Physica 14D (1985) 177–192
North-Holland, Amsterdam

THE EXISTENCE OF ARBITRARILY MANY DISTINCT PERIODIC ORBITS IN A TWO DEGREE OF FREEDOM HAMILTONIAN SYSTEM

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Received 28 February 1984

Melnikov's method is used to prove the existence of arbitrarily many elliptic and hyperbolic periodic orbits in the neighborhood of an elliptic orbit of a two degree of freedom Hamiltonian system which is 'almost integrable'. The existence of such orbits precludes the existence of analytic second integrals of a certain type. The methods used permit a detailed analysis of the way in which resonant tori break up between the KAM irrational tori which are preserved for weak coupling of two independent nonlinear oscillators.

1. Introduction: coupled pendula

In earlier work [1, 2], adaptations of Melnikov's [3] method were used to study the perturbations of homoclinic manifolds in 2 and n degree of freedom Hamiltonian systems of the form

$$H^{\varepsilon}(\boldsymbol{q}, \boldsymbol{p}) = H^0(\boldsymbol{q}, \boldsymbol{p}) + \varepsilon H^1(\boldsymbol{q}, \boldsymbol{p}), \tag{1.1}$$

where for $\varepsilon = 0$ the H^0 system is completely integrable. The simplest nontrivial case occurs for a two degree of freedom system with a direct product structure:

$$H^{\varepsilon}(q_1, p_1, q_2, p_2) = F_1(q_1, p_1) + F_2(q_2, p_2) + \varepsilon H(q_1, p_1, q_2, p_2). \tag{1.2}$$

In this case, if the F_i are positive definite near $q_i = p_i = 0$, then each (compact) energy surface $(H^0)^{-1}(h) \sim S^3$ for the unperturbed problem is foliated by a smooth family of invariant two-dimensional tori given by $F_i^{-1}(h_i)$ $(h_1 + h_2 = h)$. Such a torus is called *resonant* if it is filled with periodic orbits and *non-resonant* if the flow on it is irrational.

In the papers cited above we showed that, if a certain *Melnikov function* had simple zeros, then stable and unstable manifolds which, for $\varepsilon = 0$, coincide as a smooth homoclinic manifold, intersect transversely for $\varepsilon \neq 0$, small. In that situation the Smale–Birkhoff homoclinic theorem allows one to establish the existence of a complicated invariant set – a Smale horseshoe – for the Poincaré map defined on each constant energy hypersurface in some energy interval [4–6]. This in turn implies that no analytic integral of motion exists for the perturbed problem, other than the total energy H^{ε}. The proof of non-integrability uses

†Partially supported by NSF grant CME 80-17570.

the fact that the nonwandering Cantor set of the horseshoe map has a dense orbit, and the existence of the 'large' unperturbed homoclinic manifold is an essential starting point.

In the present paper we establish results which are somewhat weaker, but without appealing to perturbations of homoclinic orbits. Specifically, we take the example of a pair of coupled pendula with Hamiltonian

$$H^\varepsilon = \frac{p_1^2}{2} + (1-\cos q_1) + \frac{p_2^2}{2} + (1-\cos q_2) + \frac{\varepsilon}{2}(q_1-q_2)^2$$

$$\stackrel{\text{def}}{=} F_1(q_1,p_1) + F_2(q_2,p_2) + \varepsilon H^1(q_1,q_2), \tag{1.3}$$

and consider the flow in a neighborhood of the elliptic periodic orbit $(0,0,q_2(t),p_2(t))$ on the energy surfaces $H^\varepsilon = c$ for $c \in (0,2)$. (For $c = 2(q_2(t), p_2(t))$ becomes a homoclinic orbit and we essentially have the situation considered earlier.) We use the method of reduction [4, 7, 1] to restrict the motion to an energy surface and thus recast the problem as a periodically forced single degree of freedom system. Melnikov's method is then used to study the bifurcation of smooth, resonant tori for $\varepsilon = 0$ into discrete periodic motions for $\varepsilon \neq 0$ as described by Greenspan and Holmes [8] (cf. Guckenheimer and Holmes [6]). We obtain the following main theorem:

Theorem A. For any integer $N < \infty$, there exists $\varepsilon(N) > 0$ such that, for $0 < \varepsilon \leq \varepsilon(N)$, on each energy surface $H^\varepsilon = c \in (0,2)$ of (1.3), and in any neighborhood of an invariant torus for the unperturbed system, there are at least N distinct periodic orbits. Moreover, there are only a finite number of each period. Half of these orbits are hyperbolic and (for possibly smaller ε) half are elliptic.

Corollary. For sufficiently small ε there are arbitrarily many distinct periodic orbits in any neighborhood of the unperturbed elliptic periodic orbit $(0,0,q_2(t),p_2(t))$.

Remark. We can make N arbitrarily large at the cost of taking $\varepsilon(N)$ arbitrarily small.

Sections 2 and 3 of this paper are devoted to an outline of the Melnikov method and the proof of the theorem. In section 4 we conclude with some comments on the relation of this result to Kolmogorov–Arnold–Moser (KAM) theory. In fact the result we obtain is slightly stronger: we are able to compute the number of orbits of each period m in a neighborhood of each "m/n" resonant torus, which for the unperturbed system is filled with a continuous family of m-periodic orbits.

The present result is similar in spirit to that of a paper by Moser [9] in which he proved existence of *infinitely* many periodic points near an elliptic fixed point of a polynomial area preserving planar map. He also demonstrated that there were only a finite number of periodic points of each period n, and hence that no invariant circles of periodic motions could exist. This in turn implies that no analytic change of coordinates exists by which the map can be transformed into a twist map, and hence that the system is non-integrable in the sense of Poincaré and Birkhoff [4]. We note that the existence of a *finite* set of elliptic and hyperbolic periodic motions, as proved in the present paper, also precludes the existence of such an analytic coordinate change, but that it does not preclude the existence of continuous families of smooth invariant surfaces (for this one needs an infinite set of such periodic orbits). See fig. 2 in Section 4, below.

2. Melnikov's method

In this section we provide a brief review of the Melnikov method for finding periodic orbits in perturbations of integrable two degree of freedom Hamiltonian systems. For details see Holmes and Marsden [1] and Guckenheimer and Holmes [6, §§ 4.5–8]. For simplicity, we assume that the unperturbed system has the direct product structure of (1.2), although this is not necessary ([1]); moreover, we assume that the second (2-) system can be put into action angle variables by a symplectic change of coordinates

$$\begin{aligned} I_2 &= \mathscr{I}_2(q_2, p_2), \quad q_2 = \mathscr{Q}_2(I_2, \theta_2), \\ \theta_2 &= \Theta_2(q_2, p_2), \quad p_2 = \mathscr{P}_2(I_2, \theta_2), \end{aligned} \tag{2.1}$$

in a compact region Ω of the (q_2, p_2)-plane. Letting $F_2(I_2)$ denote $F_2(\mathscr{I}_2(q_2, p_2))$ the system may then be written

$$H^\varepsilon(q_1, p_1, I_2, \theta_2) = F_1(q_1, p_1) + F_2(I_2) + \varepsilon H^1(q_1, p_1, \theta_2, I_2). \tag{2.2}$$

We make the following assumptions on the functions F_1, F_2 and H^1:

(H1) F_1, F_2 and H^1 are analytic in all variables (including ε) and the F_i are positive definite for small q_i, p_i;

(H2) $\Omega(I_2) \overset{\text{def}}{=} F_2'(I_2) > 0$ for $I_2 > 0$;

(H3) F_1 has a non-degenerate minimum at $(q_1, p_1) = (0,0)$ and the unperturbed F_1, F_2 system is *isoenergetically nondegenerate* (Arnold [10, appendix 8]).

Remark. Using action angle variables $I_1 = \mathscr{I}_1(q_1, p_1), \theta_1 = \Theta_1(q_1, p_1)$ the F_1 energy can be written $F_1(I_1)$. Isoenergetic nondegeneracy implies that

$$F_1''(I_1)\left(F_2'(I_2)\right)^2 + F_2''(I_2)\left(F_1'(I_1)\right)^2 \neq 0, \tag{2.3}$$

i.e. the frequency ratio of the two oscillators changes as we move transversely across the 1-parameter family of tori in a fixed total energy surface $F_1 + F_2 = h$. (This guarantees that the unperturbed Poincaré map of the reduced system, discussed below, in a twist map; cf. Moser [9].)

The analysis proceeds in two stages. First we use *reduction* to restrict to a three-dimensional constant energy surface. Since $H = h$ is constant for the flow and F_2 is invertible by assumption H2, for small ε we can invert $H = h$ and solve for I_2 in the form

$$I_2 = \mathscr{L}^0(q_1, p_1; h) + \varepsilon \mathscr{L}^1(q_1, p_1, \theta_2; h) + \mathcal{O}(\varepsilon^2) \overset{\text{def}}{=} \mathscr{L}^\varepsilon(q_1, p_1, \theta_2; h), \tag{2.4}$$

where

$$\mathscr{L}^0 = F_2^{-1}(h - F_1(q_1, p_1)), \quad \text{and} \quad \mathscr{L}^1 = -\frac{H^1(q_1, p_1, \theta_2; \mathscr{L}^0(q_1, p_1; h))}{\Omega(\mathscr{L}^0(q_1, p_1; h))}, \tag{2.5}$$

as can be verified by expanding all the functions in Taylor series.

Having passed to the level set $H^\varepsilon = h$ and eliminated I_2, we next eliminate the variable t, conjugate to I_2. Since $\Omega(I_2) > 0$ and H^ε is not explicitly t-dependent, $\theta_2(t)$ is an increasing function (for small ε) and we can eliminate t by inverting $\theta_2(t)$ and expressing q_1 and p_1 as functions of θ_2. Letting $(\ )'$ denote $\mathrm{d}(\ )/\mathrm{d}\theta_2$, we have

$$q_1' = \dot{q}_1/\dot{\theta}_2, \quad p_1' = \dot{p}_1/\dot{\theta}_2, \tag{2.6}$$

and using (2.5) and implicit differentiation of $H^\varepsilon(q_1, p_1, \theta; \mathscr{L}^\varepsilon(q_1, p_1, \theta; h)) = h$ with respect to q_1 and p_1, we obtain the Hamiltonian evolution equations for the reduced system:

$$q_1' = -\frac{\partial \mathscr{L}^0}{\partial p_1} - \varepsilon \frac{\partial \mathscr{L}^1}{\partial p_1} + \mathcal{O}(\varepsilon^2), \quad p_1' = \frac{\partial \mathscr{L}^0}{\partial q_1} + \varepsilon \frac{\partial \mathscr{L}^1}{\partial q_1} + \mathcal{O}(\varepsilon^2). \tag{2.7}$$

For each fixed h, eqs. (2.7) take the form of a periodically perturbed planar system, since $\mathscr{L}^0$ depends only on (q_1, p_1), while $\mathscr{L}^1$ has explicit θ_2-dependence. We can therefore apply the Melnikov theory for subharmonic motions as developed by Greenspan and Holmes [8], cf. Guckenheimer and Holmes [6]. We now use assumption H3, which guarantees that we can find countably many *resonant tori* in the (q_1, p_1, θ_2) phase space of each constant energy surface surrounding the unperturbed periodic orbit $(0, 0, \Omega(\mathscr{L}^0(0, 0, h))t + \theta_0)$. Such tori are direct products of periodic orbits for the F_1 and F_2 systems and the periods of such orbits must satisfy the relationship

$$T_1 = \frac{mT_2}{n} = \frac{m 2\pi}{n\Omega(h_2)} \tag{2.8}$$

for (admissible) relatively prime integer pairs m, n. In terms of the 'new time' θ_2, the period $\overline{T}_1$ of the F_1 system must satisfy

$$\overline{T}_1 = \frac{2\pi m}{n}. \tag{2.9}$$

This in turn implies a relationship between the unperturbed energy levels (integrals) $F_1(q_1, p_1) = h_1$, $F_2(I_2) = h_2$, which, together with the constraint $h_1 + h_2 = h$, fixes a unique unperturbed resonant torus for each pair (m, n).

We pick a point $(q_1(0), p_1(0))$ on the unperturbed m/n-resonant level curve $F_1^{-1}(h_1)$ of the 1-system and a starting time $\theta_0 \in [0, 2\pi)$ and write the unperturbed solution based at this point as $(q_1(\theta), p_1(\theta), \theta_2 + \theta_0)$. Using regular perturbation theory we express the distance between the base point of the perturbed solution at θ_0 and the return point at $\theta_0 + n\overline{T}_1 = \theta_0 + 2\pi m$ as

$$d(\theta_0) = \frac{\varepsilon M(\theta_0; m, n, h)}{\| X_F(0) \|} + \mathcal{O}(\varepsilon^2), \tag{2.10}$$

where $X_F(0) = (\partial F_1/\partial p_1, -\partial F_1/\partial q_1)\,(q_1(0), p_1(0))$ is the Hamiltonian vector field at the base point, and is necessary for normalization. A short calculation using the first variational equation for (2.7) yields the *Melnikov function*

$$M(\theta_0; m, n, h) = \int_0^{2\pi m} \{\mathscr{L}^0, \mathscr{L}^1\}(q_1(\theta_2 - \theta_0), p_1(\theta_2 - \theta_0), \theta_2; h)\, \mathrm{d}\theta_2, \tag{2.11}$$

(cf. Guckenheimer and Holmes [6, chap. 4], Greenspan and Holmes [8]) and we have our main result:

Theorem 2.1. Fix $h>0, m,\ n \in \mathbb{Z}^+$ relatively prime and choose ε sufficiently small. Then, if M has j simple zeros as a function of θ_0 in $[0, 2\pi m/n)$, the resonant torus given by $(q_1, p_1, \theta_2) = (q_1(\theta_2 - \theta_0), p_1(\theta_2 - \theta_0), \theta_2)$ breaks into $2k = j/m$ distinct $2\pi m$-periodic orbits, and there are no other $2\pi m$ periodic orbits in its neighborhood.

Remarks. The integer j is necessarily an even multiple of m, so that $2k$ is even. Note that each $2\pi m$-periodic orbit pierces the Poincaré cross section at $\theta_2 = 0$ precisely m times before closing up (see below). For sufficiently small ε, precisely k of the periodic orbits are hyperbolic and k are elliptic. The coupling parameter ε must be taken sufficiently small for two reasons: (i) to guarantee ε-closeness of the perturbed solution to $(q_1(\theta), p_1(\theta), \theta + \theta_0)$; and (ii) to guarantee that the term $\varepsilon M/\|X_F\|$ in (2.10) dominates the uniformly bounded $\mathcal{O}(\varepsilon^2)$ error. This means that, in the absence of special circumstances, we must let $\varepsilon \to 0$ as $m, n \to \infty$. Hence we cannot *directly* prove the coexistence of infinitely many periodic orbits using Melnikov theory. (In previous work infinitely many periodic orbits were found *via* the Smale–Birkhoff homoclinic theorem, as a consequence of transversal homoclinic intersections.)

Sketch of proof. There are two main ingredients in the proof of this theorem, which is essentially a straightforward application of the implicit function theorem. The Melnikov function measures the 'radial' distance (in the I_1 direction) by which the perturbed orbit fails to match up. The twist guaranteed by assumption H3, ensures that, if $M(\theta_0)$ has a simple zero, implying that the radial component of the difference between base and final point vanishes, then the angular (θ_1) component can also be made to vanish nearby.

To count the number of periodic orbits for $\varepsilon \neq 0$ we count the number of zeros of M in a suitable θ_0-interval. The period of the unperturbed F_1 system is T_1, or, in terms of the 'new time' θ_2, the period of $\mathscr{L}^0$ is $2\pi T_1/T_2 = 2\pi m/n$, since $d\theta_2/dt = \Omega(I_2) = 2\pi/T_2$. Let $(q_1(\theta_2 - \theta_0), p_1(\theta_2 - \theta_0))$ denote a family of $2\pi m/n$-periodic orbits of the unperturbed $\mathscr{L}^0$ system based at 'time' $\theta_2 = 0$. As θ varies from 0 to $2\pi m/n$, the base point of the unperturbed orbit moves once around a closed circle containing (0,0). A simple zero of M at $\theta_0 = \phi$ implies that the degenerate periodic orbit $(q_1(\theta_2 - \phi), p_1(\theta_2 - \phi))$ perturbs to a nearby $2\pi m/n$ periodic orbit for ε small. Hence, j simple zeros in the interval $\theta_0 \in [0, 2\pi m/n)$ implies that precisely j/m such orbits are preserved. The reason that $j = 2km$ for some $k \in \mathbb{Z}$ is described below.

We remark that a simple translation of θ_2 enables us to rewrite (2.11) in the more convenient form

$$M(\theta_0; m, n, h) = \int_0^{2\pi m} \{\mathscr{L}^0, \mathscr{L}^1\}(q_1(\theta_2), p_1(\theta_2), \theta_2 + \theta_0; h)\, d\theta_2, \tag{2.12}$$

as in Guckenheimer and Holmes [1983, Theorem 4.6.2], but note that there is an error in the θ_0-domain given in that theorem. ■

The behavior of the perturbed system is best seen in terms of the (time 2π) Poincaré map P_ε defined on the cross section $\Sigma = \{(q_1, p_1, \theta_2, h) | \theta_2 = 0, h = \text{constant}\}$ for eq. (2.7). We consider the mth interate of this map in the neighborhood of an m/n-resonant torus, which intersects Σ in a simple closed curve Γ. Clearly P_0^m fixes every point on Γ. Since P_ε is area preserving, the image of the interior of any such closed curve Γ' under P_ε^m cannot strictly contain Γ' or be contained by Γ'; there consequently must be point of intersection between Γ' and $P_\varepsilon^m(\Gamma')$. Now these points may not be fixed for P_ε^m, since the perturbed orbits may undergo angular motions, but the $\mathcal{O}(1)$ twist assumed in (H3) guarantees that a closed curve $\Gamma_\varepsilon = \Gamma + \mathcal{O}(\varepsilon)$ *does* exist with the property that each point Γ_ε moves precisely radially under P_ε^m. The Melnikov function provides a measure of the radial motion of points at different angles θ_0 on this curve.

Fixing the twist so that the period of the unperturbed orbits *decreases* as action increases ($F_1''(I)(F_2'(I_2))^2 + F_2''(I_2)(F_1'(I_1))^2 < 0$ in (H3)), we have the situation of fig. 1, which is adapted from Arnold and Avez [11, § 20]. It is not too hard to see that, of the fixed points lying at transverse intersections of Γ_ε and $P_\varepsilon^m(\Gamma_\varepsilon)$, half are hyperbolic and half are elliptic, *if ε is small enough*. If all intersections are transverse then there are necessarily $2km$ of them. Although transverse intersections are generic, we cannot assert their existence in specific cases without computation. The Melnikov function provides the necessary computational tool.

In practice the reduced Hamiltonian $\mathscr{L}^\varepsilon \approx \mathscr{L}^0 + \varepsilon\mathscr{L}^1$ is often awkward to calculate, but fortunately a simple identity can be used:

Proposition 2.2. (Holmes and Marsden [1])

$$\{\mathscr{L}^0, \mathscr{L}^1\}(q_1(\theta_2), p_1(\theta_2), \theta_2 + \theta_0, h) = \frac{1}{[\Omega(I_2)]^2}\{F_1, H^1\}(q_1(t), p_1(t), \theta_2 + \theta_0, I_2), \tag{2.13}$$

where $I_2 = F_2^{-1}(h_2)$ is the unperturbed action of the 2-system.

Using (2.13) in (2.12) and changing the variable of integration from θ_2 to t and the variable θ_0 to t_0, using $\mathrm{d}\theta_2/\mathrm{d}t = \Omega(I_2)$, we have

$$M(t_0; m, n, h) = \frac{1}{\Omega(I_2)}\int_0^{2\pi m/\Omega(I_2)}\{F_1, H^1\}(q_1(t), p_1(t), q_2(t+t_0), p_2(t+t_0))\,\mathrm{d}t. \tag{2.14}$$

Thus, we need not even know the action angle transformation explicitly, provided that we know $(q_2(t), p_2(t))$ on the level set $F_2^{-1}(h_2)$. Finally, since the integrand is periodic with period $2\pi m/\Omega(I_2) = mT_2 = nT_1$, we can change the limits of integration to obtain

$$M(t_0; m, n, h) = \frac{T_2}{2\pi}\int_{-mT_2/2}^{mT_2/2}\{F_1, H^1\}(q_1(t), p_1(t), q_2(t+t_0), p_2(t+t_0))\,\mathrm{d}t. \tag{2.15}$$

Eq. (2.15) is the form we shall use in the next section. To establish the existence of $2k$ isolated m-periodic orbits, we must show that $M(t_0; m, n, h)$ has $2km$ simple zeros in $t_0 \in [0, T_1)$, since as t_0 varies from 0 to T_1, θ_0 varies from 0 to $2\pi T_1/T_2 = 2\pi m/n$.

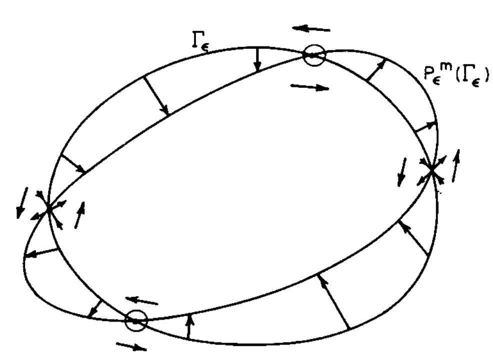

Fig. 1. The perturbed Poincaré map P_ε^m. Elliptic fixed points indicated by ○, hyperbolic points by ✕ (after Arnold and Avez [11]).

3. The proof of theorem A

In this section we perform the computations necessary to establish that the Melnikov function (2.14) for the coupled pendulum problem has a finite number of simple zeros for each relatively prime integer pair m/n (m, n odd) in some interval.

Each uncoupled pendulum has Hamiltonian

$$F_i = \frac{p_i^2}{2} + (1 - \cos q_i), \tag{3.1}$$

and the periodic orbits for energies $F_i = h_i \in (0,2)$ can be written in terms of Jacobi elliptic functions as

$$\begin{aligned} q_i(t) &= 2 \arcsin\left(\left(\frac{h_i}{2}\right)^{1/2} \operatorname{sn}\left(t \left| \frac{h_i}{2}\right.\right)\right), \\ p_i(t) &= 2\left(\frac{h_i}{2}\right)^{1/2} \operatorname{cn}\left(t \left| \frac{h_i}{2}\right.\right), \end{aligned} \tag{3.2}$$

where sn and cn are the elliptic sine and cosine functions with modulus $h_i/2$. We have based the orbits at $(q_i(0), p_i(0)) = (0, \sqrt{2h_i})$. For a resonance of order m/n, we require

$$nT_1 = mT_2, \tag{3.3}$$

and since $p_i = \mathrm{d}q_i/\mathrm{d}t$, (3.1) can be integrated to yield

$$T_i = 4K\left(\frac{h_i}{2}\right), \tag{3.4}$$

where K is the complete elliptic integral of the first kind. The resonance condition relating the energies h_1 and h_2 is therefore

$$nK\left(\frac{h_1}{2}\right) = mK\left(\frac{h_2}{2}\right). \tag{3.5}$$

We also have

$$h_1 + h_2 = h, \tag{3.6}$$

if we work in a constant (total) energy surface $H^\varepsilon = h$. Combining (3.5) and (3.6) we obtain

$$nK\left(\frac{h_1}{2}\right) = mK\left(\frac{h - h_1}{2}\right). \tag{3.7}$$

Lemma 3.1. For $h \in (0,2)$ and each $m, n \in \mathbb{Z}^+$ with $m/n \in [K(0)/K(h/2), K(h/2)/K(0)]$ eq. (3.7) has a unique solution $h_1^*(m, n, h)$. These solutions are dense in $h_1 \in (0, h)$.

Proof. $K(\alpha)$ is a monotonically increasing function with domain $\alpha \in [0,1)$, so $nK(h_1/2)$ increases on $h_1 \in [0,2)$ while $mK(h - h_1/2)$ decreases on $h_1 \in (h-2, h]$. The fact that $K(\alpha)$ increases monotonically with α (on its domain of definition), yields at most a unique intersection point. The fact that solutions cannot be found for all m, n is due to the restrictions that $mK(h/2) \geq nK(0)$ and $mK(0) \leq nK(h/2)$ for intersections to exist. Density follows from the density of the rationals m/n and the continuity of $K(\alpha)$ for $\alpha \in (0,1)$. ∎

This result implies that the unperturbed system has a dense set of resonant tori in any neighborhood of the elliptic orbit $(0,0,q_2(t),p_2(t))$ on every energy surface $h \in (0,2)$. This is, of course, the generic situation for integrable two degree of freedom systems.

We next compute the Melnikov function. Selecting a total energy $h > 0$ and an admissible integer pair (m,n) and the associated unperturbed orbits (q_i, p_i) with energies h_i, and recalling that the perturbation is $H^1 = \frac{1}{2}(q_1 - q_2)^2$ we substitute into eq. (2.15) to obtain

$$M(t_0; m,n,h) = \frac{T_2}{2\pi}\int_{-mT_2/2}^{mT_2/2} \{F_1, H^1\}(q_1(t), p_1(t), q_2(t+t_0), p_2(t+t_0))\,\mathrm{d}t$$
$$= \frac{T_2}{2\pi}\int_{-mT_2/2}^{mT_2/2} -p_1(t)(q_1(t) - q_2(t+t_0))\,\mathrm{d}t, \tag{3.8}$$

where

$$T_2 = 4K\left(\frac{h_2}{2}\right).$$

Since $p_1(t) = \dot{q}_1(t)$ is even and $q_1(t)$ is odd (see (3.2)) the integral of $p_1 q_1$ vanishes and, substituting (3.2) into (3.8) we obtain

$$M(t_0, m,n,h) = \frac{T_2}{2\pi}\int_{-mT_2/2}^{mT_2/2} 4\left(\frac{h_1}{2}\right)^{1/2} \mathrm{cn}\left(t\left|\frac{h_1}{2}\right.\right) \arcsin\left[\left(\frac{h_2}{2}\right)^{1/2} \mathrm{sn}\left(t+t_0\left|\frac{h_2}{2}\right.\right)\right]\mathrm{d}t. \tag{3.9}$$

The integral (3.9) is difficult to evaluate, and so we use indirect arguments to establish the following:

Lemma 3.2. For each $h \in (0,2)$ and each pair (m,n) of odd, relatively prime integers satisfying $m/n \in [K(0)/K(h/2), K(h/2)/K(0)]$, $M(t_0; m,n,h)$ has precisely $2m$ simple zeros in the interval $t_0 \in [0, T_1)$, where $T_1 = 4K(h_1/2)$.

Proof. We first demonstrate the existence of at least $2m$ zeros. We note that, since $p_1(t)$ is even while, for $t_0 = 0$, $q_2(t+t_0)$ is odd, (3.9) vanishes for $t_0 = 0$. Similarly, for each $t_0 = kT_2/2 = knT_1/2m$, $q_2(t+t_0)$ is also odd and so $M(knT_1/2m; m,n,h) = 0$ for all integers $k \in \mathbb{Z}$. Computing $kn/2m$ mod.1 for n, m relatively prime and *both odd*, we find precisely $2m$ such zeros (at $kn \bmod 2m = 0,1,2,\ldots,2m-1$) in the interval $0 \leq kn/2m < 1$.

We next show that the zeros found above are simple. We have

$$M'(t_0; m,n,h) = \frac{T_2}{2\pi}\int_{-mT_2/2}^{mT_2/2} \frac{\mathrm{d}}{\mathrm{d}t_0}\{\ldots\}\,\mathrm{d}t$$
$$= \frac{T_2}{2}\int_{-mT_2/2}^{mT_2/2}\left\{4\left(\frac{h_1}{2}\right)^{1/2} \mathrm{cn}_1 \frac{1}{(1-(h_2/2)\,\mathrm{sn}_1^2)^{1/2}}\left(\frac{h_2}{2}\right)^{1/2} \mathrm{cn}_2 \mathrm{d}n_2\right\}\mathrm{d}t, \tag{3.10}$$

where $\mathrm{cn}_i = \mathrm{cn}(t|h_i/2)$ etc. Using elliptic function identities, (3.10) simplifies to

$$M'(t_0; m, n, h) = \frac{T_2}{2\pi} 4\left(\frac{h_1 h_2}{4}\right)^{1/2} \int_{-mT_2/2}^{mT_2/2} \mathrm{cn}\left(t\left|\frac{h_1}{2}\right.\right) \mathrm{cn}\left(t + t_0\left|\frac{h_2}{2}\right.\right) \mathrm{d}t. \tag{3.11}$$

Setting $t_0 = 0$ (or $t_0 = kT_2/2$ for $k \in \mathbb{Z}$), the integrand of (3.11) becomes

$$\mathrm{cn}\left(t\left|\frac{h_1}{2}\right.\right) \mathrm{cn}\left(t\left|\frac{h_2}{2}\right.\right). \tag{3.12}$$

To evaluate M' we will use the Fourier series representation of the elliptic cosine:

$$\mathrm{cn}\left(t\left|\frac{h_i}{2}\right.\right) = \frac{2\pi}{(h_i/2)^{1/2} K(h_i/2)} \sum_{l=0}^{\infty} \frac{q_i^{l+1/2}}{1 + q_i^{2l+1}} \cos\left[(2l+1)\frac{\pi t}{2K(h_i/2)}\right], \tag{3.13}$$

where

$$q_i = \exp\left[-\pi K\left(1 - \frac{h_i}{2}\right)\Big/ K\left(\frac{h_i}{2}\right)\right] \tag{3.14}$$

is the elliptic nome [12]. Substituting into (3.11), using (3.12), recalling that $T_i = 4K(h_i/2)$, and interchanging the order of summation and integration, we have

$$\begin{aligned} M'\left(\frac{kT_2}{2}; m, n, h\right) &= \frac{T_2}{2\pi} \frac{256\pi^2}{T_1 T_2} \sum_{j,l}^{\infty} \frac{q_1^{j+1/2} q_2^{l+1/2}}{(1 + q_1^{2j+1})(1 + q_2^{2l+1})} \\ &\quad \times \int_{-mT_2/2}^{mT_2/2} \cos\left[(2j+1)\frac{2\pi t}{T_1}\right] \cos\left[(2l+1)\frac{2\pi t}{T_2}\right] \mathrm{d}t. \end{aligned} \tag{3.15}$$

Using orthogonality of the Fourier components, we see that the integral is non-zero if and only if

$$\frac{(2j+1)}{T_1} = \frac{(2l+1)}{T_2} \quad \text{or} \quad n(2j+1) = m(2l+1), \tag{3.16}$$

since $nT_1 = mT_2$. This condition can only be satisfied for m, n both odd and if in addition we have

$$(2j+1) = (2\mu+1)m, \quad (2l+1) = (2\mu+1)n, \quad \mu = 0, 1, 2, 3 \ldots. \tag{3.17}$$

Finally, using (3.13–14), we have

$$\begin{aligned} M'\left(\frac{kT_2}{2}; m, n, h\right) &= \frac{T_2}{2\pi} \frac{64\pi^2}{T_1 T_2} \sum_{\mu=0}^{\infty} \mathrm{sech}\left[\pi(\mu + \tfrac{1}{2}) m\tau_1\right] \mathrm{sech}\left[\pi(\mu + \tfrac{1}{2}) n\tau_2\right] \frac{mT_2}{2} \\ &= 16\pi n \sum_{\mu=0}^{\infty} \mathrm{sech}\left[\pi(\mu + \tfrac{1}{2}) m\tau_1\right] \mathrm{sech}\left[\mu(\pi + \tfrac{1}{2}) n\tau_2\right], \end{aligned} \tag{3.18}$$

where

$$\tau_i = K(1-h_i/2)/K(h_i/2) = K'(h_i/2)/K(h_i/2). \tag{3.19}$$

Since q_i and τ_i are positive, we can estimate $M'(kT_2/2; m, n, h)$ as follows:

$$\left|M'\left(\frac{kT_2}{2}; m, n, h\right)\right| \le 16\pi n \sum_{\mu=0}^{\infty} 2\,\mathrm{e}^{-\pi(\mu+1/2)m\tau_1} 2\,\mathrm{e}^{-\pi(\mu+1/2)n\tau_2}, \tag{3.20}$$

to obtain

$$\left|M'\left(\frac{kT_2}{2}; m, n, h\right)\right| \le 64\pi n\,\mathrm{e}^{-(\pi/2)m\tau_1-(\pi/2)n\tau_2}\cdot\left(1-\mathrm{e}^{(-\pi m\tau_1-\pi n\tau_2)}\right)^{-1}$$
$$= 32\pi n \operatorname{csch}\left[\frac{\pi}{2}(m\tau_1+n\tau_2)\right]. \tag{3.21}$$

In the same way we obtain

$$\left|M'\left(\frac{kT_2}{2}; m, n, h\right)\right| \ge 16\pi n \sum_{\mu=0}^{\infty} \mathrm{e}^{-\pi(\mu+1/2)m\tau_1}\,\mathrm{e}^{-\pi(\mu+1/2)n\tau_2}$$
$$= 8\pi n \operatorname{csch}\left[\frac{\pi}{2}(m\tau_1+n\tau_2)\right]. \tag{3.22}$$

In obtaining these estimates we use the fact that, for $a, b \ge 0$

$$\frac{1}{4\mathrm{e}^{a+b}} \le \frac{1}{\mathrm{e}^{a}+\mathrm{e}^{-a}}\,\frac{1}{\mathrm{e}^{b}+\mathrm{e}^{-b}} \le \frac{1}{\mathrm{e}^{a+b}}, \tag{3.23}$$

and then sum the resulting geometric series. Our bounds on $|M'(kT_2/2; m, n, h)|$ show that, for any odd $m, n < \infty$, the zeros are simple. However, we remark that to obtain *all* the resonant tori in any neighborhood of a given torus or of the elliptic orbit $(0, 0, q_2(t), p_2(t))$ we must let $m, n \to \infty$. Thus $\pi n \operatorname{csch}[(\pi/2)(m\tau_1+n\tau_2)] \sim n\,\mathrm{e}^{-(\pi/2)(mT_1+nT_2)} \to 0$ and simple arguments do not yield a uniform lower bound for (3.18) for all admissible pairs (m,n). In fact M'(and M) $\to 0$ as $m, n \to \infty$.

To compute M' as a function of t_0 we simply expand the cosines of the Fourier series of (3.13), and use $n/T_2 = m/T_1$ to obtain

$$\cos\left(\frac{2\pi(2l+1)(t+t_0)}{T_2}\right) = \cos\left(\frac{2\pi(2l+1)2t}{T_2}\right)\cos\left(\frac{2\pi(2l+1)mt_0}{nT_1}\right)$$
$$-\sin\left(\frac{2\pi(2l+1)t}{T_2}\right)\sin\left(\frac{2\pi(2l+1)mt_0}{nT_1}\right). \tag{3.24}$$

Using this expression in place of the second cosine term in the sum of the integral of (3.15) and selecting $(2l+1) = (2\mu+1)n$ (cf.(3.17)), we obtain

$$M'(t_0; m, n, h) = 16\pi n \sum_{\mu=0}^{\infty} \operatorname{sech}\left[\pi\left(\mu+\tfrac{1}{2}\right)m\tau_1\right]\operatorname{sech}\left[\pi\left(\mu+\tfrac{1}{2}\right)n\tau_2\right]\cos\frac{2\pi(2\mu+1)mt_0}{T_1}. \tag{3.25}$$

Since M' is given by a convergent Fourier series we can integrate term by term to obtain

$$\begin{aligned} M(t_0; m, n, h) &= 16\pi n \sum_{\mu=0}^{\infty} \frac{T_1}{2\pi(2\mu+1)n, m} \operatorname{sech}\left[\pi(\mu+\tfrac{1}{2})m\tau_1\right] \operatorname{sech}\left[\pi(\mu+\tfrac{1}{2})n\tau_2\right] \\ &\quad \times \sin\left(\frac{2\pi(2\mu+1)mt_0}{T_1}\right) \\ &= 8T_2 \sum_{\mu=0}^{\infty} \frac{\operatorname{sech}\left[\pi(\mu+\tfrac{1}{2})m\tau_1\right] \operatorname{sech}\left[\pi(\mu+\tfrac{1}{2})n\tau_2\right]}{(2\mu+1)} \sin\left(\frac{2\pi(2\mu+1)mt_0}{T_1}\right). \end{aligned} \tag{3.26}$$

As our earlier argument showed, for each fixed odd integer pair n, m, this function has $2m$ (simple) zeros in $t_o \in [0, T_1)$ at $t_0 = kT_1/2m \bmod T_1$, with $k = 0, 1, \ldots, 2m-1$.

Finally we show that there are only $2m$ zeros in $t_o \in [0, T_1)$. To do this it is sufficient to show that each partial sum of the Fourier series:

$$T_N(\phi) = \sum_{\mu=0}^{N} \frac{a(2\mu+1)}{2\mu+1} \sin(2\mu+1)\phi, \tag{3.27}$$

with

$$a(2\mu+1) = \operatorname{sech}\left[\frac{\pi}{2}(2\mu+1)m\tau_1\right] \operatorname{sech}\left[\frac{\pi}{2}(2\mu+1)n\tau_2\right],$$

has no zero in the interval $\phi \in (0, \pi)$. We first note that each partial sum

$$S_N(\phi) = \sum_{\mu=0}^{N} \frac{1}{2\mu+1} \sin(2\mu+1)\phi \tag{3.28}$$

of the Fourier series for $f(\phi) = \begin{cases} \pi/4, & \phi \in (0,\pi) \\ -\pi/4, & \phi \in (\pi, 2\pi) \end{cases}$ is strictly positive on $\phi \in (0, \pi)$ (and strictly negative on $(\pi, 2\pi)$). This is proved in the appendix. We next note that $0 \le a(2\mu+1) \le 1$ for all μ and that the coefficients $a(2\mu+1)$ are strictly decreasing with μ. In fact, as for the bounds established on M' above, we can obtain the estimate

$$c_1 e^{-\alpha\mu} \le \frac{a(2\mu+1)}{a(2\mu-1)} \le c_2 e^{-\alpha\mu} \tag{3.29}$$

for constants c_1, c_2, α (depending upon m, n, τ_1, τ_2 and hence h_1, h_2 and h). Now clearly $S_0(\phi)$ and $T_0(\phi) = a(1)s_0(\phi)$ are positive on $(0, \pi)$. Since $S_1(\phi)$ is likewise positive on $(0, \pi)$ it follows that

$$\begin{aligned} T_1(\phi) &= T_0(\phi) + (T_1(\phi) - T_0(\phi) \\ &= a(1)S_0(\phi) + a(3)(S_1(\phi) - S_0(\phi)) \end{aligned} \tag{3.30}$$

is also positive on $(0, \pi)$, since the 'adjustment' $T_1(\phi) - T_0(\phi)$ has the same sign as $S_1(\phi) - S_0(\phi)$ and is smaller in magnitude relative to $T_0(\phi)$ by the factor $a(3)/a(1) < 1$. Continuing inductively, we see that each partial sum

$$T_N(\phi) = T_0(\phi) + (T_1(\phi) - T_0(\phi)) + \cdots + (T_N(\phi) - T_{N-1}(\phi))$$
$$= a_0(1)S_0(\phi) + a(3)(S_1(\phi) - S_0(\phi)) + \cdots + a(2N+1)(S_N(\phi) - S_{N-1}(\phi)) \tag{3.31}$$

is positive on $(0, \pi)$, and hence we obtain our result. ■

Remark. For m and/or n even, the Melnikov function is identially zero, due to the symmetries of the unperturbed solutions (the Fourier series (3.13) contains only odd terms). Thus one obtains no information for 'even' resonances from these first order perturbations.

Equipped with lemmas 3.1–2, we can now prove theorem A.

Proof of theorem A. Select $h \in (0,2)$, in which case lemma 3.1 guarantees that there is a dense set of resonant tori surrounding the unperturbed periodic orbit $(0, 0, q_2(t), p_2(t))$, where (q_2, p_2) lie in the level set $F_2^{-1}(h)$. Lemma 3.2 shows that, for any *finite* m, n in the interval $[K(0)/K(h/2), K(h/2)/K(0)]$, with m and n both odd, $M(t_0; m, n, h)$ has precisely $2m$ simple zeros in the interval $t_0 \in [0, T_1)$. Application of theorem 2.1 and lemma 3.2 with a suitable choice of ε, then yields theorem A. ■

4. Concluding remarks

As we remarked in section 2, as we take m larger and larger, so we must let $\varepsilon \to 0$ to guarantee validity of the perturbation methods used, and to guarantee that the Melnikov function dominates the radial distance calculation (eq. (2.10)). (Recall that $M(t_0; m, n, h) \to 0$ as m or $n \to \infty$; eq. (3.26)). Thus we have the curious situation that, the smaller ε and the closer the system to the integrable limit $\varepsilon = 0$, the more 'distinct' periodic orbits can be proven to exist. As $\varepsilon \to 0$ we can demonstrate that arbitrarily many of the dense set of resonant tori have broken up. In contrast, Kolmogorov–Arnold–Moser theory ([10]) guarantees that as $\varepsilon \to 0$ the measure of the set of diophantine irrational tori preserved converges to 1. There is of course no contradiction, since the irrational tori form a nowhere dense, albeit measurable, Cantor set and the resonant orbits inhabit the gaps in this set.

For sufficiently small ε and any fixed $m, n < \infty$, the width of these gaps goes as $\mathcal{O}(\sqrt{\varepsilon})$, as can be seen from the following argument. Using the action angle variables (I_1, θ_1) for the F_1 system, for each fixed h the Hamiltonian takes the form

$$\mathscr{L}^\varepsilon(I_1, \theta_1; \theta_2) = \mathscr{L}^0(I_1) + \varepsilon \mathscr{L}^1(I_1, \theta_1; \theta_2) + \cdots, \tag{4.1}$$

so that eq. (2.7) for the reduced periodically perturbed Hamiltonian can be written

$$I_1' = \varepsilon \frac{\partial \mathscr{L}^1}{\partial \theta_1}(I_1, \theta_1; \theta_2) + \mathcal{O}(\varepsilon^2),$$
$$\theta_1' = -\omega(I_1) - \varepsilon \frac{\partial \mathscr{L}^1}{\partial I_1}(I_1, \theta_1; \theta_2) + \mathcal{O}(\varepsilon^2), \tag{4.2}$$

where $\omega(I_1) = \partial \mathscr{L}^0 / \partial I_1$. Moving to a coordinate system

$$
\begin{aligned}
I_1 &= \bar{I}_1 + \sqrt{\varepsilon}\, K, \\
\theta_1 &= -\omega(\bar{I}_1)\theta_2 + \psi = -\frac{n}{m}\theta_2 + \psi,
\end{aligned}
\tag{4.3}
$$

which fixes the resonant torus with action $\bar{I}_1 = (\mathscr{L}^0)^{-1}(h_1)$, (4.2) becomes

$$
\begin{aligned}
K' &= \sqrt{\varepsilon}\, \frac{\partial \mathscr{L}^1}{\partial \theta_1}\left(\bar{I}_1, -\frac{n\theta_2}{m} + \psi, \theta_2\right) + \mathscr{O}(\varepsilon), \\
\psi' &= -\sqrt{\varepsilon}\, \omega'(\bar{I}_1) K + \mathscr{O}(\varepsilon),
\end{aligned}
\tag{4.4}
$$

where the $\mathscr{O}(\varepsilon)$ remainder is uniformly bounded and 2π periodic in ψ and θ. We next apply the averaging theorem (Hale [14]) to (4.4) to remove the explicit 'fast' θ_2 dependence, thus obtaining the truncated system

$$
\begin{aligned}
K' &= \sqrt{\varepsilon}\, \frac{1}{2\pi n} \int_0^{2\pi m} \{\mathscr{L}^0, \mathscr{L}^1\}\left(\bar{I}_1, \psi - \frac{n\theta_2}{m}, \theta_2\right) \mathrm{d}\theta_2, \\
\psi' &= -\sqrt{\varepsilon}\, \omega'(\bar{I}_1) K.
\end{aligned}
\tag{4.5}
$$

In deriving the first component of (4.5) we have used the fact that the canonical Poisson bracket $\{\mathscr{L}^0, \mathscr{L}^1\}$ can be written

$$
\{\mathscr{L}^0, \mathscr{L}^1\} = \frac{\partial \mathscr{L}^0}{\partial I_1}\frac{\partial \mathscr{L}^1}{\partial \theta_1} - \frac{\partial \mathscr{L}^0}{\partial \theta_1}\frac{\partial \mathscr{L}^1}{\partial I_1} = \omega(I_1)\frac{\partial \mathscr{L}^1}{\partial \theta_1}
\tag{4.6}
$$

and hence that

$$
\int_0^{2\pi m} \frac{\partial \mathscr{L}^1}{\partial \theta_1}\, \mathrm{d}\theta_2 = \int_0^{2\pi m} \frac{1}{\omega(I_1)} \{\mathscr{L}^0, \mathscr{L}^1\}\, \mathrm{d}\theta_2.
\tag{4.7}
$$

We remark that the right-hand side of the first component of (4.5) is essentially the Melnikov function (2.11), cf. Guckenheimer and Holmes [6, §4.7]. Denoting this right-hand side by $\sqrt{\varepsilon} f(\psi)$, it is clear that (4.5) is an integrable Hamiltonian system with Hamiltonian

$$
\mathscr{H}(K, \psi) = \sqrt{\varepsilon}\left\{\omega'(I_1)\frac{K^2}{2} + \int_0^{\psi} f(\eta)\, \mathrm{d}\eta\right\}.
\tag{4.8}
$$

Since $\mathscr{L}^1$ is 2π-periodic in θ_1, $\int_0^{\psi} f(\eta)\, \mathrm{d}\eta$ is 2π-periodic in ψ and the maximum separation of the level curves of $\mathscr{H}$ which contain hyperbolic fixed points can be estimated as

$$
K_{\max} = \sqrt{\frac{2}{\omega'(\bar{I}_1)} \max_{\psi \in [0, 2\pi)} \left(\int_0^{\psi} f(\eta)\, \mathrm{d}\eta\right)}.
$$

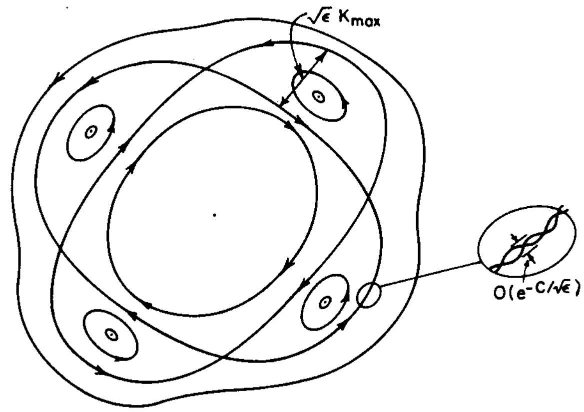

Fig. 2. Level curves of the $\mathcal{O}$ ($\sqrt{\varepsilon}$) averaged system.

(See fig. 2.) Thus from the transformation (4.3) we obtain the maximum separation in K as $\sqrt{\varepsilon}\,K_{\max}$. This argument can be made rigorous by appeal to the averaging theorem, which guarantees that solutions of the truncated, averaged system (4.5) remain within $\mathcal{O}(\sqrt{\varepsilon})$ of solutions of the full system (4.4) for 'times' of $\mathcal{O}(1/\sqrt{\varepsilon})$.

The level curves of the averaged system (4.5) in the neighborhood of a resonant closed curve therefore appear somewhat as sketched in fig. 2; note the alternating elliptic and hyperbolic fixed points. (The argument sketched above is essentially the same as the more familiar Birkhoff normal form transformation method, cf. Arnold [10] appendix 7.) The averaging theorem does not of course guarantee that the families of heteroclinic orbits connecting the hyperbolic points are preserved as smooth manifolds, rather we expect the stable and unstable manifolds of these points to intersect transversely. Unfortunately, attempts to prove this in specific cases using Melnikov methods run into an obstruction: the Melnikov function is exponentially small: of $\mathcal{O}(e^{-c/\sqrt{\varepsilon}})$. While there is a general belief that this does reflect the correct picture (e.g. Chirikov [13]), there is as yet no rigorous proof (cf. Holmes, Marsden and Scheurle [15]). We remark that, if such $\mathcal{O}(e^{-c/\sqrt{\varepsilon}})$ transverse homoclinic intersections could be proven to occur in the gaps between the irrational tori, then we would be able to establish the nonexistence of analytic second integrals for the perturbed system as in Holmes [2].

Appendix A

We are indebted to Richard Rand for the following result, which seems not to be widely known. Let

$$S_N(\phi) = \frac{4}{\pi} \sum_{\mu=0}^{N} \frac{1}{2\mu+1} \sin(2\mu+1)\phi \tag{A.1}$$

denote the Nth partial sum of the Fourier series for

$$f(\phi) = \begin{cases} 1, & 0 < \phi < \pi, \\ -1, & \pi < \phi < 2\pi. \end{cases} \tag{A.2}$$

Lemma. $S_N(\phi)$ is strictly positive on $\phi \in (0, \pi)$ for all $N \geq 0$.

Proof. Rewrite each term of (A.1) as

$$\int_0^{\phi} \cos(2\mu + 1)\eta \, d\eta \tag{A.3}$$

and interchange the order of integration and summation to obtain

$$S_N(\phi) = \frac{4}{\pi} \int_0^{\phi} \sum_{\mu=0}^{N} \cos(2\mu + 1)\eta \, d\eta. \tag{A.4}$$

Now use the fact that

$$\begin{aligned} &\sin\eta(\cos\eta + \cos 3\eta + \cdots + \cos(2N+1)\eta) \\ &\quad = \frac{\sin 2\eta}{2} + \frac{\sin 4\eta - \sin 2\eta}{2} + \cdots + \frac{\sin(2N+2)\eta - \sin 2N\eta}{2} \\ &\quad = \frac{\sin(2N+2)\eta}{2}, \end{aligned} \tag{A.5}$$

so that we can write

$$S_N(\phi) = \frac{2}{\pi} \int_0^{\phi} \frac{\sin(2N+2)\eta}{\sin\eta} \, d\eta. \tag{A.6}$$

Note that this integral converges for all N and $\eta \in (0, \pi)$; in fact it can be used for small η to compute the characteristic Gibb's Phenomena or over-shoot in the series representations of $f(\phi)$ at 0^+. For our purposes it is sufficient to note that, for each N, the integral of (A.6) is strictly positive on $(0, \pi/2)$ because the negative contributions to the integrand (for $\eta \in (\pi/(2N+2), \pi/(N+1))$, etc.) are smaller than the positive contributions (for $\eta \in (0, \pi/(2N+2))$, etc.), since $1/\sin\eta$ is monotonically decreasing on $(0, \pi/2)$. To obtain the same conclusion for $\phi \in (\pi/2, \pi)$ we merely use the symmetry properties of the sine function. ■

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