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Dynamics of a jumping particle on a staircase profile

J. Campos [a], M.J. Romero-Vallés [a], P.J. Torres [a,*], J.J.P. Veerman [b]

[a] *Universidad de Granada, Departamento de Matemática Aplicada, Av. Fuentenueva sn, 18071 Granada, Spain*
[b] *Department of Mathematics and Statistics, Portland State University, Portland, OR 97207, USA*

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Abstract

We perform a detailed analysis of the dynamics of the descent of a particle bouncing down a staircase profile under the action of gravity. In order to get interesting dynamics we make a detail analysis of the case which the particle loses momentum in the direction orthogonal to the collision plane but preserves the tangential component of the momentum. We prove that in this case all orbits are bounded and show the existence and stability of periodic solutions. The interplay between loss and gain of energy due to impacts and free falling respectively generates a rich dynamics.

1. Introduction

In general, models with impacts lead to non-smooth dynamical systems and include a wide range of applications like particle accelerators [2,12], dynamics of structures under the action of earthquakes [4] or percussion machines and print hammers [8]. A large number of references can be found in the book [1]. But the main context of this paper concerns granular flow.

The properties of granular flow poses substantial theoretical challenges while at the same time it is of considerable practical interest (see for example [3,6]). One of its most peculiar properties is that when one pours the material on a pile, the slope of the resulting mound varies between two angles, the so-called angle of repose and the maximal angle of stability, that are typically only a few degrees apart [5]. The usual view is that this characteristic behavior is in fact a collective phenomenon: a consequence of the interaction between many particles. However, in [9] (summarized in [11]) a model consisting of a single particle falling down an inclined staircase was conjectured to have a very similar characteristic behavior and partial results in that direction were obtained.

In essence, this conjecture reduces to the statement that if the inclination of the staircase is below a certain angle ϕ_0 a falling particle eventually stops, and if it is more than $\phi_1 > \phi_0$ the particle accelerates indefinitely (its velocity is unbounded). We refer to [11] for a more detailed discussion of the status of this conjecture and a precise statement of the partial results.

The proof of the full conjecture (or its falsification) is elusive because we lack the mathematical tools necessary to fully understand the dynamics of the model. A simpler model that allows a much more complete analysis of the dynamics was

* Corresponding author. Tel.: +34 958242941; fax: +34 958248596.
E-mail address: ptorres@ugr.es (P.J. Torres).

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formulated in [10] (and elaborated in [11]). In this model the particle does not bounce up when it hits the staircase but rather slides forward to the end of the ramp. However this model does not exhibit the required trichotomy.

The present model was formulated in order to gain a better understanding of the dynamics of a particle that bounces up when it hits the staircase. The simplification here is that the staircase is not inclined (its ramps are horizontal). As a result the horizontal component of the velocity of the falling particle is a constant of the motion for those parameter values where the interesting dynamics occurs. Thus we study the motion of a particle of unitary mass falling down a staircase under the action of gravity (see Fig. 1). If the height and length of the steps are a and b respectively, then the motion is governed by

$$\left.\begin{array}{l} x''(t) = 0 \\ y''(t) = -g \end{array}\right\}, \quad \text{if } \frac{y(t)}{a} > -E\left[\frac{x(t)}{b}\right] \tag{1.1}$$

$$\left.\begin{array}{l} x'(t^+) = e_t x'(t^-) \\ y'(t^+) = -e_n y'(t^-) \end{array}\right\}, \quad \text{if } \frac{y(t)}{a} = -E\left[\frac{x(t)}{b}\right] \tag{1.2}$$

where $E[x]$ is the integer part of x and g is the gravity constant. System (1.1) models the simple parabolic flight between impacts and (1.2) rules the change of velocity in each impact. The numbers $e_t, e_n \in [0,1]$ are usually known as *the coefficients of restitution*. As a whole, we have an impact system that is piecewise integrable.

A change of variables ($\tilde{y} = \frac{y}{a}$, $\tilde{x} = \frac{x}{b}$, $\tilde{t} = \sqrt{\frac{2g}{a}}t$, then drop the tilde) yields a simplification:

$$\left.\begin{array}{l} x''(t) = 0 \\ y''(t) = -1/2 \end{array}\right\}, \quad \text{if } y(t) > -E[x(t)] \tag{1.3}$$

$$\left.\begin{array}{l} x'(t^+) = e_t x'(t^-) \\ y'(t^+) = -e_n y'(t^-) \end{array}\right\}, \quad \text{if } y(t) = -E[x(t)] \tag{1.4}$$

Denote the initial position and velocity of the particle at take-off by (u, v, z), see Fig. 1. During its flight to the next impact, the orbit of the particle is given by

$$\begin{aligned} x(t) &= ut + 1 - z, \\ y(t) &= -\frac{t^2}{4} + vt. \end{aligned} \tag{1.5}$$

Here z is the distance from the point of impact to the border of the ramp and u, v are the horizontal and vertical velocities, respectively.

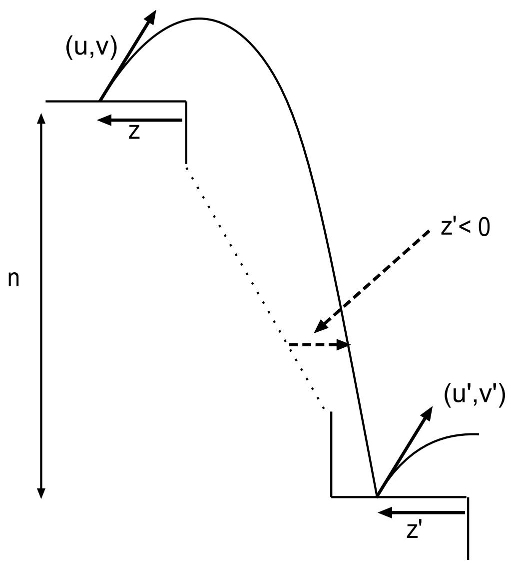

Fig. 1. The discrete model.

The number of steps between two successive impacts is called the *jump number* and will be denoted by n. The number n is the first natural number such that z' is positive. If $z \geqslant 4uv$ it is easy to see that the particle bounces again in the same step, so $n = 0$. In other case, let us denote t^* as the time at which the solution crosses the line $y = 1 - x$. By calculation,

$$-\frac{(t^*)^2}{4} + vt^* = -ut^* + z$$

so its largest root is

$$t^* = 2\left(v + u + \sqrt{(v+u)^2 - z}\right).$$

If $z \leqslant 4uv$ this is a real number and, $\Gamma[x]$ being the ceiling function (i.e., the smallest integer greater than x), the number of steps is

$$n = \Gamma[x(t^*)] - 1 = \Gamma\left[2u\left(v + u + \sqrt{(v+u)^2 - z}\right) - z\right]$$

and the time of impact is

$$t_{\text{imp}} = 2\left(v + \sqrt{v^2 + n}\right). \tag{1.6}$$

Now, a simple substitution gives

$$\begin{cases} u' = e_t u, \\ v' = e_n\sqrt{v^2 + n}, \\ z' = z + n - 2u\left(v + \sqrt{v^2 + n}\right), \\ n = 0, \quad \text{if } z \geqslant 4uv, \\ n = \Gamma\left[2u\left(v + u + \sqrt{(v+u)^2 - z}\right) - z\right], \quad \text{otherwise.} \end{cases} \tag{1.7}$$

Note that the dynamical system of Eq. (11) in [9] is obtained by coordinate transform which is singular if the inclination (or κ) of the staircase is zero. To undo the singularity substitute $\tilde{u} = \kappa u$ in Eq. (11) of [9]. If subsequently one sets $\kappa = 0$, Eq. (1.7) above is recovered.

2. Some global aspects of the discrete model

In this section we will analyze the global structure of solutions of Eq. (1.7) when $(e_t, e_n) \in [0,1]^2$. Let us observe that this equation is of the type

$$\begin{pmatrix} u' \\ v' \\ z' \end{pmatrix} = F\begin{pmatrix} u \\ v \\ z \end{pmatrix},$$

where F: $[0,\infty)^2 \times [0,1] \to [0,\infty)^2 \times [0,1]$ is a discontinuous map. Let us observe that we have skipped the cases $u < 0$ since the particle would go up the staircase. Furthermore $v < 0$ is physically impossible for an initial condition. The case $v = 0$ does not have clear meaning in the continuous time model but we include it for completeness.

2.1. Boundedness of the solutions

When one tries to understand the global behavior of a dynamical system, it is helpful to know whether the solutions are bounded. The main theorem of the actual section deals about this property and a exhaustive analysis for all (reasonable) values of the parameters will be done. Let us start proving this useful result.

Proposition 2.1. *Assume that $e_t e_n < 1$. If $(u_{k_0}, v_{k_0}, z_{k_0})$ is an initial condition satisfying*

$$z_{k_0} \geqslant \frac{4u_{k_0}v_{k_0}}{1 - e_t e_n}. \tag{2.8}$$

Then $n_k = 0$ for all $k \geqslant k_0$.

Proof. Working with (1.7) and assuming (2.8) is easy to see that $n_{k_0} = 0$, so $u_{k_0+1} = e_t u_{k_0}$, $v_{k_0+1} = e_n v_{k0}$ and

$$z_{k_0+1} = z_{k_0} - 4u_{k_0}v_{k_0} \geqslant \frac{4u_{k_0+1}v_{k_0+1}}{1 - e_t e_n}.$$

The proof follows by induction. □

Remark 2.1. If $e_t = e_n = 1$ then the previous proposition does not hold. On the contrary, if there exists some k_0 such that $n_k = 0$ for every $k \geqslant k_0$ then we would have that

$$z_k = z_{k_0} - 4(k - k_0)u_{k_0}v_{k_0},$$

and then z_k cannot always be positive, a contradiction. Therefore, n_k must be non-zero infinitely often.

Before we state the main result of the section, let us note that to study boundedness we only need to know the behavior of v_k, since u_k follows a geometrical law with ratio e_t.

Theorem 2.1. *All solutions of (1.7) are bounded if and only if $e_t e_n < 1$. Moreover for any initial condition $u_0 > 0$, $v_0 > 0$, $z_0 \in [0, 1]$ we have the following table*

	$e_t < 1$	$e_t = 1$
$e_n < 1$	$\lim v_k = 0$	v_k *is bounded*
$e_n = 1$	$\exists k_0\ \forall k \geqslant k_0 : v_{k+1} = v_k$	$\lim v_k = \infty$

Proof. One easily sees from Eq. (1.7) that when $e_t = 1$ and $e_n = 1$ there is a constant of the motion:

$$I_k = u_k^2 + v_k^2 - \sum_{i=0}^{k} n_i = \text{const.}$$

Recall that $u_k = u_0 > 0$ is also a constant of the motion. Now Remark 2.1 proves that $\sum_{i=0}^{k} n_i$ is divergent and thus v_k must be divergent as well.

In the remainder of the proof $e_t e_n < 1$. Since $\Gamma(x) \leqslant x + 1$, one gets that

$$n_k \leqslant 4u_k^2 + 4u_k v_k + 1 \tag{2.9}$$

and defining $w_k := u_k v_k$, we obtain

$$w_{k+1} \leqslant e_t e_n \sqrt{w_k^2 + 4u_k^4 + 4u_k^2 w_k + u_k^2}.$$

As u_k is decreasing we can choose a positive constant δ such that $u_k \leqslant \delta$. So w_k is a subsolution of the scalar dynamical system

$$w_{k+1} = e_t e_n \sqrt{w_k^2 + 4\delta^4 + 4\delta^2 w_k + \delta^2},$$

which has a fixed point $w^* > 0$. Using now that the right hand side of the last equation is increasing in w is standard to prove that subsolutions are bounded by solutions and in consequence $\limsup_{k\to\infty} w_k \leqslant w^*(\delta)$. Now the affirmation made for the case $e_t = 1$, $e_n < 1$ follows easily since u_k is constant and w_k bounded.

It only remains to check the properties for $e_t < 1$, taking into account that $\lim u_k = 0$ in those cases. To do that, we will use that the constant δ can be chosen arbitrary small for k large. One then sees that $w^*(\delta)$ defined in the previous paragraph can be taken arbitrary small. This implies that $\lim w_k = 0$.

The next step is to show that n_k is zero for $k \geqslant k_0$. Arguing by contradiction, let us suppose that n_k is not eventually zero. Then one uses Proposition 2.1 to get that $z_k < \frac{4u_k v_k}{1-e_t e_n} = \frac{4w_k}{1-e_t e_n}$. Now since $\lim w_k = 0$, it follows that $\lim z_k = 0$, so using the equation for z of (1.7), $\lim n_k = 0$, because n_k is bounded from (2.9). This is a contradiction. The proof finishes since $k \geqslant k_0$, $v_k = e_n^{k-k_0} v_{k_0}$. □

Remark 2.2. As consequence of the proof we can also give the behavior of the jump number. If $e_t < 1$ or else if $e_t = 1$ and $e_n = 0$, then n_k must be zero for k large. Whereas if both restitution coefficients are equal to one then u_k is constant, v_k diverges and from (1.7) also n_k tend to infinity. The case $e_t = 1$ and $e_n \in (0, 1)$ will be dealt with below.

2.2. Periodic bouncing solutions

This section is devoted to the study of the periodic solutions of (1.7). We will concentrate on the case that $e_t = 1$ and $e_n \in (0,1)$. In the other cases a relatively straightforward analysis shows that the unique periodic solutions are fixed points. These fixed points can be easily computed and are shown in the following table:

	$e_t \in [0,1)$	$e_t = 1$
$e_n = 0$	$(0,0,z^*)$	$(u^*,0,z^*)$
$e_n \in (0,1)$	$(0,0,z^*)$	To be considered later
$e_n = 1$	$(0,v^*,z^*)$	$(u^*,0,z^*)$ and $(0,v^*,z^*)$

where $u^* \geqslant 0$, $v^* \geqslant 0$, $z^* \in [0,1]$ are constants. This can be seen as a consequence of Theorem 2.1, since in those cases we know the asymptotic behavior of the solution.

From now on we will consider $e_t = 1$ and denote $e_n := e \in (0,1)$. Note that u_k becomes now a constant that will be called u, so the solutions can be described only with two components $\{v_k, z_k\}$.

We will from here on use the *bouncing* to describe a solution that touches the staircase in a discrete sequence of points. On the contrary a sliding solution may touch the staircase in intervals. We will say that an orbit is periodic if it corresponds to periodic solutions of (1.7). There is no problem in referring to such orbits as corresponding to periodic solutions of (1.3) and (1.4), since we can rewrite those in terms of moving reference frame. This moving reference frame is centered on the staircase but moves with velocity whose horizontal component equals u with respect to the old "laboratory frame". The aforementioned solutions are truly periodic in the new coordinates.

We begin by studying the simplest case which occurs when the sequence $\{v_k, z_k\}$ is constant. These periodic solutions are said of type (n), if n the number of steps jumped by the particle between two consecutive impacts.

Theorem 2.2. *Assume $e_t = 1$ and $e_n = e \in (0,1)$. There exists a periodic solution of (1.7) of type (n) if and only if*

$$u = \frac{1}{2}\sqrt{\frac{n(1-e)}{1+e}}. \tag{2.10}$$

Remark. The parameters values for which these orbits exist (for $n \in \{1,2,3\}$ only) are shown as solid curves in Fig. 2.

Proof. Given initial conditions (v_0, z_0), the first iteration is

$$v_1 = e\sqrt{v_0^2 + n},$$
$$z_1 = z_0 + n - 2u\left(v_0 + \sqrt{v_0^2 + n}\right).$$

By imposing $v_1 = v_0$ in the first equation, v_0 is uniquely determined by

$$v_0 = e\sqrt{\frac{n}{1-e^2}}.$$

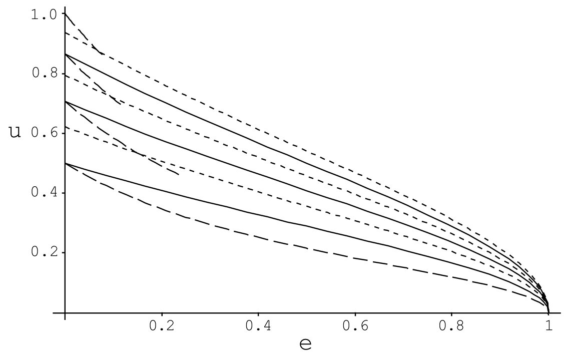

Fig. 2. Periodic bouncing solutions.

Analogously, taking $z_1 = z_0$ in the second equation we get

$$u = \frac{1}{2}\sqrt{\frac{n(1-e)}{1+e}}$$

after some computations. To finish the proof, it remains to verify the compatibility condition, namely that the number of steps in the jump is in fact n. Note that n is defined as the first positive integer such that $z_1 \geqslant 0$ and it is not restrictive to assume $z_1 = 0$, since the solution is invariant by horizontal translation. Taking into account the definition of z_1, the question is reduced to verify that

$$n - 1 - 2u\left(v_0 + \sqrt{v_0^2 + n - 1}\right) < 0.$$

By substituting the values of v_0 and u, the left-hand side of this inequality defines the following function of e

$$f(e) = n - 1 - \frac{n}{1+e}\left(e + \sqrt{1 - \frac{1-e^2}{n}}\right).$$

If $n = 1$, it is trivial to see that $f(e) < 0$ for all $e \in (0,1)$. If $n \geqslant 2$ then $f(0) = n - 1 - \sqrt{n(n-1)} < 0$ and $f'(e) < 0$ for every $e \in (0,1)$. In consequence, $f(e) < 0$ for all $e \in (0,1)$. □

We are going to analyze in the following solutions of $(0,n)$ type. These are periodic solutions that have two impacts on the same step and then jump n steps down, and so on periodically. The presence of these solutions (for $n \in \{1,2,3,4\}$) is indicated in Fig. 2 in long dashed curves.

Theorem 2.3. *Assume $e_t = 1$, $e_n \equiv e \in (0,1)$. There exists a solution of (1.7) of type $(0,n)$ if and only if*

$$u = \frac{\sqrt{n(1-e^4)}}{2(1+e)^2} \tag{2.11}$$

for $e \in (0, e_n^*)$, where $e_1^* = 1$ and $\{e_n^*\}$ is a sequence converging to zero as n tends to infinity.

Proof. Suppose that (v_0, z_0) is the initial condition and that the first impact is in the same step, so

$$v_1 = ev_0, \quad z_1 = z_0 - 4uv_0.$$

Now, the solution jumps n steps after the following impact, so

$$v_2 = e\sqrt{v_1^2 + n} = e\sqrt{e^2v_0^2 + n},$$
$$z_2z_1 + n - 2u(v_1 + \sqrt{v_1^2 + n}) = z_0 - 4uv_0 + n - 2u(ev_0 + \sqrt{e^2v_0^2 + n}).$$

By imposing that $v_2 = v_0$, $z_2 = z_0$ we get

$$v_0 = e\sqrt{\frac{n}{1-e^4}},$$
$$u = \frac{n}{2\left((2+e)v_0 + \sqrt{e^2v_0^2 + n}\right)}.$$

Then a simple substitution of the obtained value of v_0 in the second equation provides condition (2.11) after some algebra.

As in the previous proof, it remains to verify compatibility condition. In this case, it reduces to verify that if the jump number equals $n-1$, then $z_2 < 0$ because we can assume that $z_1 = 0$ without loss of generality. This means that

$$z_0 = 4uv_0 = \frac{2ne}{(1+e)^2}.$$

Considering z_2 as a function of the jump number n, we have

$$z_2(n-1) = n - 1 - 2u\left(ev_0 + \sqrt{e^2v_0^2 + n - 1}\right)n - 1 - \frac{ne^2}{(1+e)^2} - \frac{\sqrt{n}}{(1+e)^2}\sqrt{e^4 + n - 1}.$$

For $n = 1$, it is evident that $z_2(0) < 0$ for all $e \in (0,1)$. Thus, there exists a periodic solution of type $(0,1)$ for all $e \in (0,1)$.

For a general $n \geqslant 2$, by defining the function

$$F(e,n) = \sqrt{n(e^4 + n - 1)} + e^2 - 2e(n-1) - (n-1),$$

it is easy to verify that $z_2(n-1)<0$ if and only if $F(e,n)>0$. Note that

$$\frac{\partial F}{\partial e}=\frac{2e^3\sqrt{n}}{\sqrt{e^4+n-1}}+2e-2(n-1)<\frac{2(3-n)\sqrt{n}}{\sqrt{e^4+n-1}}\leqslant 0 \quad \text{if } n\geqslant 3.$$

We will study first the case $n\geqslant 3$, leaving $n=2$ for a separate study. Now F is strictly decreasing in e. Besides, $F(0,n)<0$ and $F(1,n)>0$, so in consequence for every $n\geqslant 3$ there exists $e_n^*\in(0,1)$ such that $z_2(n-1)<0$ if $e\in(0,e_n^*)$ and $z_2(n-1)>0$ if $e\in(e_n^*,1)$. By definition, $F(e_n^*,n)=0$.

For the particular case $n=2$, it is possible to compute explicitly: $z_2=\frac{1-e^2+2e-\sqrt{2(1+e^4)}}{(1+e)^2}<0$ if and only if $e\in(0,e_2^*)$ with $e_2^*=1/(2+\sqrt{5})$.

Finally, let us show that the sequence $\{e_n^*\}$ is strictly decreasing. Note that e_n^* is the unique root of $F(e,n)=0$ in the interval $(0,1)$. By dividing $F(e,n)=0$ by $(n-1)(e+1)$, we get the equation

$$e^3+3e^2+(3-4n)e+1=0.$$

In other words, e_n^* is the unique root belonging to $(0,1)$ of

$$n=\frac{(e+1)^3}{4e}.$$

These values are easily computable numerically. An elementary analysis of the monotonicity intervals of the right-hand side as a function of e shows that the sequence $\{e_n^*\}$ is strictly decreasing. □

The last kind of solutions considered here are those of type $(n+1,n)$. They are the solutions such that the particle jumps $n+1$ steps before the first impact, then jumps n steps. Their presence is indicated in Fig. 2 by short dashed curves for $n\in\{1,2,3\}$.

Theorem 2.4. *Assume that $e_t=1$ and $e_n=e\in(0,1)$. Eq. (1.7) admits a solution of type $(n+1,n)$ (where $n\in\{1,2,3,\ldots\}$) if and only if*

$$u=\frac{(2n+1)\sqrt{1-e^4}}{2(e+1)\left[\sqrt{n(e^2+1)+1}+\sqrt{n(e^2+1)+e^2}\right]}. \tag{2.12}$$

Proof. The first iteration of the dynamical system is

$$v_1=e\sqrt{v_0^2+n+1},$$
$$z_1=z_0+n+1-2u\left(v_0+\sqrt{v_0^2+n+1}\right).$$

The next iteration is

$$v_2=e\sqrt{v_1^2+n}=e\sqrt{e^2v_0^2+n(e^2+1)+e^2},$$
$$z_2=z_1+n-2u\left(v_1+\sqrt{v_1^2+n}\right).$$

As in the previous proof, it can assume without loss of generality that $z_0=0$. Now, by imposing that $v_2=v_0$, we get

$$v_0=\frac{e}{\sqrt{1-e^4}}\sqrt{n(e^2+1)+e^2}.$$

On the other hand, if $z_2=0$ then

$$u=\frac{(2n+1)e}{2(e+1)\left(v_0+e\sqrt{v_0^2+n+1}\right)}$$

and by substituting the expression of v_0 in the previous equation we get Eq. (2.12).

The last point is to check the compatibility conditions (again z_1 is written as a function of n)

$$z_1(n)=n-2u\left(v_0+\sqrt{v_0^2+n}\right)<0$$

and

$$z_1(n+1) = n+1-2u\left(v_0+\sqrt{v_0^2+n+1}\right) \geqslant 0.$$

This can be done (after some computations) by substituting the expressions for u and v_0. □

The set

$$B = \{(e,u) : \text{there exists periodic solutions of (1.7)}\}$$

can be seen as a very complicated set composed by infinitely many branches of analytic curves.

Theorem 2.5. *For most parameter values* (e,u), *Eq.* (1.7) *does not have periodic bouncing solutions.*

Proof. We are going to show that for each word $(m_0,\ldots,m_{k-1})$ (periodically repeated), the subset of B corresponding to solutions whose jump numbers are defined by this sequence is contained in the graph of an analytic function u: $(0,1)\to IR$ and therefore has (2-dimensional Lebesgue) zero measure. Since B consists of a countable collection of such curves, it has zero measure.

Let $(m_0,\ldots,m_{k-1})$ be a given word. Then, by (1.7) we obtain

$$v_{i+1}^2 = e^2(v_i^2+m_i).$$

So

$$v_k^2 = e^{2k}v_0^2 + e^2 m_{k-1} + e^4 m_{k-2} + \cdots + e^{2(k-1)}m_2 + e^{2k}m_0.$$

Now if we impose that v_k must be equal to v_0, we can solve for v_0, and therefore obtain $v_1,\ldots,v_{k-1}$ as functions of the parameter e. Using (1.7) again for z,

$$z_{k+1} = z_k + m_k - 2u\left(v_k(e) + \frac{1}{e}v_{k+1}(e)\right).$$

So

$$z_k = z_0 + m_0 + \cdots + m_{k-1} - 2u\left(v_0 + \cdots + v_{k-1} + \frac{1}{e}(v_1 + \cdots + v_k)\right).$$

By imposing $z_k = z_0$ and solving for u one obtains

$$u(e) = \frac{e\sum_{j=0}^{k-1} m_j}{2(e+1)\sum_{j=0}^{k-1} v_j},$$

which is the announced function. □

We finish this section by studying the stability of periodic solutions. We need the following definition:

We say that a discrete solution $\{v_k, z_k\}$ is *strict* if for all k we have that $z_k > 0$. In other words, a solution is strict if the particle never lands on a point where $z = 0$.

Theorem 2.6. *Every strict bouncing periodic solution of* (1.7) *is Lyapunov stable* (*but not asymptotically stable*).

Proof. Let $(v_0^*, z_0^*,), (v_1^*, z_1^*), \ldots, (v_{k-1}^*, z_{k-1}^*)$ be the successive iterations for a strict bouncing periodic solution with associated (periodic) word $(m_0,\ldots,m_{k-1})$.

We are going to prove that if a solution of (1.7) is near enough to the above periodic solution, then it jumps exactly the same number of steps and they are close in the future. Since $\Gamma(x)$ is continuous whenever x is not an integer, we have the following. There exists $\delta > 0$ such that if $|v_0 - v_0^*| < \delta$ and $|z_0 - z_0^*| < \delta$, then the solution with initial conditions (v_0, z_0) jumps exactly $m_0,\ldots,m_{k-1}$ steps on the first k impacts.

To continue the proof we need first a lemma:

Lemma 2.1. *If two solutions have the same associated word until the* kth *impact, then they stay close in the next impact. More precisely, if* $\{v_k, z_k\}$ *and* $\{v_k^*, z_k^*\}$ *are such solutions, then the following inequalities are true for all* k

(a) $|v_k - v_k^*| \leqslant e^k |v_0 - v_0^*|$,

(b) $|z_k - z_k^*| \leqslant |z_0 - z_0^*| + 4ue\dfrac{1-e^k}{1-e}|v_0 - v_0^*|$.

Proof. Part (a) holds since the function $v \to e\sqrt{v^2+n}$ is Lipschitz continuous with constant e.

On the other hand, part (b) follows from an easy calculation based on the estimate

$$|z_k - z_k^*| \leqslant |z_{k-1} - z_{k-1}^*| + 2u\left(|v_k - v_k^*| + \frac{1}{e}|v_{k+1} - v_{k+1}^*|\right). \qquad \square$$

To continue the proof of the theorem, let $\delta > 0$ be given by Lemma 2.1 and take $\varepsilon > 0$ such that

$$\left(4u\frac{e}{1-e} + 1\right)\varepsilon < \delta. \tag{2.13}$$

Let us suppose in addition that the we are near the periodic solution in the initial condition, that is

$$|v_0 - v_0^*| < \varepsilon, \quad |z_0 - z_0^*| < \varepsilon.$$

Note that, in particular, $\varepsilon < \delta$. Taking into account Lemma 2.1, we get that $\{v_n, z_n\}$ and $\{v_n^*, z_n^*\}$ have the same configuration until the kth impact. Moreover, using (2.13) and applying the same lemma, the solutions are close at the kth impact. The lemma follows by iteration. $\square$

Remark. For non-strict periodic bouncing solutions one observes that small perturbations of the initial values may change the jump number by an integer and after it, the corresponding solution looks totally different. Such a solution is (Lyapunov) unstable.

3. The continuous time model

In this section we will consider the original model (1.3) and (1.4) because it presents some interesting particularities that cannot be obtained directly from the discrete model (1.7).

Let us start noting that x' is always non-negative. We are interested in the case where x' is strictly positive. (In the case $x' < 0$ the particle will temporarily go up the stair, but it is easy to see that eventually it must go down. Once it goes down, it will continue to do so.) The case $x' = 0$ is a trivial case that is included for completeness since it appears after the first impact when $e_t = 0$.

Secondly let us observe that if $v_k = 0$ for some k in a solution of the discrete equation (1.7) then there is no jump after it. So it is not clear what the meaning is of the discrete solution after that. In fact, if $e_n = 0$ and $e_t \in (0,1]$ there is no consistent way to continue a non-constant solution $(x(t), y(t))$ after the first impact. This forces us to not consider $e_n = 0$ since $v_k = 0$ for any $k > 0$. Moreover any attempt to extend this possible cases $e_n = 0$ gives a trivial solution.

In the following we consider $e_n \in (0,1]$. In this case a solution of the continuous time model can be described by a discrete sequence of impact positions, velocities and impact times. If the impact times accumulate, the task is how to continue the solution. Recall that if no such accumulation occurs, the solution is a *bouncing* solution.

When $e_t \in [0,1)$, then from Remark 2.2 the jump number n_k is zero for k large, so that the particle eventually remains at the some step. Depending on e_n there are several possibilities. If $e_n = 1$ then the times between consecutive impacts do not tend to zero as we see in (1.6) so all non-constant solutions are bouncing. However, if the restitution coefficient $e_n \in (0,1)$, the times between two consecutive impacts decrease geometrically and the sequence of impact t_k converges to some time t_s. Since

$$\lim u_k = 0,$$
$$\lim v_k = 0,$$

the solution of the system (1.3) and (1.4) has to be continued as a constant function and the particle stops. We will refer to it as a *halting solution*.

When $e_t = 1$ and $e_n \in (0,1)$ it is also possible that for some initial conditions, $n_k = 0$ for k large (see Proposition 2.1). We have again that the sequence of impact times tends to t_s but we only have $\lim v_k = 0$. The x component remains unchanged and y is continued as a constant until the next step, when the particle takes off. We will refer to this kind of solutions as a *sliding solution*.

Sliding solutions always exist, because of Proposition 2.1. From Section 2.2 there are bouncing solutions for certain parameter values, so we see that sliding and bouncing solutions can co-exist.

Of course, this is not true for $e_t = e_n = 1$ where again the times between consecutive impacts do not tend to zero, so all non-constant solutions are of bouncing type.

Let us clarify the meaning of Theorem 2.1 in relation with this model. It is clear that if the particle eventually stops or remains bouncing on some step, then the corresponding solution of the continuous time model is bounded. Otherwise, the solution will be unbounded because the vertical component decreases each time it goes down one step.

However, Theorem 2.1 ensures the boundedness of the solutions in this case, unless $e_t = e_n = 1$. In order to interpret this boundedness for the corresponding continuous time solutions, we choose a reference system that moves together with the stair axis. The boundedness given by the discrete solutions corresponds with the boundedness of the solutions referred to this system.

3.1. Periodic sliding solutions

We say that a solution of (1.3) and (1.4) is periodic if it is periodic in a reference system that moves together with the stair axis, or more simply if $y(t) + E[x(t)]$ is periodic. From the above discussion it is easy to conclude that $e_t = 1$ and $e_n \in (0,1)$ is the unique case which needs to be analyzed. For the other values, there are no periodic solutions because either the solutions die on a step or the relative height diverges. Therefore, we assume that $e_t = 1$ and $e_n \in (0,1)$ in the following. We denote the constant horizontal velocity by a constant u, which must be strictly positive.

As we said before, there exists two types of periodic solutions: bouncing and sliding. The first ones can be described completely by the discrete model and that has already been done. However, to study sliding periodic solutions we also need to use the discrete model but it must be done delicately. Let us explain how to describe a sliding periodic solution. Suppose first that there are k_0 impacts before an accumulation of impacts. Then, it is possible to follow the solution for the discrete system which gives a sequence $\{v_k, z_k\}$ for $k = 0, \ldots, k_0$ and a word $(n_0, \ldots, n_{k_0-1})$.

Assume now that v_k, z_k verify Eq. (2.8), so $n_k = 0$ for every $k \geqslant k_0$. We collapse this as 0^∞ and refer to it as an *accumulation tail*. This is the most delicate moment because this assertion implies that v becomes zero, so the discrete model becomes insufficient. Moreover, to be in concordance with the continuous solutions also z becomes zero because the particle slides until the border of the step. If we would only look now the discrete model to calculate the next jump number, as $z \geqslant 4uv$, this must be zero forever and the particle would not never fall down again. But this is not the hoped behavior, so it is necessary to use the other formula to calculate this jump number and then it is possible to continue by using the discrete model.

Let us now pay attention to periodic sliding solutions of type $(n, 0^\infty)$. These solutions jump n steps and then there is an accumulation of impacts, after which the solution slides until the border and the motion is repeated.

Theorem 3.1. *Assume $e_t = 1$ and $e_n \in (0,1)$. Eq. (1.7) admits sliding solutions of type $(n, 0^\infty)$ if and only if the following inequality holds*

$$\frac{\Gamma(4u^2)}{4u^2} \leqslant \left(\frac{1+e}{1-e}\right)^2.$$

Proof. Since the solution is periodic and sliding, we can start with the initial condition $v_0 = 0$, $z_0 = 0$, upon which the particle free falls n steps until the next impact.

$$\begin{aligned} v_1 &= e\sqrt{n}, \\ z_1 &= n - 2u\sqrt{n}, \\ n &= \Gamma(4u^2). \end{aligned}$$

Now, the accumulation of impacts takes place in the same step. Using a recursive procedure one gets that

$$v_2 = ev_1, \ldots, v_k = e^{k-1}v_1$$

and

$$z_2 = z_1 - 4uv_1(1+e), \ldots, z_k = z_1 - 4uv_1\frac{1-e^{k-1}}{1-e}.$$

Then, the impacts accumulate at

$$z_\infty = \lim_{k\to\infty} z_k = z_1 - 4\frac{uv_1}{1-e} = n - 2u\sqrt{n}\frac{1+e}{1-e}.$$

It remains to prove that $z_\infty \in [0,1)$. The condition $z_\infty < 1$ is evident since $z_\infty < z_1 < 1$. On the other hand, $z_\infty \geqslant 0$ is equivalent after some algebra to

$$n \geqslant 4u^2\left(\frac{1+e}{1-e}\right)^2.$$

As $n = \Gamma(4u^2)$, we get the proposed condition. □

We observe that the theorem immediately implies that the set of parameter values for which there are sliding solutions of type $(n, 0^{\infty})$ is given by the following union (see Fig. 3)

$$\cup_{n\in\mathbb{N}}\left\{(e,u)\in(0,1)\times\mathbb{R}^{+}\left|\frac{\sqrt{n-1}}{2}<u\leqslant\frac{\sqrt{n}}{2}\left(\frac{1-e}{1+e}\right)\right.\right\}.$$

With similar arguments one can show that there are other sliding regions. For example, regions $(2,1,0^{\infty})$, $(1,1,0^{\infty})$ are plotted in Fig. 3. However, not all configuration are possible, being $(1,2,0^{\infty})$ an example. The problem of describing completely the possible configurations seems to be difficult.

Nevertheless, as one can see in the next theorem, there exists a region without periodic sliding solutions. On the other hand, numerical simulations show that the region where there exists at least one periodic sliding solution presents a complex fractal-like structure. It can be observed in Fig. 4, where the solutions of type $(n, 0^{\infty})$ are excluded.

Theorem 3.2. *Assume* $e_t = 1$ *and* $e_n \equiv e \in (0,1)$*. A necessary condition for existence of periodic sliding solution of* (1.3) *and* (1.4) *is*

$$u < \frac{1-e}{4e}.$$

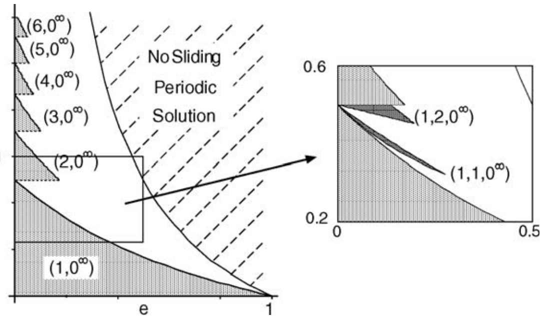

Fig. 3. Sliding periodic solutions.

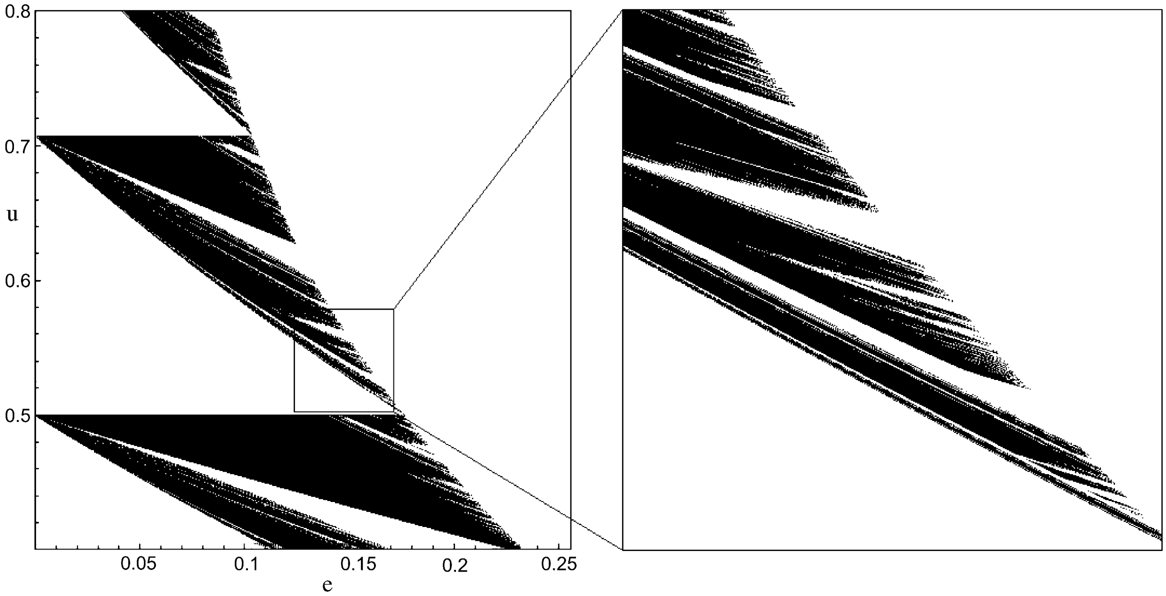

Fig. 4. Numerical computations of sliding periodic solutions. The left-hand frame corresponds to $(e,u) \in [0, 0.25] \times [0.4, 0.8]$. The zoom is done in the region $(e,u) \in [0.12, 0.17] \times [0.5, 0.6]$.

Proof. Take a periodic sliding solution and let us label (v_0, z_0) and (v_1, z_1) two adjacent impacts with jump number $n > 0$ just before an accumulation tail. Then $v_1 = e\sqrt{v_0^2 + n} \geqslant e$ and

$$1 \geqslant z_1 \geqslant \frac{4uv_1}{1-e} \geqslant \frac{4ue}{1-e} \quad \Rightarrow \quad u \leqslant \frac{1-e}{4e}.$$

Now we show that the equality is not possible. In such case, we would have $n = 1$ and $v_0 = 0$ necessarily. Then, we are after an accumulation tail and so $z_0 = 0$. Now we calculate $z_1 = 1 - 4u < 1$. Inserting this in the first equation above we see that at least one of the inequalities must be strict. □

The exclusion region for sliding solutions given by the previous result is depicted in Fig. 3.

Remark 3.1. If we take into account the Theorem 2.5 and the non-existence of periodic sliding solutions whenever $u \geqslant \frac{1-e}{4e}$, we can affirm that there exist parameter values (e, u) without any type of periodic solutions.

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