A Remark on the Prime Number Theorem

J. J. P. Veerman*, F. M. Tangerman†

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Abstract

In this note, we state various somewhat unusual formulations (Theorems 1.3 and 2.2) of the Prime Number Theorem (PNT), that serve to underline how remarkably intertwined the multiplicative structure of integers is with the transcendental number e. Theorem 2.2 concerns the limiting behavior of the difference of the two Chebyshev functions. While close to certain inequalities in the literature and easy to prove, we have been unable to find this particular limit in the literature.

We then use this Theorem to devise a numerical test of the Riemann Hypothesis (RH) which is very successful in the range we tested and consistent with the truth of RH. We show however that it follows from a theorem of Hardy and Littlewood (which is independent of the truth of the RH), that a numerical test like the one proposed must eventually fail, see Theorem 3.1! This conveys how subtle and resistant to numerical attack the RH is.

KEYWORDS: Prime Number Theorem, Riemann Hypothesis

1 Introduction

In this work, a sum or product over p means a sum or product over the primes in $\mathbb{N}$.

Definition 1.1 *The first Chebyshev function $\theta(x)$ is defined as*

$$\theta(x) := \sum_{p\leq x} \ln p = \ln\left(\prod_{p\leq n} p\right) .$$

The second Chebyshev function $\psi(x)$ is defined as

$$\psi(x) := \sum_{p\leq x} \left\lfloor \frac{\ln x}{\ln p} \right\rfloor \ln p = \ln\left(\,\mathrm{lcm}\left(\{1,2,\cdots,n\}\right)\right) .$$

*veerman@pdx.edu, Math, Portland State University, Portland, OR, USA.
†fmtangerman@gmail.com, FM Tangerman LLC, Hartford CT, USA.

Notice that $e^{\theta(x)}$ is a divisor of $e^{\psi(x)}$: every prime in the prime factorization of $\operatorname{lcm}(\{1,2,\cdots,\lfloor x\rfloor\})$ occurs in the product $\prod_{p\leq x} p$. The difference is that in the latter they occur with a power 1.

Recall that the prime number theorem (PNT) states:

Theorem 1.2 (PNT) *Let $\pi(x)$ denote the number of primes less than or equal to x.*

$$\lim_{x\to\infty} \frac{\pi(x)\ln x}{x} = 1\,.$$

The Chebyshev functions play a central role in most common proofs of the prime number theorem (see for example [7], [1]). This involves showing the following result.

Theorem 1.3 *The following statements hold:*

$$\lim_{x\to\infty} e^{\frac{\psi(x)}{x}} = \lim_{x\to\infty} \left(\operatorname{lcm}(\{1,2,\cdots,\lfloor x\rfloor\})\right)^{1/x} = e \qquad \text{and} \qquad \lim_{x\to\infty} e^{\frac{\theta(x)}{x}} = \lim_{x\to\infty} \left(\prod_{p\leq x} p\right)^{1/x} = e\,.$$

Furthermore, each of these statements readily implies the PNT and vice versa.

To illustrate how remarkable these two statements are, compare them to two other products of natural numbers (which are easily verified using Stirling's formula):

$$\lim_{x\to\infty} \left(\prod_{i\leq x} i\right)^{1/x} = \infty \qquad \text{and} \qquad \forall d>1\,:\, \lim_{x\to\infty} \left(\prod_{i^d\leq x} i^d\right)^{1/x} = 1\,.$$

An even more striking result is stated in Theorem 2.2.

It is here that we need to introduce the Riemann zeta function. The following expansions converge and are equal for $\operatorname{Re} z > 1$:

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z} = \prod_{p \text{ prime}} (1-p^{-z})^{-1}\,.$$

The zeta function is the analytic continuation to all complex numbers $z\neq 1$ (it has a simple pole at 1).

Conjecture 1.4 (Riemann Hypothesis) *[4] All non-real zeros of $\zeta(z)$ lie on the line $\operatorname{Re} z = \frac{1}{2}$.*

The machinery employed to prove the PNT uses that there are no zeroes on $\operatorname{Re} z \geq 1$. For $\operatorname{Re} z > 1$, this can be proved by standard complex analytic methods, but the proof of the absence of zeros with $\operatorname{Re} z = 1$ is considered a very weak version of the RH. It was achieved by [2] and [5] who thereby gave the first proof of the PNT. The (now) standard proofs of the PNT all follow a similar strategy. First show that

$$\int_1^{\infty} \frac{\theta(x)-x}{x^{1+z}}\,dx = -\frac{\zeta'(z)}{z\zeta(z)} - \frac{1}{z-1} - \frac{1}{z}\sum_p \frac{\ln p}{p^z(p^z-1)}\,, \tag{1.1}$$

or, equivalently, that

$$\int_1^\infty \frac{\psi(x)-x}{x^{1+z}}\,dx = -\frac{\zeta'(z)}{z\zeta(z)} - \frac{1}{z-1}\,. \tag{1.2}$$

A combinatorial argument establishes that (a) $\theta(x)/x$ is bounded (and $\psi(x)/x$ as well). The right hand side is then manipulated to show that (b) ζ has no zeroes[1] on the line $\mathrm{Re}\, z = 1$ and thus the right hand side has an analytic continuation to $\mathrm{Re}\, z \geq 1$ with a simple pole at 1. Finally, a complex analytic (Tauberian) theorem yields that (a) and (b) imply that $\lim_{x\to\infty}\theta(x)/x = 1$ (or $\lim_{x\to\infty}\psi(x)/x = 1$). The latter is equivalent to the prime number, as mentioned[2]. A detailed description of this proof can be found in [6].

2 Can the PNT Say More about the RH?

We now turn the tables and try to leverage the prime number theorem to glean more information about the location of the zeroes of the zeta function.

Lemma 2.1 *Let $x \geq 2$, then $\theta(x^{1/2}) + \theta(x^{1/3}) \leq \psi(x) - \theta(x) \leq \left(\frac{1}{2}\pi(x^{1/2}) + \frac{1}{2}\pi(x^{1/3})\right)\ln x$.*

Proof. Clearly, $\prod_{p\leq n} p$ is a factor of $\mathrm{lcm}\,(\{1,2,\cdots,n\})$. We consider two methods to bound the quotient.

First, for every prime $p \leq n^{1/2}$, lcm gets a factor p extra. Furthermore, for every prime $p \leq n^{1/3}$, lcm gets another factor p extra, and so forth. Thus we see that

$$\frac{\mathrm{lcm}\,(\{1,2,\cdots,n\})}{\left(\prod_{p\leq n} p\right)} = \left(\prod_{p\leq n^{1/2}} p\right)\cdots\left(\prod_{p\leq n^{1/\ell}} p\right),$$

where ℓ equals $\lfloor \log_2 n \rfloor$. Taking logarithms, we obtain a well-known equality (and a lower bound)

$$\psi(x) - \theta(x) = \sum_{k=2}^{\ell} \theta(x^{1/k}) \geq \theta(x^{1/2}) + \theta(x^{1/3})\,.$$

For the second method, we re-order the factors in the above quotient as follows.

for all p in	the extra factor is in
$\left(x^{1/3}, x^{1/2}\right]$	$\left(x^{1/3}, x^{1/2}\right]$
$\left(x^{1/4}, x^{1/3}\right]$	$\left(x^{2/4}, x^{2/3}\right]$
$\left(x^{1/5}, x^{1/4}\right]$	$\left(x^{3/5}, x^{3/4}\right]$
$\left(x^{1/6}, x^{1/5}\right]$	$\left(x^{4/6}, x^{4/5}\right]$

and so on. For simplicity, denote $\pi(x^{1/k})$ by π_k. We then obtain

$$\frac{\mathrm{lcm}\,(\{1,2,\cdots,n\})}{\left(\prod_{p\leq n} p\right)} \leq n^{\frac{1}{2}(\pi_2-\pi_3)} \cdot n^{\frac{2}{3}(\pi_3-\pi_4)} \cdot n^{\frac{3}{4}(\pi_4-\pi_5)} \cdots,$$

[1] This is where the weak RH comes in!

[2] In fact, the equality $\theta(n) \sim n$ literally means that the sum of the lengths of the intervals $[p, p+\ln p)$ for primes $p \leq n$ roughly equals n. Thus on average the distance between successive primes p and p' roughly equals $\ln p$.

where π_k with $k > \lfloor \log_2 x \rfloor$ is zero. Taking logarithms again and then re-arranging a bit, we get:

$$\begin{aligned}\psi(x) - \theta(x) &\leq \left(\frac{1}{2}\pi_2 + \pi_3\left(\frac{2}{3} - \frac{1}{2}\right) + \pi_4\left(\frac{3}{4} - \frac{2}{3}\right)\cdots\right)\ln x \\ &\leq \left(\frac{1}{2}\pi_2 + \pi_3\left(\frac{2}{3} - \frac{1}{2} + \frac{3}{4} - \frac{2}{3} + \frac{4}{5} - \frac{3}{4} + \cdots\right)\cdots\right)\ln x = \left(\frac{1}{2}\pi_2 + \frac{1}{2}\pi_3\right)\ln x\end{aligned}$$

This finishes the proof. ∎

Theorem 2.2 *The following statement holds (and is equivalent with the PNT):*

$$\lim_{x\to\infty} e^{\frac{\psi(x)-\theta(x)}{\sqrt{x}}} = \lim_{x\to\infty}\left(\frac{\operatorname{lcm}\left(\{1,2,\cdots,\lfloor x\rfloor\}\right)}{\prod_{p\leq x} p}\right)^{1/\sqrt{x}} = e\,.$$

Proof. Divide the statement of Lemma 2.1 by $x^{1/2}$. By Theorem 1.3, the limit of the left hand side equals 1. The PNT states that $\lim_{x\to\infty} \pi_2 \ln x / x^{1/2} = 2$. Therefore, the fact that the limit of the right hand side equals 1 is precisely the PNT. The final result is obtained by exponentiating these inequalities. ∎

The function $\psi(x) - \theta(x)$ appears to be (for a number theoretical function) *shockingly* well-behaved at least for values on the order of 10^7. In fact, Lemma 2.1 implies that for every $\epsilon > 0$, we can choose x sufficiently large so that

$$(1-\epsilon)(x^{1/2} + x^{1/3}) < \psi(x) - \theta(x) < (1+\epsilon)(x^{1/2} + 1.5x^{1/3})\,.$$

In the indicated range of x, however, $\psi(x) - \theta(x)$ appears almost indistinguishable from $x^{1/2} + x^{1/3}$, as indicated Figure 2.1. It seems plausible that for higher values of x, one can get ever better approximations by including more terms in Lemma 2.1.

Lemma 2.3 *Let* $Q(z) := \int_1^\infty \frac{q(x)-x}{x^{1+z}}\,dx$*. If* $|q(x) - x| = O\left(x^{\mu+\epsilon}\right)$ *for all* $\epsilon > 0$*, then* $Q(z)$ *is analytic on* $\operatorname{Re} z > \mu$*.*

Proof. We chop up the integral to get a summation. So

$$Q(z) = \sum_{n\geq 1} \int_n^{n+1} \frac{q(x)-x}{x^{1+z}}\,dx\,.$$

Choose a compact disk D in $\operatorname{Re} z > \mu$. Now choose $\epsilon > 0$ small enough so that on D, $\operatorname{Re} z \geq \mu + 2\epsilon$. By hypothesis, we have that there is a $k_\epsilon > 0$ such that $|q(x) - x| \leq k_\epsilon\, x^{\mu+\epsilon}$. Thus

$$\left|\int_n^{n+1} \frac{q(x)-x}{x^{1+z}}\,dx\right| \leq \int_n^{n+1} \left|\frac{q(x)-x}{x^{1+z}}\right| dx \leq k_\epsilon\, n^{\mu - \operatorname{Re} z + \epsilon - 1}\,.$$

Therefore sums of the latter are uniformly convergent on D, and that means that $Q(z)$ is analytic on $\operatorname{Re} z > \mu$. ∎

We see that the only way to violate the RH is as follows.

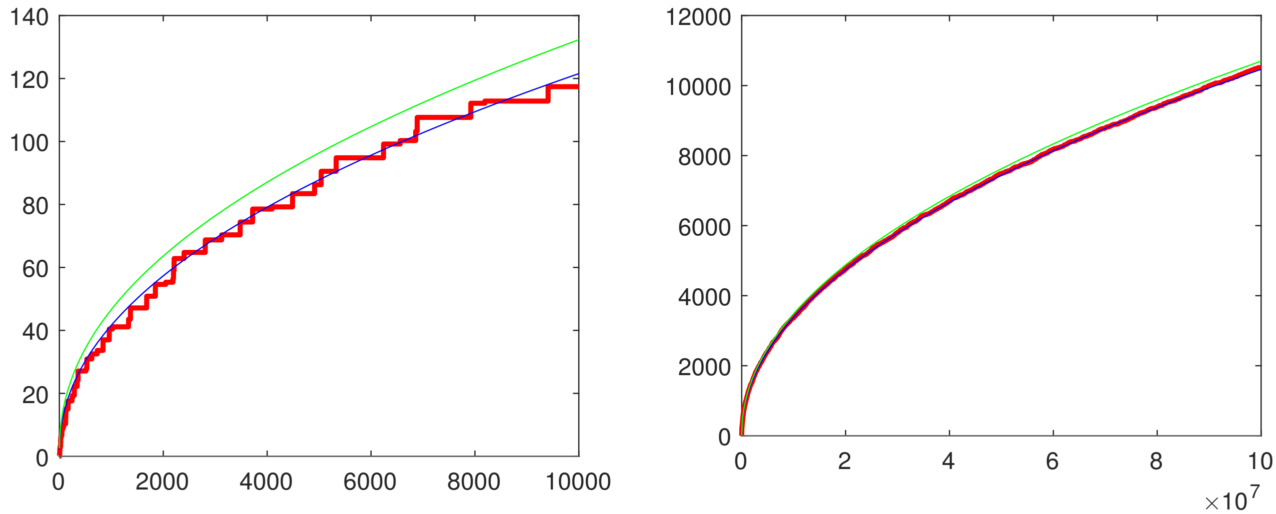

Figure 2.1: *The functions* $\psi(x) - \theta(x)$ *in red,* $x^{1/2} + x^{1/3}$ *in blue, and* $x^{1/2} + 1.5x^{1/3}$ *in green. Left:* x *in* $[1, 10^4]$*. Right:* x *in* $[1, 10^8]$.

Corollary 2.4 *If the RH is false, then* $\left\{\frac{\psi(n)-n}{\theta(n)-n}\right\}_{n=1}^{\infty}$ *has a subsequence that converges to 1.*

Proof. In view of Lemma 2.3 and equations (1.1) and (1.2), if one of $\theta(x) - x$ or $\psi(x) - x$ is $O(\sqrt{x})$, then $\zeta(z)$ cannot have zeroes in $\operatorname{Re} z > 1/2$ and the Riemann hypothesis is implied! Theorem 2.2 implies that $\psi(x) - \theta(x)$ is $O(\sqrt{x})$, and so if $\psi(x) - x$ or $\theta(x) - x$ is much larger than $\sqrt{x}$ in absolute value, the other one must be close to it. ■

3 Promising Results and the 'Escape' of the RH

We plot $\theta(x) - x$ and $\psi(x) - x$ in Figure 3.1. From Corollary 2.4, it is clear that to refute the RH, both red and blue curves in that figure need to have the same sign and their quotient needs to approach 1 infinitely often. That doesn't seem to be the case at the scale considered here. One might optimistically be inclined to interpret the results as supportive of the RH. Indeed, from Figure 3.2, it appears that, except for (finitely many) small x, the quotient $\frac{\psi(x)-x}{\theta(x)-x}$ stays well below 0.5 ! Choose any $\epsilon > 0$. Theorem 2.2 allows us to replace $\psi(x)/\sqrt{x}$ by $(1 + \epsilon(x))\theta(x)/\sqrt{x}$, where for x large enough $|\epsilon(x)|$ is smaller than ϵ.

$$\frac{\theta(x) + (1 + \epsilon(x))\sqrt{x} - x}{\theta(x) - x} = 1 + \frac{(1 + \epsilon(x))\sqrt{x}}{\theta(x) - x}\,, \tag{3.1}$$

It is easy to see that if the quotient in (3.1) is less than 1/2, then

$$-2(1 + \epsilon(x))\sqrt{x} < \theta(x) - x < 0\,,$$

which would imply the RH. It is thus clear that (at these scales) we are *very far* from refuting the Riemann Hypothesis. It is not a close call. Even more interesting: the pictures plotted on very

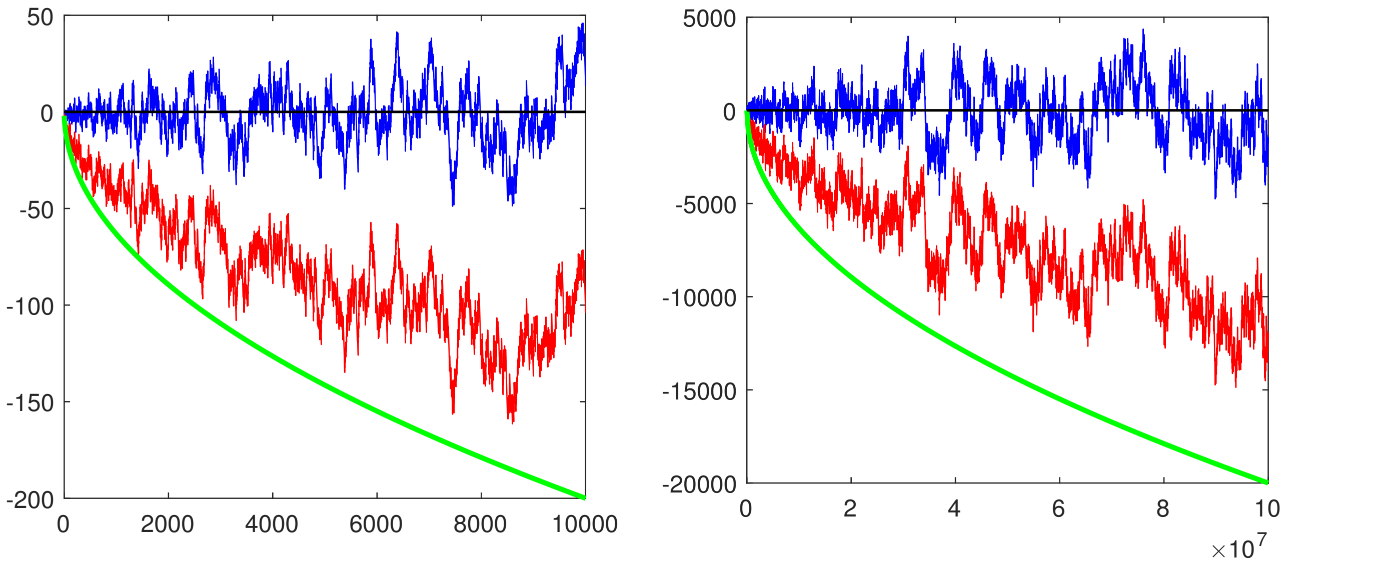

Figure 3.1: *Left: the functions* $\psi(x) - x$ *in blue and* $\theta(x) - x$ *in red for* x *in* $[1, 10^4]$*. We added* $-2\sqrt{x}$ *in green. Right: the same for* x *in* $[1, 10^8]$*.*

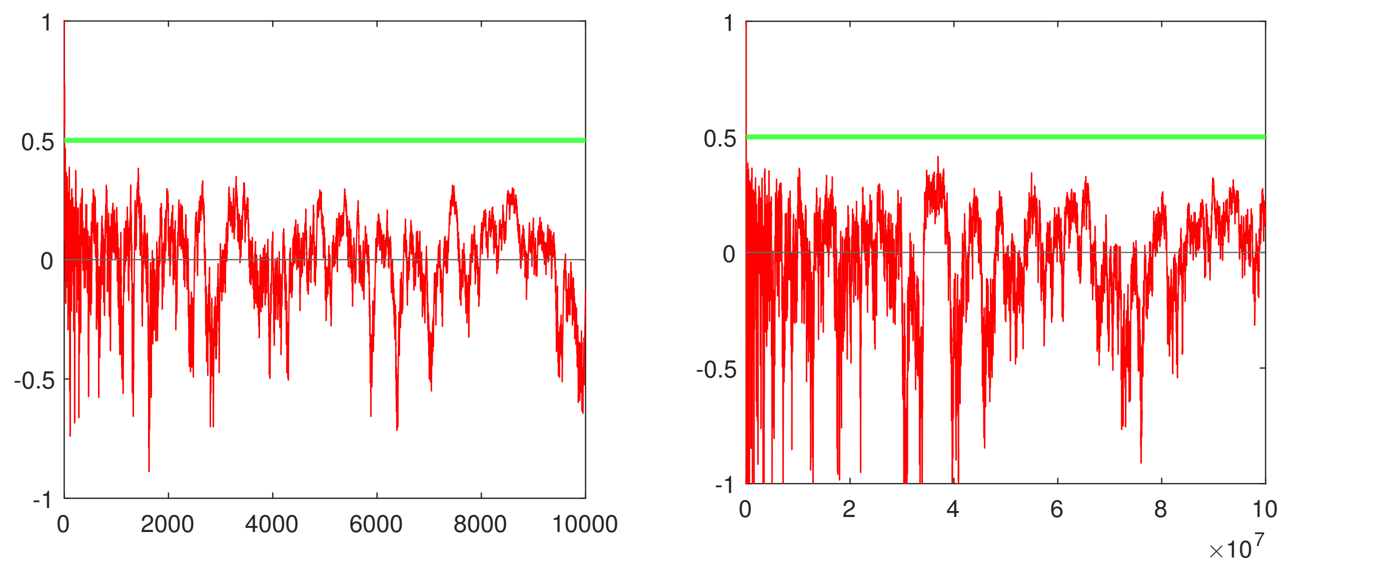

Figure 3.2: *Left:* $\frac{\psi(x)-x}{\theta(x)-x}$ *for* x *in* $[1, 10^4]$*. Right: for* x *in* $[1, 10^8]$*. We added* $y = 0.5$ *in green.*

different scales, look almost indistinguishable. That might lead one to intuit that perhaps the pattern continues.

Finally, here is how the RH defeats us: Hardy and Littlewood [3] proved the following result.

Theorem 3.1 $|\psi(x) - x| \neq o(\sqrt{x} \ln \ln \ln x)$.

Let us reflect on this for a moment. The Hardy-Littlewood result itself does not come close to refuting the RH, because $\psi(x) - x \neq o(\sqrt{x} \ln \ln \ln x)$ does *not* exclude $\psi(x) - x = O(x^{0.5+\epsilon})$ for all $\epsilon > 0$. But it *does* imply that for all $K > 0$, indeed, $|\psi(x) - x| > K\sqrt{x}$ holds for infinitely many positive integers.

Our numerics are completely insufficient to capture that and thus present no evidence whatsoever one way or the other! To make things worse, since we now know that $\psi(x) - x \neq O(\sqrt{x})$ holds infinitely often, we are forced to test for

$$\forall\epsilon > 0\ , \quad \psi(x) - x = O(x^{0.5+\epsilon})\ ,$$

which seems numerically impossible[3].

4 Final Remarks

The story told here is not uncommon in number theory. To name just one — very similar — example: the Mertens function $M(x)$ is defined as the count of *square free* integers with an even number of distinct prime factors *minus* those with an odd number of distinct prime factors. In the late 19th century, Mertens conjectured that $M(x)$ has the property that $|M(x)|/\sqrt{x} \leq 1$. As in our example, this would imply the RH. In 1985, Odlyzko and Te Riele proved that $|M(x)|/\sqrt{x}$ is greater than 1 for for infinitely many integers x. Exactly how large $|M(x)|/\sqrt{x}$ can grow is still unknown. More details of this story can be found in [8].

References

[1] T. Apostol, *Introduction to Analytic Number Theory*, Springer, 1976.

[2] J. Hadamard, *Sur la distribution des zéros de la fonction ζ(s) et ses conséquences arithmétiques*, Bulletin de la Société Mathématique de France 24, 199 – 220, 1896.

[3] G. H. Hardy, J. E. Littlewood, *Contributions to the Theory of the Riemann Zeta-Function and the Theory of the Distribution of Primes*, Acta Mathematica, 41 (1916) pp. 119 - 196.

[4] B. Riemann, *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse*, Monatsberichte der Berliner Akademie, November 1859.

[5] C. J. de la Vallée Poussin, *Recherches analytiques de la théorie des nombres premiers*, Annales de la Societe Scientifique de Bruxelles, 21B, 351 – 368, 1896.

[6] J. J. P. Veerman, *Numbers from all Angles*, accepted for publication, Springer Nature, 2025.

[7] D. Zagier, *Newman's short proof of the prime number theorem*, The American Mathematical Monthly 104 (1997), 705 – 708.

[8] Wikipedia, *Mertens' Conjecture*, retrieved September 2025.

[3]It is also foolish to hope for a numerical check of Theorem 3.1, because $\ln\ln\ln(10^{200 \text{ million}})$ is still less than 3.