**Communications in Mathematical Physics manuscript No.**
(will be inserted by the editor)

# On Brillouin Zones

**J. J. P. Veerman[1], Mauricio M. Peixoto[2], André C. Rocha[3], Scott Sutherland[4]**

[1] Mathematics Department, Pennsylvania State University, University Park, Pennsylvania, USA. E-mail: veerman@math.psu.edu
[2] Instituto de Matemática Pura e Aplicada, Rio de Janeiro, Brazil. E-mail: peixoto@impa.br
[3] Departamento de Matemática, Universidade Federal de Pernambuco, Recife, Brazil.
[4] Institute for Mathematical Sciences and Mathematics Department, State University of New York, Stony Brook, New York, USA. E-mail: scott@math.sunysb.edu

Received: 7 August 1998/ Accepted: 22 March 2000

**Abstract:** Brillouin zones were introduced by Brillouin [Br] in the thirties to describe quantum mechanical properties of crystals, that is, in a lattice in $\mathbb{R}^n$. They play an important role in solid-state physics. It was shown by Bieberbach [Bi] that Brillouin zones tile the underlying space and that each zone has the same area. We generalize the notion of Brillouin zones to apply to an arbitrary discrete set in a proper metric space, and show that analogs of Bieberbach's results hold in this context.

We then use these ideas to discuss focusing of geodesics in spaces of constant curvature. In the particular case of the Riemann surfaces $\mathbb{H}^2/\Gamma(k)$ $(k = 2, 3, \text{ or } 5)$, we explicitly count the number of geodesics of length $t$ that connect the point $i$ to itself.

## 1. Introduction

In solid-state physics, the notion of Brillouin zones is used to describe the behavior of an electron in a perfect crystal. In a crystal, the atoms are often arranged in a lattice; for example, in NaCl, the sodium and chlorine atoms are arranged along the points of the simple cubic lattice $\mathbb{Z}^3$. If we pick a specific atom and call it the origin, its **first Brillouin zone** consists of the points in $\mathbb{R}^3$ which are closer to the origin than to any other element of the lattice. This same zone can be constructed as follows: for each element $a$ in the lattice, let $L_{0a}$ be the perpendicular bisecting plane of the line between $0$ and $a$ (this plane is called a Bragg plane). The volume about the origin enclosed by these intersecting planes is the first Brillouin zone, $b_1(0)$. This construction also allows us to define the higher Brillouin zones as well: a point $x$ is in $b_n$ if the line connecting it to the origin crosses exactly $n-1$ planes $L_{0a}$, counted with multiplicity.

This notion was introduced by Brillouin in the 1930s ([Br]), and plays an important role in solid-state theory (see, for example, [AM,Jo2,Ti]). The construction which gives rise to Brillouin zones is not limited to consideration of crystals, however. For example, in computational geometry, the notion of the Voronoi cell corresponds exactly to the first Brillouin zone described above (see [PS]). We shall also see below how, after suitable generalization, this construction coincides with the Dirichlet domain of Riemannian geometry, and in many cases, with the focal decomposition introduced in [Pe1] (see also [Pe3]).

With some slight hypotheses (see Sect. 2), we generalize the construction of Brillouin zones to any discrete set $S$ in a path-connected, proper metric space $X$. We generalize the Bragg planes above as mediatrices, defined here.

**Definition 1.1.** *For $a$ and $b$ distinct points in $S$, define the* **mediatrix** *(also called the* **equidistant set** *or* **bisector***) $L_{ab}$ of $a$ and $b$ as:*

$$L_{ab} = \{x \in X \mid d(x,a) = d(x,b)\}\,.$$

Now choose a preferred point $x_0$ in $S$, and consider the collection of mediatrices $\{L_{x_0,s}\}_{s\in S}$. These partition $X$ into Brillouin zones as above: roughly, the $n^{th}$ Brillouin zone $B_n(x_0)$ consists of those points in $X$ which are accessible

Stony Brook IMS Preprint #1998/7
*June 1998*
*revised December 1999*

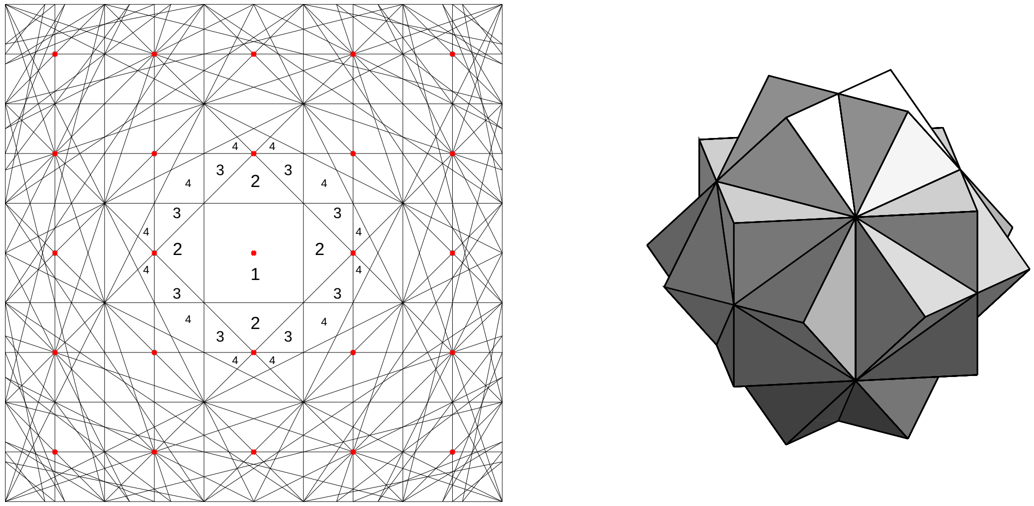


**Fig. 1.1.** On the left are the Brillouin zones for the lattice $\mathbb{Z}^2$ in $\mathbb{R}^2$. On the right is the outer boundary of the third Brillouin zone for the lattice $\mathbb{Z}^3$ in $\mathbb{R}^3$.

from $x_0$ by crossing exactly $n-1$ mediatrices. (There is some difficulty accounting for multiple crossings— see Def. 2.6 for a precise statement.)

One basic property of the zones $B_n$ is that they tile the space $X$:

$$\bigcup_{x_i \in S} B_n(x_i) = X \qquad \text{and} \qquad B_n(x_0) \cap B_n(x_1) \text{ is small} .$$

Here, with some extra hypotheses, "small" means of measure zero. Furthermore, again with some extra hypothesis, each zone $B_n$ has the same area. (This property was "obvious" to Brillouin.) Both results were proved by Bieberbach in [Bi] in the case of a lattice in $\mathbb{R}^2$. Indeed, he proves (as we do) that each zone forms a fundamental set for the group action of the lattice. His arguments rely heavily on planar Euclidean geometry, although he remarks that his considerations work equally well in $\mathbb{R}^d$ and can be extended to "groups of motions in non-Euclidean spaces". In [Jo1], Jones proves these results for lattices in $\mathbb{R}^d$, as well as giving asymptotics for both the distance from $B_n$ to the basepoint, and for the number of connected components of the interior of $B_n$. In Sect. 2, we show that the tiling result holds for arbitrary discrete sets in a metric space. If the discrete set is generated by a group of isometries, we show that each $B_n$ forms a fundamental set, and consequently all have the same area (see Prop. 2.10).

We now discuss the relationship of Brillouin zones and focal decomposition of Riemannian manifolds.

If $x_1(t)$ and $x_2(t)$ are two solutions of a second order differential equation with $x_1(0) = x_2(0)$ and there is some $T \neq 0$ so that $x_1(T) = x_2(T)$, then the trajectories $x_1$ and $x_2$ are said to **focus** at time $T$. One can ask how the number of trajectories which focus varies with the endpoint $x(T)$— this gives rise to the concept of a **focal decomposition** (originally called a **sigma decomposition**). This concept was introduced in [Pe1] and has important applications in physics, for example when computing the semiclassical quantization using the Feynman path integral method (see [Pe3]). There is also a connection with the arithmetic of positive definite quadratic forms (see [Pe2, KP, Pe3]). Brillouin zones have a similar connection with arithmetic, as can be seen in Sect. 4 as well as [Pe3].

More specifically, consider the two-point boundary problem

$$\ddot{x} = f(t, x, \dot{x}), \qquad x(t_0) = x_0, \qquad x(t_1) = x_1, \qquad x, t, \dot{x}, \ddot{x} \in \mathbb{R}.$$

Associated with this equation, there is a partition of $\mathbb{R}^4$ into sets $\Sigma_k$, where a point $(x_0, x_1, t_0, t_1)$ is in $\Sigma_k$ if there are exactly $k$ solutions which connect $(x_0, t_0)$ to $(x_1, t_1)$. This partition is the focal decomposition with respect to the boundary value problem. In [PT], several explicit examples are worked out, in particular the fundamental example of the pendulum $\ddot{x} = -\sin x$. Also, using results of Hironaka ([Hi]) and Hardt ([Ha]), the possibility of a general, analytic theory was pointed out. In particular, under very general hypotheses, the focal decomposition yields an analytic Whitney stratification.

Later, in [KP], the idea of focal decomposition was approached in the context of geodesics of a Riemannian manifold $M$ (in addition to a reformulation of the main theorem of [PT]). Here, one takes a basepoint $x_0$ in the manifold

$M$: two geodesics $\gamma_1$ and $\gamma_2$ **focus** at some point $y \in M$ if $\gamma_1(T) = y = \gamma_2(T)$. This gives rise to a decomposition of the tangent space of $M$ at $x$ into regions where the same number of geodesics focus.

In order to study focusing of geodesics on a manifold $(M, g)$ with metric $g$ via Brillouin zones, we do the following. Choose a base-point $p_0$ in $M$ and construct the universal cover $X$, lifting $p_0$ to a point $x_0$ in $X$. Let $\gamma$ be a smooth curve in $M$ with initial point $p_0$ and endpoint $p$. Lift $\gamma$ to $\tilde{\gamma}$ in $X$ with initial point $x_0$. Its endpoint will be some $x \in \pi^{-1}(p)$. The metric $g$ on $M$ is lifted to a metric $\tilde{g}$ on $X$ by setting $\tilde{g} = \pi^* g$. Under the above conditions, the group $G$ of deck transformations is discontinuous and so $\pi^{-1}(p_0) \subset X$ is a discrete set. One can ask how many geodesics of length $t$ there are which start at $p_0$ and end in $p$, or translated to $(X, \tilde{\gamma})$, this becomes: *How many mediatrices $L_{x_0,s}$ intersect at $x$, as $s$ ranges over $\pi^{-1}(p_0)$?*

Notice that if the universal cover of $M$ coincides with the tangent space $TM_x$, the focal decomposition of [KP] and that given by Brillouin zones will be the same. If the universal cover and the tangent space are homeomorphic (as is the case for a manifold of constant negative curvature), the two decompositions are not identical, but there is a clear correspondence. However, if the universal cover of the manifold is not homeomorphic to the tangent space at the base point, the focal decomposition and that given by constructing Brillouin zones in the universal cover are completely different. For example, let $M$ be $\mathbb{S}^n$, and let $x$ be any point in it. The focal decomposition with respect to $x$ gives a collection of nested $n-1$-spheres centered at $x$; on each of these infinitely many geodesics focus (each sphere is mapped by the exponential to either $x$ or its antipodal point). Between the spheres are bands in which no focusing occurs. (See [Pe3]). However, using the construction outlined in the previous paragraph gives a very different result. Since $\mathbb{S}^n$ is simply connected, it is its own universal cover. There is only one point in our discrete set, and so the entire sphere $\mathbb{S}^n$ is in the first zone $B_1$.

The organization of this paper is as follows. In Sect. 2, we set up the general machinery we need, and prove the main theorems in the context of a discrete set $S$ in a proper metric space.

Section 3 explores this in the context of manifolds of constant curvature. The universal cover is $\mathbb{R}^n$, $\mathbb{S}^n$, or $\mathbb{H}^n$, and the group $G$ of deck transformations is a discrete group of isometries (see, for example, [doC]). The discrete set $S$ is the orbit of a point not fixed by any element of $G$ under this discontinuous group. It is easy to see that the mediatrices in this case are totally geodesic spaces. From the basic property explained above, one can deduce that the $n^{th}$ Brillouin zone is a fundamental region for the group $G$ in $X$.

In Sect. 4, we calculate exactly the number of geodesics of length $t$ that connect the origin to itself in two cases: the flat torus $\mathbb{R}^2/\mathbb{Z}^2$ and the Riemann surfaces $\mathbb{H}^2/\Gamma(p)$, for $p \in \{2, 3, 5\}$. While these calculations could, of course, be done independent of our construction, we find that the Brillouin zones help visualize the process.

In the final section, we give a nontrivial example in the case of a non-Riemannian metric, and mention a connection to the question of how many integer solutions there are to the equation $a^k + b^k = n$, for fixed $k$.

*Acknowledgement.* It is a pleasure to acknowledge useful conversations with Federico Bonetto, Johann Dupont, Irwin Kra, Bernie Maskit, John Milnor, Chi-Han Sah, and Duncan Sands. Part of this work was carried out while Peter Veerman was visiting the Center for Physics and Biology at Rockefeller University and the Mathematics Department at SUNY Stony Brook; the authors are grateful for the hospitality of these institutions.

## 2. Definitions and Main Results

In this section, we prove that under very general conditions, Brillouin zones tile (as defined below) the space in which they are defined, generalizing an old result of Bieberbach [Bi]. With stronger assumptions, we prove that these tiles are fairly well-behaved sets (see Prop. 2.13).

*Notation.*Throughout this paper, we shall assume $X$ is a path connected, proper (see below) metric space (with metric $d(\cdot, \cdot)$). We will make use of the following notation:

- Write an open $r$-neighborhood of a point $x_0$ as $N_r(x_0) = \{x \in X \mid d(x_0, x) < r\}$.
- Define the circumference as $C_r(x_0) = \{x \in X \mid d(x_0, x) = r\}$.
- The closed disk of radius $r$, denoted by $D_r(x_0) = \{x \in X \mid d(x_0, x) \leq r\}$, is their union.

**Definition 2.1.** *A metric space $X$ is* **proper** *if the distance function $d(x,\cdot)$ is a proper map for every fixed $x \in X$. In particular, for every $x \in X$ and $r > 0$, the closed ball $D_r(x)$ is compact. Such a metric space is also sometimes called a* **geometry** *(see [Ca]).*

Note if $X$ is proper, path-connected metric space, it is locally compact and complete. By the Hopf-Rinow Theorem, the converse also holds if $X$ is a geodesic metric space, also called a "length space" (see [Gr]). A metric space is a length space if the distance between any two points coincides with the infimum of the lengths of curves joining them. Although the notions do not quite coincide, metrically consistent spaces (defined below) are closely related to length spaces.

**Definition 2.2.** *The space $X$ is called* **metrically consistent** *if, for all $x$ in $X$, all $R > r > 0$ in $\mathbb{R}$ with $r$ sufficiently small, and for each $a \in C_R(x)$, there is a $z \in C_r(x)$ satisfying $N_{d(z,a)}(z) \subseteq N_R(x)$ and $C_{d(z,a)}(z) \cap C_R(x) = \{a\}$.*

Metric consistency ensures some regularity properties, which we need to use only in Proposition 2.13. We note that every Riemannian metric space is metrically consistent.

Any mediatrix $L_{a,b}$ separates $X$, that is: $X \setminus L_{ab}$ contains at least two components (one containing the point $a$ and the other $b$). Another regularity condition that we will sometimes want is for the complement of $L$ to have *exactly* two components, and for $L$ to be minimal:

**Definition 2.3.** *We say that the mediatrix $L_{ab}$ is* **minimally separating** *if for any subset $\hat{L} \subset L_{ab}$ with $\hat{L} \neq L_{ab}$, the set $X - \hat{L}$ has one component.*

We will use the notation

$$L_{0a}^- = \{x \in X \mid d(0,x) - d(a,x) < 0\} \text{ and } L_{0a}^+ = \{x \in X \mid d(0,x) - d(a,x) > 0\}$$

for the two components of $X \setminus L_{0a}$; we will sometimes omit the subscripts and just use $L^+$ and $L^-$. Note that a minimally separating set $L$ is contained both in the closure of $L^-$ and in the closure of $L^+$. To see this, let $V$ be an open set contained in $L$. Then $L^- \cup V$ and $L^+$ are disjoint open sets. Consequently, $L \setminus (L \cap V)$ separates $X$, which contradicts the minimality of $L$.

Usually, there will be a discrete set of points $S = \{x_i\}_{i \in I}$ in $X$ which will be of interest. By discrete we mean that any compact subset of $X$ contains finitely many points of $S$. Note that if $\liminf\limits_{a,b \in S} d(a,b) > 0$, then $S$ is discrete.

**Definition 2.4.** *We say a proper, path connected metric space $X$ is* **Brillouin** *if it satisfies the following conditions:*

*1: $X$ is metrically consistent.*
*2: For all $a$, $b$ in $X$, the mediatrices $L_{ab}$ are minimally separating sets.*

The second condition in the above definition may be weakened to apply only to those mediatrices $L_{ab}$ where $a$ and $b$ are in $S$. In this case, we will say that $X$ **is Brillouin over** $S$, if it is not obvious from the context.

*Example 2.5.* Equip $\mathbb{R}^2$ with the "Manhattan metric", that is, $d(p,q) = |p_1 - q_1| + |p_2 - q_2|$. The Manhattan metric is not metrically consistent: a circle $C_r(p)$ is a diamond of side length $r\sqrt{2}$ centered at $p$, and the definition fails because $C_{d(z,a)}(z) \cap C_R(x)$ is a segment rather than a point. Neither are the mediatrices minimally separating: if the coordinates of a point $a$ are equal, then $L_{0a}$ consists of a line segment and two quarter-planes (see Fig. 2.1). Even if the discrete set $S$ contains no such points, we can still run into strange situations. For example, the mediatrices $L_{(0,0),(2,4)}$ and $L_{(0,0),(4,6)}$ both contain the ray $\{(t,1) \mid t \geq 4\}$ (Fig. 2.2). But, if we are careful, we can avoid this. If $(0,0)$ is the basepoint, we must have that for all pairs $(a_1,a_2)$ and $(b_1,b_2)$ in $S$, $a_1 - a_2 \neq b_1 - b_2$. For example, take $S$ to be an irrational lattice such as $\{(m, n\sqrt{2}) \mid m,n \in \mathbb{Z}\}$. (From this example, we see that to do well in Manhattan, one should be carefully irrational.) It is interesting to note that while this example is not metrically consistent and hence not Brillouin, all the conclusions of this section (in particular, Prop. 2.13) still hold.

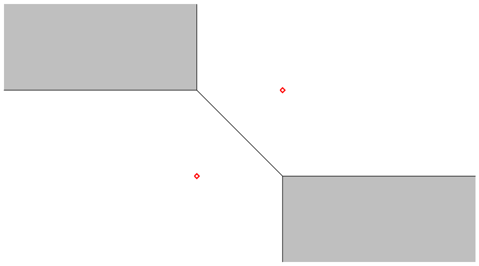


**Fig. 2.1.** The set $L_{(0,0),(a,a)}$ contains two quarter-planes.

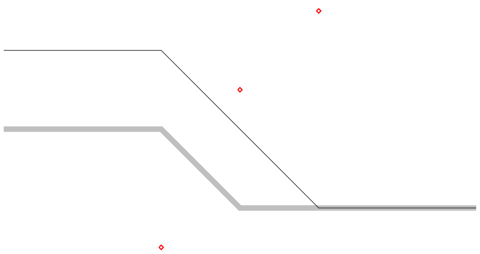


**Fig. 2.2.** $L_{(0,0),(4,6)}$ (thin solid line) and $L_{(0,0),(2,4)}$ (thick grey line) have open segments in common.

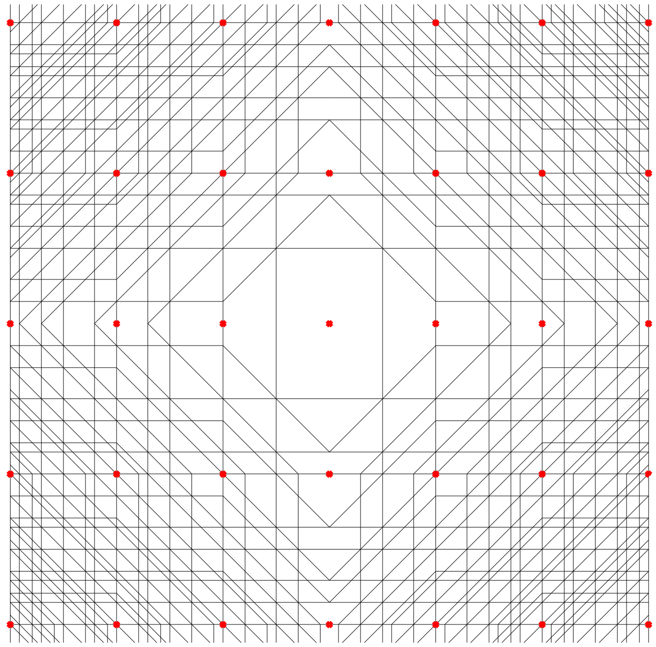


**Fig. 2.3.** The mediatrices $L_{0a}$ for $\mathbb{R}^2$ with the Manhattan metric and $a$ in the lattice $\left\{(m, n\sqrt{2})\right\}$.

As mentioned in the introduction, for each $x_0 \in S$, the mediatrices $L_{x_0 a}$ give a partition of $X$. Informally, those elements of the partition which are reached by crossing $n-1$ mediatrices from $x_0$ form the $n^{th}$ Brillouin zone, $B_n(x_0)$. This definition is impractical, in part because a path may cross several mediatrices simultaneously, or the same mediatrix more than once. Instead, we will use a definition given in terms of the number of elements of $S$ which are nearest to $x$. In many cases, this definition is equivalent to the informal one. See the remarks at the end of this section for more details. We use the notation $\#(S)$ to denote the cardinality of the set $S$.

**Definition 2.6.** *Let $x \in X$, let $n$ be a positive integer, $n \leq \#(S)$, and let $r = d(x, x_0)$. Then define the sets $b_n(x_0)$ and $B_n(x_0)$ as follows:*

- $x \in b_n(x_0) \iff \#(N_r(x) \cap S) = n-1$ *and* $C_r(x) \cap S = \{x_0\}$.
- $x \in B_n(x_0) \iff \#(N_r(x) \cap S) = m$ *and* $\#(C_r(x) \cap S) = \ell \geq 1,$ *where* $l, m \in \mathbb{Z}^+$ *with* $m+1 \leq n \leq m+\ell$.

Here the point $x_0$ is called the **base point**, and the set $B_n(x_0)$ is the $n^{th}$ Brillouin zone with base point $x_0$. Note that in the second part, if $m = n-1$ and $\ell = 1$, then $x \in b_n(x_0)$. So $b_n(x_0) \subseteq B_n(x_0)$. Note also that the complement of $b_n(x_0)$ in $B_n(x_0)$ consists of subsets of mediatrices (see Def. 1.1). Note also that $b_n(x_0)$ is open and that $B_n(x_0)$ is closed. Finally, observe that for fixed $x_0$ the sets $b_n(x_0)$ are disjoint, but the sets $B_n(x_0)$ are not.

The following lemma, which follows immediately from Def. 2.6, explains a basic feature of the zones, namely that they are concentric in a weak sense. This property is also apparent from the figures.

**Lemma 2.7.** *Any continuous path from $x_0$ to $B_n(x_0)$ intersects $B_{n-1}(x_0)$.*

The Brillouin zones actually form a covering of $X$ by non-overlapping closed sets in various ways. This is proved in parts. The next two results assert that the zones $B$ cover $X$, but the zones $b$ do not. The first of these is an immediate consequence of the definitions. The second is more surprising and ultimately leads to Corollary 3.5, the generalization of Bieberbach's "equal area" result.

**Lemma 2.8.** *For fixed n the Brillouin zones tile $X$ in the following sense:*

$$\bigcup_i B_i(x_n) = X \quad \text{and} \quad b_i(x_n) \cap b_j(x_n) = \emptyset \quad \textit{if } i \neq j.$$

*In addition, $B_i(x_n) \cap b_j(x_n) = \emptyset$ if $i \neq j$.*

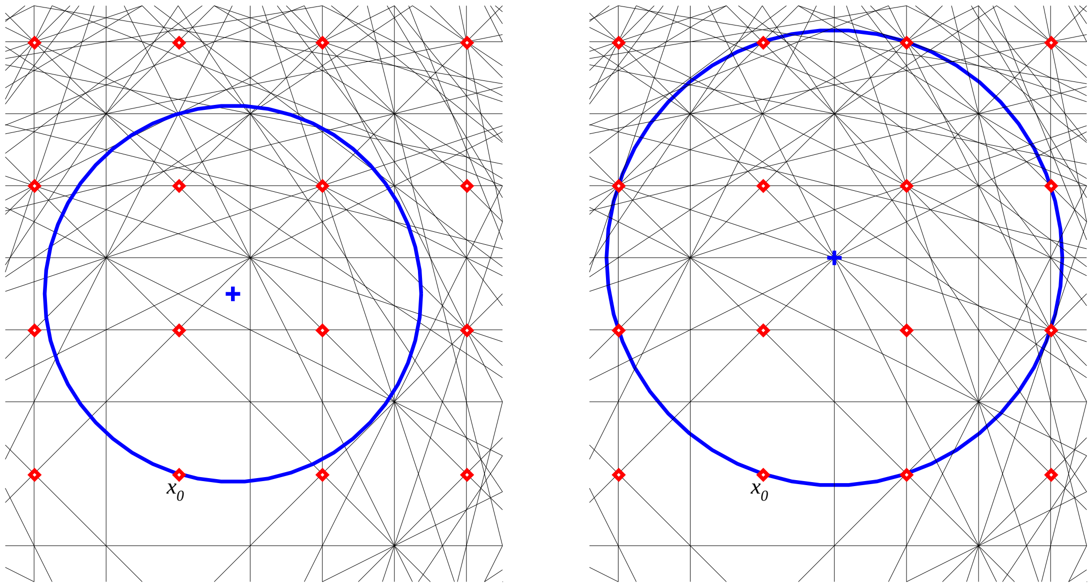



**Fig. 2.4.** Here we illustrate the definition of the sets $b_n(x_0)$ and $B_n(x_0)$ for the lattice $\mathbb{Z}^2$ in $\mathbb{R}^2$. In both pictures, the circle $C_{d(x,x_0)}(x)$ is drawn, and the basepoint $x_0$ lies in the center of the square at the lower left. On the left side, the point $x$ (marked by a small cross) lies in $b_5$, and $\#(N_r(x) \cap S) = 4$, while $x_0$ is the only point of $S$ on the circle. On the right, we have $m = 4$ and $\ell = 8$, so $x$ lies in all of the sets $B_5, B_6, \ldots, B_{12}$.

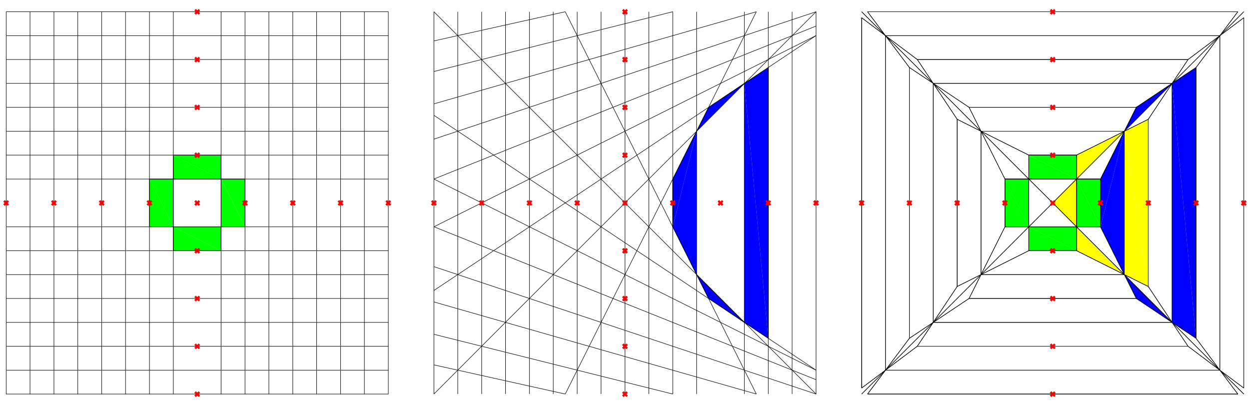


**Fig. 2.5.** This example illustrates Lemma 2.8 and Theorem 2.9. Let $S$ be the discrete set $\{(m,0)\} \cup \{(0,n)\}, m,n \in \mathbb{Z}$ in the Euclidean plane. On the left is the tiling given by $B_i(0,0)$ and in the middle is the tiling by $B_i(2,0)$. In both cases, $b_2$ is shaded. On the right is the tiling given by $B_2(x_i)$ as in Thm. 2.9. The sets $b_2(0,0)$, $b_2(1,0)$, and $b_2(2,0)$ have been shaded. Note that this $S$ does not correspond to a group, nor does it satisfy the hypotheses of Prop. 2.10, because there are no isometries which permute $S$ and do not fix the origin.

**Theorem 2.9.** *Let $X$ be a proper, path-connected metric space and let $S = \{x_i\}_{i \in I}$ be a discrete set. Then, for fixed $n \leq \#(S)$, the sets $\{B_{n_0}(x_i)\}_{i \in I}$ tile $X$ in the following sense:*

$$\bigcup_i B_n(x_i) = X \qquad \text{and} \qquad b_n(x_i) \cap b_n(x_j) = \emptyset \quad \textit{if } i \neq j.$$

*Proof.* First, we show that for any fixed $n > 0$ and each $x \in X$, there is an $x_i \in S$ with $x \in B_n(x_i)$. Re-index $S$ so that if $S = \{x_1, x_2, x_3, \ldots\}$ and $i < j$, then $d(x, x_i) \leq d(x, x_j)$. This can be done; since $S$ is a discrete subset and closed balls $D_c(x_i)$ are compact, the subsets of $S$ with $d(x, x_i) \leq c$ are all finite. Let $r_i = d(x, x_i)$. We will show that $x \in B_n(x_n)$.

Note that $r_n \geq r_{n-1}$. Suppose first that $r_n > r_{n-1}$, then $N_{r_n}(x) \cap S$ contains exactly $n-1$ points, and $x_n \in C_{r_n}(x) \cap S$. Thus $x \in B_n(x_n)$. Note that if $r_{n+1} > r_n$, then we would have $x \in b_n(x_n) \subset B_n(x_n)$.

If, on the other hand, $r_n = r_{n-1}$, then there is a $k > 0$ so that $r_n = r_{n-1} = \ldots = r_{n-k}$, and so $\#(N_{r_n}(x) \cap S) = n - k - 1 \leq n - 1$. But then $\#(C_{r_n}(x) \cap S) \geq k + 1$, and hence $x \in B_n(x_n)$ as desired.

For the second part, we show that $b_n(x_i) \cap b_n(x_j) = \emptyset$. If not, then there is a point $x$ in their intersection. If $r_i = r_j$, then $x_i = x_j$, because by the definition of $b_n(x_k)$, $\{x_k\} = C_{r_k}(x) \cap S$. If not, then $r_i < r_j$. In this case, $x_i \in D_{r_i}(x) \subset N_{r_j}(x)$ . Thus, since $\#(N_{r_i}(x) \cap S) = n-1$, $N_{r_j}(x)$ must contain at least $n$ points of $S$, a contradiction. □

The next result indicates how this notion of tiling is related to the notion of a fundamental set.

**Proposition 2.10.** *Let $S$ be a discrete set in a metric space $X$ as in Thm. 2.9. Suppose that for each $x_i$ in $S$ there is an isometry $g_i$ of $X$ such that $g_i(x_0) = x_i$, $g_i$ permutes $S$ and the only $g_i$ which leaves $x_0$ fixed is the identity. Then there is a set $F$ (the fundamental set), satisfying:*

$$b_n(x_0) \subseteq \quad F \quad \subseteq B_n(x_0) \qquad \text{with}$$

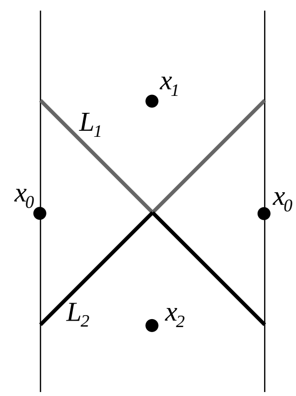
$$\bigcup_i g_i(F) = X \quad \text{and} \quad g_i(F) \cap g_j(F) = \emptyset \ (i \neq j).$$

*Proof.* Suppose that $x \in b_n(x_0)$. From Def. 2.6 and the fact that the $g_i$ are isometries, we see that this is equivalent to $g_i(x) \in b_n(x_i)$. Thus $g_i(b_n(x_0)) = b_n(x_i)$. Now apply Theorem 2.9. A similar reasoning proves the statement for $B_n(x_0)$. □

*Remark 2.11.* The fundamental set $F$ is not necessarily connected. Also, note that it follows from this proposition that $B_i(x_0)$ is scissors congruent to $B_j(x_0)$ (see [Sah] for a discussion of scissors congruence). In particular, this implies immediately that the $B_i$ all have the same area. Note that this result does not hold if $S$ is not generated by a group of isometries. See, for example, Fig. 2.5.

In many examples, $B_n$ is the closure of $b_n$. However, this need not always be the case, even if we assume the space is Brillouin, as the example below shows. We will give additional, more involved examples in a forthcoming work.

*Example 2.12.* Let $X$ be the flat cylinder obtained by identifying opposite sides of the strip $\{z \mid -1 \leq \text{Re}\,(z) \leq 1\}$ in the usual way. We will denote points in the cylinder by a corresponding complex number. Let $x_0 = 1$, $x_1 = i$, and $x_2 = -i$. Each mediatrix $L_i$ is a topological circle consisting of a pair of segments meeting at right angles. The first zone $b_1(x_0)$ is the part of the cylinder where $|\text{Im}\,(z)| < |\text{Re}\,(z)|$, and $B_1(x_0)$ is the closure of $b_1$. The second zone is the complement of $b_1$ in the cylinder, and $b_2$ is its interior. However, $B_3 = \{0\}$ and $b_3$ is empty. Note that in this example, $B_3$ is contained in the closures of $b_1$ and $b_2$.




Despite the fact that the zones $B_i$ are not always the closure of their interiors, if $X$ is a Brillouin space, the $B_i$ are still fairly well behaved sets, as the next proposition shows.

**Proposition 2.13.** *If $X$ is Brillouin over S, then*

*(i) Interior points of $B_n(x_0)$ are in $b_n(x_0)$.*
*(ii) $B_n(x_0)$ is contained in the closure of $b_1(x_0) \cup \cdots \cup b_n(x_0)$.*

*Proof.* Without loss of generality, we can restrict our attention to $B_n(x_0)$, which we will denote $B_n$ throughout the proof. Since $b_n \subset B_n$, with $b_n$ open and $B_n$ closed, it is obvious that $\overline{b_n} \subseteq B_n$.

Let $x$ be a point in $B_n \setminus b_n$. By Definition 2.6, $x \in B_{m+1} \cap B_{m+2} \cap \ldots \cap B_{m+\ell}$, with $\ell \geq 2$ and $m+1 \leq n \leq m+\ell$. The point $x$ lies on the intersection of $\ell - 1$ mediatrices, that is, $C_{d(x,x_0)}(x) \cap S$ consists of $\ell$ points.

Suppose $x$ is an interior point of $B_{m+d}$ for some $d \in \{1, \cdots \ell\}$. Let $V$ be an arbitrary, small neighborhood of $x$, so that $V \subset B_{m+d}$. Continuity of the metric allows us to choose $y \in V$ such that $N_{d(y,x_0)}(y) \cap S$ contains $m$ points, and using metric consistency we can ensure that $C_{d(y,x_0)}(y) \cap S$ contains exactly one point, namely $x_0$. Thus, we have $y \in b_{m+1}$.

Suppose $x_s \neq x_0$ is a point in $C_{d(x,x_0)}(x) \cap S$. By the same reasoning as above, $V$ must contain a point $z$ such that $N_{d(z,x_s)}(z) \cap S$ contains $m$ points, and $C_{d(z,x_s)}(z) \cap S = \{x_0\}$. Thus $d(z, x_0) > d(z, x_s)$ and so $N_{d(z,x_0)}(z) \cap S$ contains at least $m + 1$ points. This implies that $z \in B_{m+2} \cup \cdots \cup B_{m+\ell}$.

Since $y$ and $z$ are in $V \subset B_{m+d}$, we have that for some $d \geq 1$, $B_{m+d} \cap b_{m+1}$ and $B_{m+d} \cap (B_{m+2} \cup \cdots B_{m+\ell})$ are both non-empty. In view of Lemma 2.8 this is a contradiction.

To prove the second statement, we start again by observing that if $x$ is a point in $B_n \setminus b_n$, then $x \in B_{m+1} \cap B_{m+2} \cap \ldots \cap B_{m+\ell}$. Exactly as above, we note that any neighborhood of $x$ contains points of $b_{m+1}$. □

*Remark 2.14.* In practice, using Definition 2.6 directly can be unwieldy. It is typically easier to identify the various $b_n$ using the informal definition, counting the number of mediatrices crossed by a path which starts at $x_0$. Suppose $X$ is such that between $x_0$ and any point of $b_n$, one can find a path $\gamma$ so that if $L_i$ and $L_j$ are distinct mediatrices, then $\gamma \cap L_i \neq \gamma \cap L_j$. In this case, it follows immediately that a point is in $b_n$ if and only if such a path crosses exactly $n - 1$ mediatrices. If the path $\gamma$ crosses the same mediatrix more than once, we must use a signed notion of crossing. This allows us to account only for those crossings which are essential.

However, such a process is not always possible— we can not always push a path off a point where several mediatrices intersect. One way around this is to adjust the definition of “cross”.

As in [Pe1], we assign to each point $x$ its Brillouin index:

$$\beta(x) \equiv \max \{n \mid x \in B_n(x_0)\} .$$

From Lemma 2.8, we see that this is a well defined function which is constant on $b_n(x_0)$. If $L = L_{x_0,x_s}$ is a mediatrix, we say that $\gamma$ crosses $L$ if $\gamma(1) \in L^+_{x_0,x_s}$, the component containing $x_s$. (Recall that $\gamma(0) \in L^-_{x_0,x_s}$ by definition.) Notice that this definition only makes sense if $X - L$ has two components, which is always the case if $X$ is Brillouin. With this definition of “cross”, then there is always a path $\gamma$ from $x$ to $x_0$ which crosses exactly $n - 1$ mediatrices if and only if $x \in b_n(x_0)$.

## 3. Brillouin Zones in Spaces of Constant Curvature

In this section $X$ will be assumed to be one of $\mathbb{R}^n$, $\mathbb{S}^n$, or $\mathbb{H}^n$, all equipped with the standard metric, and let $G$ be a discontinuous group of isometries of $X$. Denote the quotient $X/G$ with the induced metric by $(M, g)$. Then the construction of lifting to the universal cover, as outlined in the introduction, applies naturally to $(M, g)$. In this section we describe focusing of geodesics in $(M, g)$ by Brillouin zones in $X$. The discrete set $S$ is given by the orbit of a chosen point in $X$ (which we will call the origin) under the group of deck-transformations $G$. The fact that the Brillouin zones are fundamental sets is now a direct corollary of Prop. 2.10.

The regularity conditions of Def. 2.4 are easily verified in the present context. We do this first.

**Lemma 3.1.** *If $X$ is either $\mathbb{R}^n$, $\mathbb{S}^n$, or $\mathbb{H}^n$, then a mediatrix $L_{ab}$ in $X$ is an $(n - 1)$-dimensional, totally geodesic subspace consisting of one component, and $X - L_{ab}$ has two components.*

*Proof.* This is easy to see if we change coordinates by an isometry of $X$, putting $a$ and $b$ in a convenient position, say as $x$ and $-x$. The mediatrix $L_{x,-x}$ is easily seen to satisfy the conditions (in the case of $\mathbb{S}^n$, it is the equator, and for the others, it is a hyperplane). The conclusion follows. □

**Proposition 3.2.** *All such spaces $X$ are Brillouin (see Definition 2.4).*

*Proof.* As remarked before, the first condition is satisfied for any Riemannian metric. The second condition is also easy. It suffices to observe that the subspaces of Lemma 3.1 are minimally separating. □

*Remark 3.3.* Note that in the Riemannian case, mediatrices always cross transversally. If $L_{0a}$ and $L_{0b}$ coincide in an open set, then their tangent spaces also coincide at some point. Uniqueness of solutions of second order differential equations then implies $L_{0a} = L_{0b}$.

Recall that a metric space $X$ is called *rigid* if the only isometry which fixes each point of a nonempty open subset of $X$ is the identity. It is not hard to see that $\mathbb{S}^n$, $\mathbb{H}^n$, and $\mathbb{R}^n$ are rigid spaces. See [Ra] for more details of rigid metric spaces and for the proof of the following result. Recall that the stabilizer in $G$ of a point $x \in X$ consists of those elements of $G$ that fix $x$.

**Proposition 3.4.** *Let $G$ be a discontinuous group of isometries of a rigid metric space $X$. Then there exists a point $y$ of $X$ whose stabilizer $G_y$ consists of the identity.*

We now return to Brillouin zones as defined in the last section. Recall that $G$ is a group of isometries of $X$ that acts discontinuously on points in $X$. Let $x_0$ be a point in $X$ whose stabilizer under the action of $G$ is trivial. For any $x \in X$, let $[x_0, x]$ be a geodesic segment of minimal length whose endpoints are $x_0$ and $x$. Then $B_n(x_0)$, the $n^{th}$ Brillouin zone relative to $x_0$, is the set of points $x$ in $X$ such that the geodesic segment $[x_0, x]$ intercepts exactly $n-1$ mediatrices $L_{x_0,y}$, where $y$ is in the orbit of $x_0$ under the group $G$. Proposition 2.10 immediately implies the most important fact about Brillouin zones in this setting.

**Corollary 3.5.** *Let $X$ be $\mathbb{R}^n$, $\mathbb{S}^n$, or $\mathbb{H}^n$, and let $G$ be a discontinuous group of isometries of $X$. Let $x_0 \in X$ be such that its stabilizer $G_{x_0}$ under $G$ is trivial. Then for every positive integer $n$, the $n^{th}$ Brillouin zone $B_n(x_0)$ is a fundamental set for the action of $G$ on points in $X$. Its boundary is the union of pieces of totally geodesic subspaces and equals the boundary of its interior.*

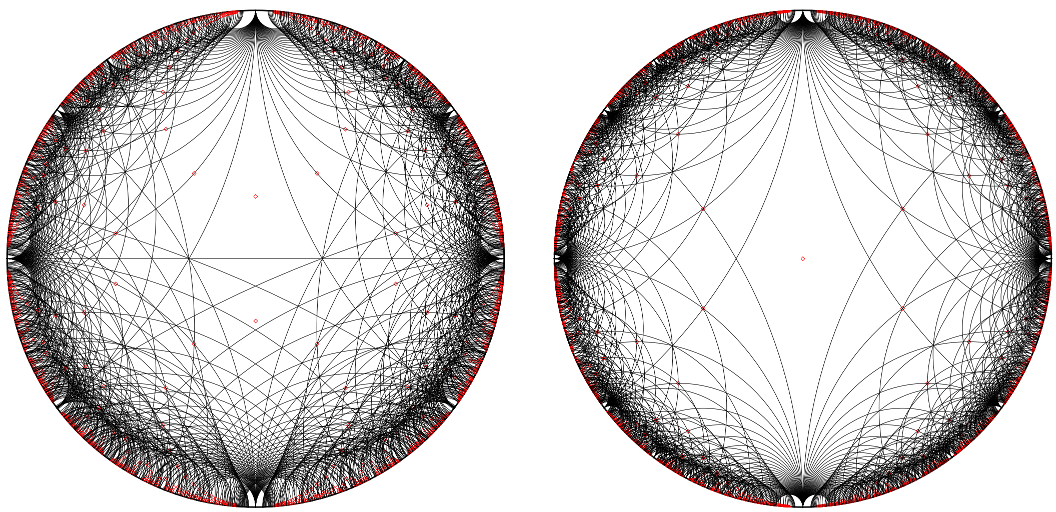


**Fig. 3.1.** Brillouin zones for $PSL(2, \mathbb{Z})$ in the hyperbolic disk. We have transported the "usual" upper half-plane representation using the map $z \mapsto \frac{iz+1}{z+i}$. On the left are the sets $B_n(\frac{i}{4})$, which give fundamental sets as in Cor. 3.5. On the right, 0 is taken as a basepoint. Since the origin has a non-trivial stabilizer, the corresponding Brillouin zones give a double cover of the fundamental sets.

*Remark 3.6.* The above corollary is the generalization of Bieberbach's main result on Brillouin zones [Bi]. The first Brillouin zone $B_1(0)$ is the usual Dirichlet fundamental domain for the action of $G$. Furthermore, even when $G_{x_0}$ is not trivial, $B_n(x_0)$ is a $k$-fold cover of a fundamental region.

As pointed out in the introduction, the number of geodesics that focus in a certain point is counted in the lift. So if a given point $x \in X$ is intersected by $n$ mediatrices, it is reached by $n+1$ geodesics of length $d(0, x)$ emanating from the reference point (the origin). In the next section, we give more specific examples of this.

Finally, we state a conjecture.

*Conjecture 3.7.* Let $(X, \tilde{g})$ be the universal cover of a $d$-dimensional smooth Riemannian manifold $(M, g)$ as described in the construction. For a generic metric $g$ on $M$, no more than $d$ mediatrices intersect in any given point $y$ of $X$.

This conjecture acquires perhaps even more interest (and certainly more structure), when one restricts the collection of metrics on $M$ to conformal ones ([Mas]). A result in this direction for $M = \mathbb{R}^2/\mathbb{Z}^2$ can be found in [Jo1].

## 4. Focusing in Two Riemannian Examples

In this section, we give two examples (one of them new as far as we know) of focusing. Suppose that at $t = 0$ geodesics start emanating in all possible directions from a point. At certain times $t_1, t_2, ....$, we may see geodesics returning to that point. We derive expressions for the number of geodesics returning at $t_n$ in two cases. First, as an introductory example we will discuss this for the case of the flat, square torus $M = \mathbb{R}^2/\mathbb{Z}^2$ (a more complete discussion of this example can be found in [Pe3]). Second, we will deal with a much more unusual example, namely $M = \mathbb{H}^2/\Gamma(k)$, where $\Gamma(k)$ is a subgroup of $PSL(2,\mathbb{Z})$ called the principal congruence subgroup of level $k$ (defined in more detail below). We note that it seems to be considerably harder to count geodesics that focus in points other than our basepoint.

Before continuing, consider the classical problem of counting $R_g(n)$, the number of solutions in $\mathbb{Z}^2$ of

$$p^2 + q^2 = n.$$

Let

$$n = 2^\alpha \prod_{i=1}^{k} p_i^{\beta_i} \prod_{j=1}^{\ell} q_i^{\gamma_i}$$

be the prime decomposition of the number $n$, where $p_i \equiv 1 (\mathrm{mod}\, 4)$ and $q_i \equiv 3 (\mathrm{mod}\, 4)$. The following classical result of Gauss (see, for example, [NZM]) will be very useful.

**Lemma 4.1.** *$R_g(n)$ is zero whenever $n$ is not an integer, or any of the $\gamma_i$ is odd. Otherwise,*

$$R_g(n) = 4 \prod_{i=1}^{k} (1 + \beta_i).$$

*Example 4.2.* Choose an origin in $M = \mathbb{R}^2/\mathbb{Z}^2$ and lift it to the origin in $\mathbb{R}^2$. Our discrete set $S$ is then $\mathbb{Z}^2$. Let $\rho_x(t)$ be the number of geodesics of length $t$ that connect the origin to the point $x \in M$.

**Proposition 4.3.** *In the flat torus $\mathbb{R}^2/\mathbb{Z}^2$, the number of geodesics of length $t$ that connect any point to itself is $\rho_0(t) = R_g(t^2)$.*

*Proof.* Notice that by definition geodesics of length $t$ leaving from the origin in $\mathbb{R}^2$ reach the points contained in $C_t(0)$. Only if $t^2$ is an integer does this circle intersect points of $\mathbb{Z}^2$. Because of the homgeneity of the flat, square torus, it does not matter where we choose the origin. □

*Example 4.4.* We now turn to the next example. Recall that $PSL(2,\mathbb{Z})$ can be identified with the group of two by two matrices with integer entries and determinant one, and with multiplication by $-1$ as equivalence. For each $k$, the group $\Gamma(k)$ is the subgroup of $PSL(2,\mathbb{Z})$ given by

$$\Gamma(k) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z}) \,\middle|\, a \equiv d \equiv 1 \,(\mathrm{mod}\, k),\ b \equiv c \equiv 0 \,(\mathrm{mod}\, k) \right\}.$$

This group has important applications in number theory. The action of $\Gamma(k)$ on $\mathbb{H}^2$ is given by the Möbius transformations

$$g(z) = \frac{az+b}{cz+d} \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(k).$$

We point out that for $k = 2$, 3, or 5, the surface $\mathbb{H}^2/\Gamma(k)$ is a sphere with 3, 4, or 12 punctures (see [FK]).

We will find it more convenient to work in the hyperbolic disk $\mathbb{D}^2$, which is the universal cover of $\mathbb{H}^2/\Gamma(k)$. We shall choose a representation of $\Gamma(k)$ in the disk so that $i \in \mathbb{H}^2$ corresponds to the origin. This will allow us to determine the focusing of the geodesics which emanate from $i$. Note that the surface $\mathbb{H}^2/\Gamma(k)$ has special symmetries with respect to $i$: for example, $i$ is the unique point fixed by the order 2 element of $PSL(2,\mathbb{Z})$.

**Lemma 4.5.** *The action of the fundamental group of the surface $\mathbb{H}^2/\Gamma(k)$ can be represented as*

$$\left\{ \begin{pmatrix} r-is & p+iq \\ p-iq & r+is \end{pmatrix} \,\middle|\, \begin{array}{c} p^2+q^2+1 = r^2+s^2 \\ r+p \equiv 1 \,(\mathrm{mod}\ k), \quad r-p \equiv 1 \,(\mathrm{mod}\ k) \\ s+q \equiv 0 \,(\mathrm{mod}\ k), \quad s-q \equiv 0 \,(\mathrm{mod}\ k) \end{array} \right\},$$

*acting on $\mathbb{D}^2$. We shall denote this particular representation as the group $\Gamma_\odot(k)$.*

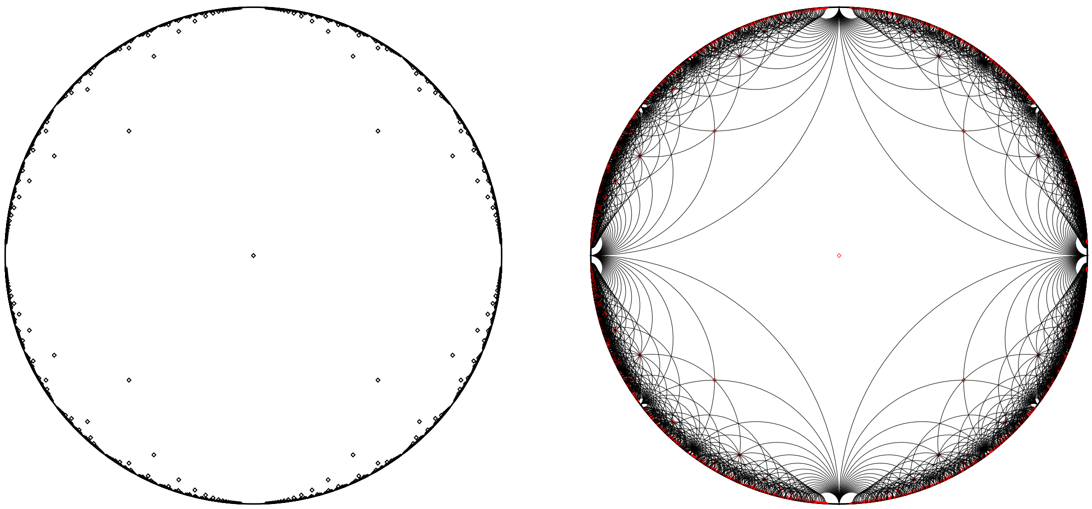


**Fig. 4.1.** The orbit of $i$ under $\Gamma(2)$ transported to the hyperbolic disk, and the corresponding Brillouin zones. Each zone $B_n$ forms a fundamental domain for a 3-punctured sphere.

*Proof.* Following the conventions in [Be], define

$$\phi : \mathbb{D}^2 \to \mathbb{H}^2, \quad \phi(z) = i\frac{z+1}{-z+1}.$$

Push back the transformation $g \in \Gamma(k)$ from $\mathbb{H}^2$ to $\mathbb{D}^2$ by $g \to \phi^{-1} g \phi$ to obtain a representation of $g \in \Gamma(k)$ as a transformation acting on $\mathbb{D}^2$. The matrix representation of this transformation is given by:

$$A_g = \begin{pmatrix} \frac{a+d}{2} + i\frac{b-c}{2} & \frac{a-d}{2} - i\frac{b+c}{2} \\ \frac{a-d}{2} + i\frac{b+c}{2} & \frac{a+d}{2} - i\frac{b-c}{2} \end{pmatrix},$$

where $\det A_g = 1$, since this matrix is conjugate to $g$, whose determinant is equal to $1$. Let

$$\begin{aligned} p &= (a-d)/2 & q &= -(b+c)/2 \\ r &= (a+d)/2 & s &= -(b-c)/2 \end{aligned}$$

and $A_g$ now written as

$$A_g = \begin{pmatrix} r - is & p + iq \\ p - iq & r + is \end{pmatrix}.$$

Here the numbers $p, q, r, s$ are in $\mathbb{Z}$ and must satisfy the following congruence conditions:

$$\begin{aligned} r + p &\equiv 1 \,(\mathrm{mod}\ k), & r - p &\equiv 1 \,(\mathrm{mod}\ k), \\ s + q &\equiv 0 \,(\mathrm{mod}\ k), & s - q &\equiv 0 \,(\mathrm{mod}\ k). \end{aligned}$$

Since the determinant of $A_g$ is equal to $1$, we must also have

$$p^2 + q^2 + 1 = r^2 + s^2.$$

□

We need another auxiliary result before we state the main result of this section.

**Lemma 4.6.** *Let $(p, q)$ and $(r, s)$ be two points in $\mathbb{Z}^2$ such that the integers $A = p^2 + q^2$ and $B = r^2 + s^2$ are relatively prime, and let $\varphi$ be a rotation fixing the origin. Now $\varphi(p, q) = (p', q')$ and $\varphi(r, s) = (r', s')$ are in $\mathbb{Z}^2$ if and only if $\varphi$ is a rotation by an integer multiple of $\pi/2$.*

*Proof.* Let $c$ be the cosine of the angle of rotation. We have

$$c = \frac{p'p + q'q}{A} = \frac{r'r + s's}{B}.$$

Thus if $p'p + q'q$ and $r'r + s's$ are not both equal to zero,

$$\frac{p'p + q'q}{r'r + s's} = \frac{A}{B}.$$

Because $A$ and $B$ are relatively prime and surely $|p'p + q'q|$ is less than or equal to $A$, and similarly for $B$, we have that

$$p'p + q'q = \pm A \ \text{ and } \ r'r + s's = \pm B.$$

This implies the result. $\square$

Now we define a counter just as before. Choose a lift of $M = \mathbb{D}^2/\Gamma_{\odot}(k)$ so that $0 \in M$ lifts to $0 \in \mathbb{D}^2$. Let $\gamma_x(t)$ be the number of geodesics of length $t$ that connect the origin to the point $x \in M$.

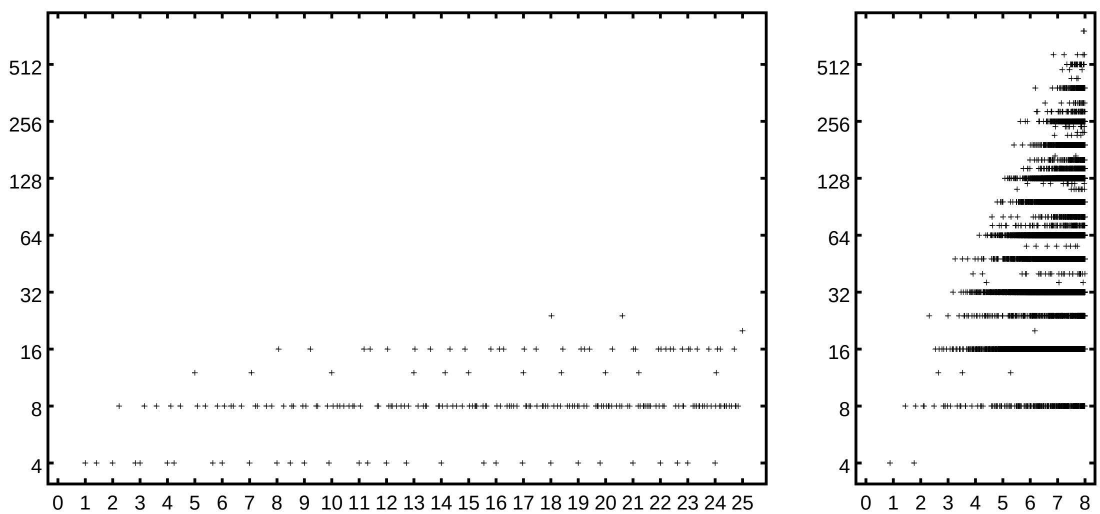



**Fig. 4.2.** The non-zero values of $\rho_0(t)$ for $t \leq 25$ (left) and $\gamma_0(t)$ for $t \leq 8$ (right), which count how many geodesics of length $t$ connect the origin to itself in the $\mathbb{R}^2/\mathbb{Z}^2$ and $\mathbb{D}^2/\Gamma_{\odot}(2)$, respectively.

**Theorem 4.7.** *In the surface $\mathbb{H}^2/\Gamma(k)$, the number of geodesics of length $t$ which connect the point $i \in \mathbb{H}^2$ to itself is given by*

$$\begin{array}{ll} \frac{1}{4}R_g(\cosh^2 t - 1)R_g(\cosh^2 t) & \textit{for } k = 2, \\ \frac{1}{4}R_g\left(\frac{\cosh^2 t - 1}{9}\right)R_g(\cosh^2 t) & \textit{for } k = 3, \\ \frac{1}{4}R_g\left(\frac{\cosh^2 t - 1}{25}\right)R_g(\cosh^2 t) & \textit{for } k = 5. \end{array}$$

*Note that in all cases, the number is nonzero only if* $\cosh^2 t \in \mathbb{N}$.

*Proof.* We shall work in the disk, rather than in $\mathbb{H}^2$. Let $S$ be the orbit of $0 \in \mathbb{D}^2$ under $\Gamma_{\odot}(k)$. Then the number of such geodesics is exactly the number of distinct points of $S$ which lie on the circle $C_t(0)$ of radius $t$ and centered at the origin.

If $x \in S$, then by Lemma 4.5, it is of the form $\frac{p+iq}{r+is}$ with $p$, $q$, $r$, $s$ integers satisfying $p^2 + q^2 = r^2 + s^2 - 1$. Let $n$ be their common value, that is,

$$n = p^2 + q^2 = r^2 + s^2 - 1.$$

We will first count the number of 4-tuples $(p, q, r, s)$ that solve this equation, momentarily ignoring the congruence conditions.

Note that the point $x$ has Euclidean distance to the origin given by

$$|x|_e^2 = \frac{p^2+q^2}{r^2+s^2} = \frac{n}{n+1}.$$

The hyperbolic length of the geodesic which connects $x$ to the origin is $\operatorname{arctanh}(|x|_e)$. Consequently, $\gamma_0(t)$ is only non-zero when $t = \operatorname{arctanh}\sqrt{n/(n+1)}$, or, equivalently, when $n = \cosh^2 t - 1$.

To count the number of intersections of $C_t(0)$ with $S$ for these values of $t$, observe that we can use Gauss' result to count the number of pairs $(p,q)$ such that $p^2+q^2=n$. This number is given by $R_g(n)$. For each such pair $(p,q)$, we have a number of choices to form

$$x = \frac{p+iq}{r+is}.$$

By the above, this number is equal to $R_g(n+1)$. Thus, $\gamma_0(t)$ is at most $R_g(n)R_g(n+1)$.

However, we have over-counted: some of our choices for $p,q,r,s$ represent the same point $x \in S$, and some of them may not satisfy the congruence conditions, which we have so far ignored. We will first account for the multiple representations, and then account for the congruence relations.

Let $p,q,r,s \in \mathbb{Z}$ be as above, giving a point $x = \frac{p+iq}{r+is}$ which is at distance $t = \operatorname{arctanh}\sqrt{n/(n+1)}$ from the origin. If we multiply the numerator and denominator of $x$ by $e^{i\theta}$, then $x$ will remain unchanged. Because of the requirement that $p^2+q^2 = r^2+s^2-1 = n$, this is the only invariant, and by Lemma 4.6, $\theta$ must be a multiple of $\frac{\pi}{2}$ for the numerator and denominator to remain Gaussian integers. We see that in our counting, we have represented our point $x$ in 4 different ways:

$$x = \frac{p+iq}{r+is} = \frac{-q+ip}{-s+ir} = \frac{-p-iq}{-r-is} = \frac{q-ip}{s-ir},$$

meaning we have over-counted by a factor of at least 4.

Now we account for the congruence conditions.

First, consider the case $k=2$. Note that $q+s \equiv 0 \pmod 2$ if and only if $p+r \equiv 1 \pmod 2$, because $p^2+q^2+1 = r^2+s^2$, so we need only check this one condition. If the representation $\frac{p+iq}{r+is}$ fails to satisfy our parity condition, then $q+s \equiv 1 \pmod 2$ and consequently $p+r \equiv 0 \pmod 2$. This means that the representation $\frac{-q+ip}{-s+ir}$ of this same point does satisfy the parity conditions, giving exactly

$$\frac{1}{4} R_g(\cosh^2 t - 1) R_g(\cosh^2 t)$$

distinct points of $S$ at distance $t$ from the origin.

If $k=3$, then since $k$ is odd, the congruence conditions on $p,q,r,$ and $s$ imply that

$$r \equiv 1 (\operatorname{mod} 3) \quad \text{and} \quad p \equiv q \equiv s \equiv 0 (\operatorname{mod} 3).$$

Note that the equation

$$p^2+q^2 = n \quad \text{and} \quad p \equiv q \equiv 0 \pmod 3$$

will be satisfied exactly $R_g(n/3^2)$ times. (Recall that if $n$ is not divisible by 9, then $R_g(n/9)$ is 0.)

For fixed $n$, let $(p,q)$ be any one of the solutions. We need to decide how many solutions the equation

$$r^2+s^2 = n+1 \quad \text{with} \quad r \equiv 1 \pmod 3 \quad \text{and} \quad s \equiv 0 \pmod 3$$

admits. The solution of the first equation implies that 3 divides $n$. Thus $r^2+s^2 \equiv 1 \pmod 3$. Consequently, we have 4 choices $\operatorname{mod} 3$ for the pair $(r,s)$, namely $(0,1)$, $(1,0)$, $(0,2)$, and $(2,0)$.

Let $(p,q,r,s) \in \mathbb{Z}^2 \times \mathbb{Z}^2$ be any solution to $n = p^2+q^2 = r^2+s^2-1$ with $p \equiv q \equiv 0 (\operatorname{mod} 3)$. For each choice of $(p,q)$, we have exactly $R_g(n+1)$ choices of $(r,s)$. Now let $R$ denote the product of the rotations by $\pi/2$ on each of the components of $\mathbb{Z}^2 \times \mathbb{Z}^2$. Using Lemma 4.6, we see that all such solutions can be obtained from just one by applying $R$ repeatedly. It is easy to check that each quadruple of solutions thus constructed runs exactly once through the above list. Since precisely one out of the four associated solutions is compatible with the conditions, the total number of solutions is exactly:

$$\frac{1}{4} R_g\left(\frac{n}{9}\right) R_g(n+1).$$

Using the relationship between the Euclidean distance and the Poincaré length as before gives the result.

If $k=5$, the proof for $k=3$ can be literally transcribed to obtain the result. $\square$

*Remark 4.8.* Note that the above results do not hold if $k$ is not one of the cases mentioned. The primary difficulty is that for prime $k \geq 7$, there are solutions which are not related by applying the rotation $R$. However, the argument does give an upper bound of $\frac{1}{4}R_g\left((\cosh^2 t - 1)/k^2\right) R_g(\cosh^2 t)$ for $\mathbb{H}^2/\Gamma(k)$ when $k$ is an odd prime. Note that the surface $\mathbb{H}^2/\Gamma(k)$ is of genus 0 if and only if $k \leq 5$ (see [FK]).

## 5. Non-Riemannian Examples

The present context is certainly not restricted to Riemannian metrics. As an indicator of this we now discuss a different set of examples.

Let $k$ be a positive number greater than one. Equip $\mathbb{R}^2$ with the distance function

$$\| x - y \| = \left(|x_1 - y_1|^k + |x_2 - y_2|^k\right)^{1/k}$$

and let the discrete set $S$ be given by $\mathbb{Z}^2$. For $k$ not equal to 2, this is not a Riemannian metric, yet all conclusions of Sect. 2 hold. In particular, each Brillouin zone forms a fundamental domain. Note that determining the zones by inspecting the picture requires close attention!

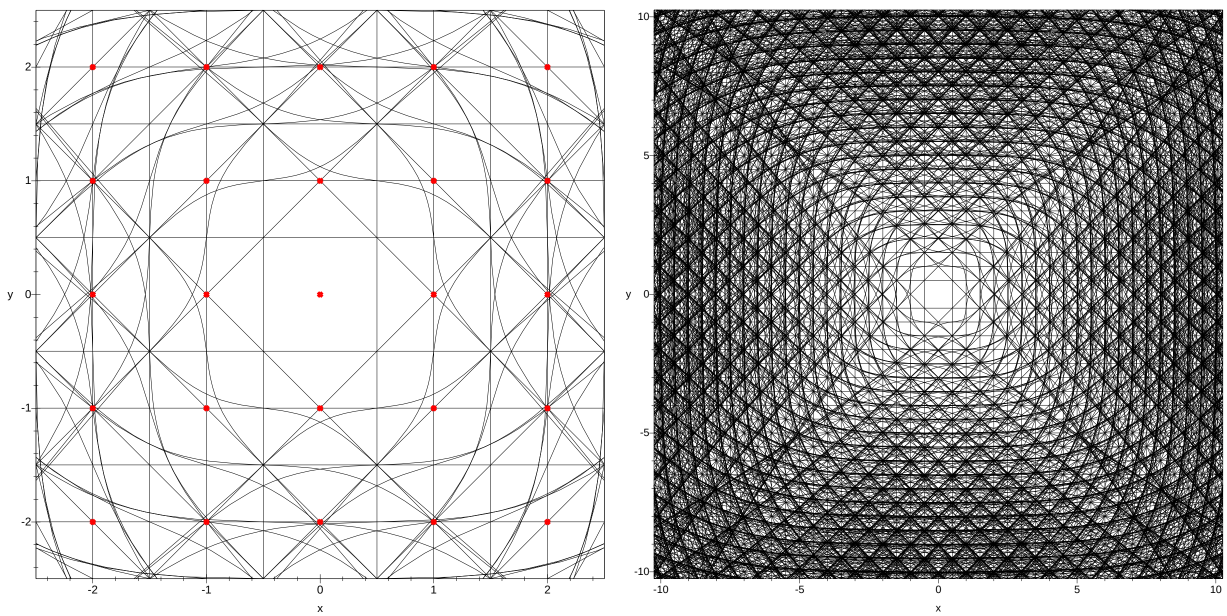



**Fig. 5.1.** Brillouin zones for the lattice $\mathbb{Z}^2$ in $\mathbb{R}^2$ with the metric $\left(|x_1 - y_1|^4 + |x_2 - y_2|^4\right)^{1/4}$. See also Fig. 2.3 and Example 2.5, which deal with the case $k = 1$, the "Manhattan metric".

Now the problem of determining $C_t(0) \cap S$ for any given $t$ is unsolved for general $k$. In fact, even for certain integer values of $k$ greater than 2, it is not known whether $C_t(0) \cap S$ ever contains at least two points that are not related by the symmetries of the problem. For $k = 4$, the smallest $t$ for which $C_t(0) \cap S$ has at least two (unrelated) solutions is given by

$$t^4 = 133^4 + 134^4 = 158^4 + 59^4.$$

However, for $k \geq 5$, it unknown whether this can happen at all (see [SW]).

There are some things that can be said, however. In the situation where $k \geq 3$, the mediatrices intersect the coordinate axes only in irrational points or in multiples of $1/2$. For if $x = (p/q, 0)$ is a point of a mediatrix $L_{(0,0),(a_1,a_2)}$, we have

$$|p|^k = |p - qa_1|^k + |a_2 q|^k \quad (p \neq 0, q \neq 0) \quad .$$

By Fermat's Last Theorem, this has no solution unless either $p = qa_1$ or $a_2 = 0$. In the first case, $\frac{p}{q} = \pm a_2$, which can only occur if the lattice point is of the form $(a_2, \pm a_2)$. If $a_2 = 0$, then $\frac{p}{q} = \frac{a_1}{2}$. In particular, there is no nontrivial focusing along the axes.

To compute Fig. 5.1, we took advantage of the smoothness of the metric. Not all metrics are sufficiently smooth for this procedure to work. Even for Riemannian metrics, in general the distance function is only Lipschitz, which will not be sufficiently smooth.

For each $a = (a_1, a_2) \in \mathbb{Z}^2$, define a Hamiltonian:

$$H_a(x) = \| x - a \| - \| x \| .$$

The mediatrix $L_{0a}$ corresponds to the level set $H_a(x) = 0$. Because $H_a(x)$ is smooth, we have uniqueness of solutions to Hamilton's equations. In the current situation, where the dimension is two, the level set consists of one orbit. Thus, one can produce the mediatrix by numerically tracing the zero energy orbits of the above Hamiltonian.

As mentioned above, for a general Riemannian metric, the distance function is only Lipschitz. This means we have no guarantee that the solutions of the above differential equation are unique. Indeed, there are examples of multiply connected Riemannian manifolds with self-intersecting mediatrices, as will be shown in a forthcoming work.

## References


[AM] Ashcroft, N. W., and Mermin, N. D.: *Solid State Physics*. New York: Holt, Rhinehart, and Winston, 1976

[Be] Beardon, A. F.: *The Geometry of Discrete Groups*. Berlin–Heidelberg–New York: Springer-Verlag, 1983

[Bi] Bieberbach, L.: Über die Inhaltsgleichheit der Brillouinschen Zonen. Monatshefte für Math. und Phys. **48** 509–515, (1939)

[Br] Brillouin, L.: *Wave Propagation in Periodic Structures*. New York: Dover, 1953

[Ca] Cannon, J.: The Theory of Negatively Curved Spaces and Groups. In: *Ergodic Theory, Symbolic Dynamics, and Hyperbolic Spaces*. (Ed: Bedford, Keane, & Series). Oxford: Oxford University Press, 1991

[doC] do Carmo, M.: *Riemannian Geometry*. Basel–Boston: Birkhäuser, 1992

[FK] Farkas, H., and Kra, I.: Automorphic Forms for Subgroups of the Modular Group. II: Groups Containing Congruence Subgroups. J. d'Analyse Math. **70** 91-156, (1996)

[Gr] Gromov, M.: *Metric Structures for Riemaniannian and Non-Riemannian Spaces*. Basel–Boston: Birkhäuser, 1999

[Ha] Hardt, R. M.: *Stratifications of real analytic mappings and images*. Inv. Math. **28** 193–208, (1975)

[Hi] Hironaka, H.: Subanalytic sets. Number theory. In: *Algebraic geometry and commutative algebra in honor of Y. Akizuki*, Tokyo: Kinokuniya Publications, 1973, pp. 453–493

[Jo1] Jones, G. A.: Geometric and Asymptotic Properties of Brillouin Zones in Lattices. Bull. Lond. Math. Soc. **16** 241–263, (1984)

[Jo2] Jones, H.: *The Theory of Brillouin Zones and Electronic States in Crystals*. Amsterdam: North-Holland, 1975

[KP] Kupka, I. A. K., and Peixoto, M. M.: On the Enumerative Geometry of Geodesics. In: *From Topology to Computation*, (Ed: Marsden & Shub), Berlin–Heidelberg–New York: Springer, 1993, pp. 243–253

[Mas] Maskit, B.: personal communication

[NZM] Niven, I., Zuckerman, H. S., and Montgomery, H. L.: *An Introduction to the Theory of Numbers*. fifth edition, New York: John Wiley and Sons, 1991

[Pe1] Peixoto, M. M.: On end point boundary value problems. J. Differential Equations **44** 273–280, (1982)

[Pe2] Peixoto, M. M.: Sigma décomposition et arithmétique de quelqes formes quadratiques définies positives. In: R. Thom Festschift volume: *Passion des Formes* (Ed: M. Porte), Paris: ENS Editions, 1994, pp. 455–479

[Pe3] Peixoto, M. M.: Focal Decomposition in Geometry, Arithmetic, and Physics. In: *Geometry, Topology and Physics*, (Ed: Apanasov, Bradlow, Rodrigues, & Uhlenbeck), Berlin–New York: de Gruyter & Co., 1997, pp. 213–231.

[PT] Peixoto, M. M., and Thom, R.: Le point de vue énumératif dans les problèmes aux limites pour les équations différentielles ordinaires. I: Quelques exemples, C. R. Acad. Sc. Paris I **303** 629–632 (1986); Erratum. C. R. Acad. Sc. Paris I **307** 197–198 (1988); II: Le théoréme. C. R. Acad. Sc. Paris I **303** 693–698 (1986)

[PS] Preparata, F. P., and Shamos, M. I.: *Computational Geometry*. Berlin–Heidelberg–New York: Springer, 1985

[Ra] Ratcliffe, J. G.: *Foundations of Hyperbolic Manifolds*. Berlin–Heidelberg–New York: Springer, 1994.

[Sah] Sah, C. H.: *Hilbert's Third Problem: Scissors Congruence*. Research Notes in Mathematics 33, London: Pitman, 1979

[SW] Skinner, C. M., and Wooley, T. D.: Sums of $k$-th Powers. J. Reine Angew. Math. **462** 57–68, (1995)

[Ti] Tinkham, M.: *Group Theory and Quantum Mechanics*. New York: McGraw-Hill, 1964.


Communicated by A. Connes