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4.9 : http://web.pdx.edu/~veerman/SurveyLaplacian.pdf
5.1 : http://web.pdx.edu/~veerman/ChemReacNets.pdf
5.2 : http://web.pdx.edu/~veerman/Statistics_of_Certain_1D_Maps.pdf
5.4 : in preparation, http://web.pdx.edu/~veerman/publ20.html
7.5 : unpublished, http://web.pdx.edu/~veerman/herman.pdf