Cauchy distributions for the integrable standard map

Anastasios Bountis*
Department of Mathematics, Nazarbayev University, Nur–Sultan, Kazakhstan

J. J. P. Veerman†
²*Department of Mathematics and Statistics, Portland State University, Portland, OR, USA*

Franco Vivaldi‡
³*School of Mathematical Sciences, Queen Mary, University of London, London E1 4NS, UK*
(Dated: September 3, 2020)

We consider the integrable (zero perturbation) two–dimensional standard map, in light of current developments on ergodic sums of irrational rotations, and recent numerical evidence that it might possess non-trivial q-Gaussian statistics. Using both classical and recent results, we show that the phase average of the sum of centered positions of an orbit, for long times and after normalization, obeys the Cauchy distribution (a q-Gaussian with $q = 2$), while for almost all individual orbits such a sum does not obey any distribution at all. We discuss the question of existence of distributions for KAM tori.

I. INTRODUCTION

In the last fifteen years, there has been a growing number of numerical investigations suggesting that chaotic orbits of conservative (primarily Hamiltonian) systems are characterized by q–Gaussian statistics, whose index varies from $q = 1$ (Gaussian) in 'wide chaotic seas' to $1 < q < 3$ in 'thin chaotic layers' close to invariant tori (see [2, ch. 8] for details).

More recently, numerical evidence has shown that, as the perturbation parameter $\varepsilon > 0$ decreases, chaotic orbit distributions of the Chirikov-Taylor standard map (a paradigm of two–degree–of–freedom conservative flows)

$$(x, y) \mapsto (x + y, y + \varepsilon \sin(2\pi x)) \,(\mathrm{mod}\ 1) \tag{1}$$

change from Boltzmann-Gibbs (Gaussian) to Tsallis (q-Gaussian) statistics [3]. At present, for $\varepsilon \neq 0$, this claim cannot be rigorously verified nor challenged.

The purpose of this Letter is to elucidate the ergodic properties of the standard map for $\varepsilon = 0$ by applying the theory of ergodic sums of irrational rotations. This contribution fits within the broader study of regular motions on two-dimensional invariant tori and its connection with their (irrational) rotation number (see [4] and the collection of articles in [5]).

For $\varepsilon = 0$ the momentum y in (1) is constant, the map is integrable, and the phase space foliates into invariant circles (tori), parametrized by y. Let $y = \alpha$ represent one such circle, with $\alpha \notin \mathbb{Q}$, and consider the distributional properties (if any) of the ergodic sums

$$S_t(x, \alpha) = \sum_{k=0}^{t-1} h(f_\alpha^k(x)), \qquad h(x) = \{x\} - \frac{1}{2}, \qquad f_\alpha(x) = \{x + \alpha\} \tag{2}$$

where $\{\cdot\}$ denotes the fractional part, within the interval $[0, 1)$ with end-points identified.

We will show that the distributional properties of (2) for $\varepsilon = 0$ can be rigorously determined by applying both classical and recent results of ergodic theory that deal with number-theoretic questions. These are theorems that focus on the convergence in distribution of the sums $S_t(x, \alpha)$, *appropriately centralized and normalized.* As $t \to \infty$, our main findings are (details will be given below):

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*Electronic address: anastasios.bountis@nu.edu.kz
†Electronic address: veerman@pdx.edu
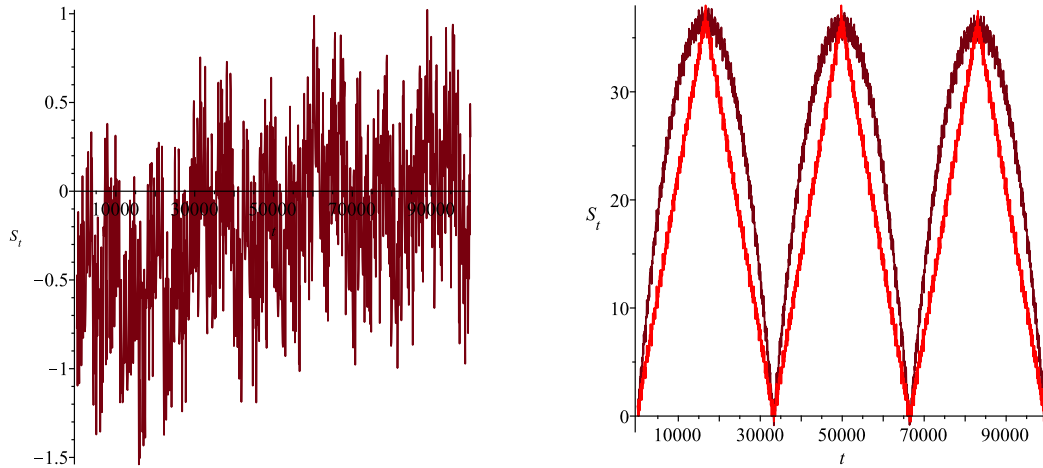‡Electronic address: f.vivaldi@maths.qmul.ac.uk

FIG. 1: Left: The sequence S_t, with initial condition $x = 0$ and $t \leqslant 10^5$. Left: $\alpha = (\sqrt{5} - 1)/2$, with continued fractions coefficients $[1, 1, 1, \ldots]$. Right: $\alpha = \pi - 3$ with continued fractions $[7, 15, 1, 293, 1, \ldots]$. The isolated large coefficient causes large fluctuations with an approximate periodicity of 33102, the denominator of the corresponding convergent. The red curve corresponds to the observable $-\text{sign}(h(x))/2$, see (7).

1. *The phase average of S_t converges to a Cauchy distribution (a q-Gaussian with $q = 2$).*

2. *For almost all initial points (x, α), S_t does not converge to any distribution.*

3. *If α is a badly approximable irrational[1], then S_t converges in distribution to a Gaussian (central limit theorem).*

The first statement follows from a classical theorem by Kesten (1960); the second is due to Dolgopyat & Sarig (2018); the third to Bromberg and Ulcigrai (2018).

Based on the above, we shall proceed to examine recent numerical evidence by Tirnakli & Tsallis [1] on the sum (2) for the map (1) with $\epsilon = 0$. More specifically, they sought to identify its statistical distribution within the family of q-Gaussians with zero mean

$$P_q(s) = \frac{\sqrt{\beta}}{C_q} \exp_q(-\beta s^2) \quad \text{where} \quad \exp_q(x) = [1 + (1 - q)x]^{\frac{1}{1-q}} \tag{3}$$

where β is a parameter and C_q is the normalization coefficient [6]. The Cauchy distribution of Theorem 1 that we obtain for this case corresponds to $q = 2$, with $C_2 = \pi$. In spite of approximately 8×10^{14} iterations of the map (2×10^8 initial points, each iterated 2^{22} times), the numerical value proposed in [1] $q \approx 1.935$ agrees with our theoretical prediction, within only one significant digit (see Appendix A).

II. BACKGROUND THEOREMS

Let us begin with some general observations:

1. The distributional properties of S_t depend sensitively on the type of averaging performed. Indeed, in (2) one may randomise any combination of x, α, and t, each sampled from a uniform distribution. In this regard, there are several types of limit theorems, stated below.

2. The behaviour of S_t for individual orbits is dictated by the arithmetical properties of α. Bounded continued fraction coefficients improve convergence (Figure 1, left), while the presence of large coefficients in the continued

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[1] with bounded continued fraction coefficients

fraction expansion cause large fluctuations (Figure 1, right). Recall that almost all real numbers have unbounded coefficients [7, Theorem 29].

3. The properties of S_t in (2) depend on the choice of the observable h. Thus, if h is sufficiently smooth, then S_t is bounded for almost all pairs (x, p) [8] (see also [9, Appendix A]). Bounded ergodic sums may occur also with non-smooth observables [10], but not for our choice of h.

Let us now recall various limit theorems in [9, 11], adopting the corresponding terminology.

For distributions along individual orbits, we fix the initial conditions and randomise t. The ergodic sum S_t of a map f and observable h satisfies a *temporal distributional limit theorem* if there is a centralizing sequence U_t, a normalizing sequence $V_t \to \infty$, and a (non-constant) random variable Y such that for all real y

$$\lim_{T \to \infty} \frac{1}{T} \# \Big\{ t \in \{0, \dots, T-1\} \ : \ \frac{S_t(x, \alpha) - U_T}{V_T} \leqslant y \Big\} = F_Y(y), \tag{4}$$

where $\#$ denotes the cardinality and F_Y is the cumulative distribution function of Y. Equivalently, the random variable $(S_t - U_T)/V_T$, where t is uniformly distributed among the first T iterates, converges in distribution to Y, as T goes to infinity. In general, the quantities U_T, V_T, and Y will depend on the initial point, as well as on the function h.

In a *spatial distributional limit theorem* one randomizes x instead of t, and modifies (4) as follows:

$$\lim_{T \to \infty} \frac{1}{T} \mu \Big\{ x \in [0, 1) \ : \ \frac{S_T(x, \alpha) - U_T}{V_T} \leqslant y \Big\} = F_Y(y), \tag{5}$$

where μ is the Lebesgue measure. (In a more general setting, the unit interval is replaced by the phase space of f, with invariant measure μ.) Unlike in (4), here only S_T is considered, the earlier values of the sum being ignored.

In the above limit theorems α is kept fixed, being regarded as a parameter. If we randomize α then the limit theorems (spatial or temporal) are said to be *annealed*. For the integrable standard map, a limit theorem resulting from a phase average is of the annealed spatial type.

Let us recall the first limit theorem of this kind, regarding rotations, due to Kesten [12]:

Theorem 1 [12]. *If (x, α) is uniformly distributed on $\mathbb{T}^2$, then the distribution of $S_T(x, \alpha)/\ln T$ converges as $T \to \infty$ to a Cauchy distribution: for some ρ and all y we have*

$$\lim_{T \to \infty} \frac{1}{T} \mu \Big\{ (x, \alpha) \in \mathbb{T}^2 \ : \ \frac{S_T(x, \alpha)}{\ln T} \leqslant y \Big\} = \frac{\rho}{\pi} \int_{-\infty}^{y} \frac{\mathrm{d}x}{1 + \rho^2 x^2}, \tag{6}$$

where μ is the Lebesgue measure.

Thus, phase averaging and logarithmic scaling yield the Cauchy distribution. Kesten, in fact, gives a formula for the value of ρ, which we compute in Appendix B below to find $\rho = 4\pi$. Comparing the right-hand-side of (6) with (3) for $q = 2$, we find $\beta = \rho^2$, hence $P(0) = \rho/\pi = 4$.

Recently, Dolgopyat & Sarig [9, Theorem 2.1] established a temporal version of Kesten's theorem, randomising t and α instead of x and α. The distribution is again Cauchy, with the same logarithmic scaling but a different constant: $\rho = 3\pi\sqrt{3}$.

We now turn to the existence of temporal distributions for individual orbits. The initial condition (x, α) is fixed, with $\alpha \notin \mathbb{Q}$, and we attempt to extract a distribution from the terms of the sequence (2). The following result (Dolgopyat & Sarig, 2018) shows that without α-averaging, almost surely, no temporal distribution exists.

Theorem 2 [13, Theorem 1.2]. *Let h be a piecewise smooth function of zero mean. Then there is a set of full measure $\Lambda \subset \mathbb{T}^2$ such that, if $(x, \alpha) \in \Lambda$ then the ergodic sum of $h \circ f_\alpha$ does not satisfy a temporal distributional limit theorem on the orbit of x.*

The observable h in (2) satisfies the assumptions of this theorem. Thus, for a generic initial condition (x, α), the sequence (2) for the integrable standard map does not admit any distributional limit, which is our second statement in Section I. This is because there are different scaling limits on different subsequences, due to the presence of large continued fractions coefficients The latter appear at random in the continued fraction expansion of most numbers, due to the ergodic properties of Gauss' map (see [14, section 3.2] and [13, Section 1.3]). Thus, distributional data extracted from a typical orbit are intrinsically unstable, and this problem cannot be fixed by choosing appropriate centralizing and normalizing sequences in (4).

The capricious nature of the ergodic sums (2) is best illustrated by fixing α and randomizing the initial condition. For any fixed time t, the appropriately normalized S_t of (2) gives a well-defined density $\nu(z, t, \alpha)$, symmetric around the origin (see [15]).

In Figure 2 we plot the density $\nu(z,t,\alpha)$ as a function of z, for $\alpha = \mathrm{e} - 2 = [1, 2, 1, 1, 4, 1, 1, 6, \ldots]$, with unbounded (although still regular) continued fractions coefficients. As t changes, we observe significant variations, ranging from near-uniformity, if t is the denominator of a convergent, to exotic shapes for other values of t. For a typical α these variations cannot be tamed by an averaging process.
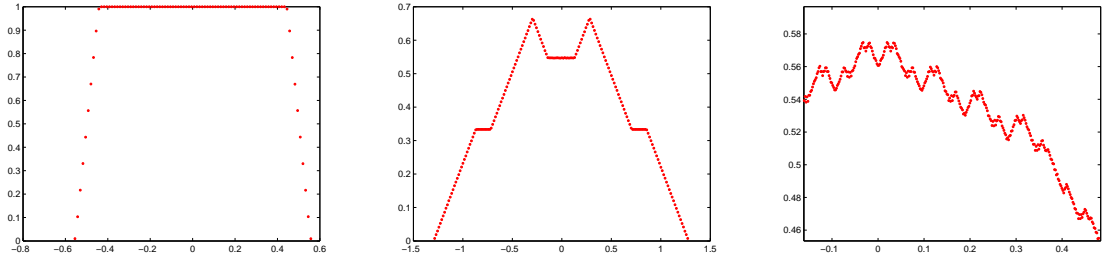

FIG. 2: Numerical approximations of the density $\nu(z,t,\alpha)$ for $\alpha = \mathrm{e} - 2$, and $t = 1001$ (left), $t = 213$ (centre), and a detail for $t = 334$ (right). Only the first value of t is a denominator of a convergent of $\mathrm{e} - 2$.

To obtain temporal limit theorems for individual orbits, one must restrict α to a zero measure set over which the fluctuations of S_t can be controlled. In this direction, a *temporal central limit theorem* has been proved very recently by Bromberg and Ulcigrai (see below) who considered the sum S_t corresponding to the much–studied observable [10, 12, 16, 17]

$$h_2(x) = \chi_I(x) - \gamma, \tag{7}$$

where $I = [0, \gamma)$ is an interval of length γ and χ_I is the characteristic function of I. (For $\gamma = 1/2$ we have $h_2(x) = -\mathrm{sign}(h(x))/2$, see Figure 1, right.) The zero–measure set chosen here consists of the *badly approximable* values of α, i.e., those with bounded continued fraction coefficients. This set is 'large' in that it has Hausdorff dimension 1 [18].

Theorem 3 [19, Theorem 1.1]. *Let α be a badly approximable irrational number. For every γ badly approximable with respect to α, every x, and every real y we have*

$$\frac{1}{T} \# \left\{ 0 \leqslant t < T \;:\; \frac{S_t(\alpha, \gamma, x) - U_t}{V_t} \leqslant y \right\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2} \mathrm{d}x$$

for some sequences $U_t(\alpha, x, \gamma)$ and $V_t(\alpha, \gamma) \to \infty$.

The restriction on γ is a (mild) diophantine condition (see [19] for details). This theorem is our third statement in Section I. It generalizes previous results by Beck [20, 21], who dealt with the special case of quadratic irrational α (irrational roots of a quadratic polynomial with integer coefficients), rational γ, and $x = 0$. Under these constraints, he was able to obtain a good description of centralizing and normalizing sequences: $U_t = U \ln t$ and $V_t = V \sqrt{\ln t}$, for some constants U and V.

III.   CONCLUDING REMARKS

In this paper we have considered the problem of the statistical properties of regular motions in the zero perturbation limit of the two–dimensional standard map. We remark at the outset, regarding the sensitivity of ergodic sums on the choice of the observable, that great care must be taken when assessing the relevance of the theory reviewed above to more general settings (cf. [13, section 2]).

Our main finding is that orbits on tori of the integrable two–dimensional standard map are described by the Cauchy distribution. We conjecture that this may hold more generally for integrable conservative systems. A second conclusion that follows from our findings is that the almost certain non-existence of distributions for individual orbits that holds for a large class of observables [9, section 1] is expected to apply to KAM tori as well.

Regarding the existence and type of distributions resulting from averaging over KAM tori, we note that the latter are parametrized by a positive measure Cantor set of rotation numbers, selected from the unit interval via a perturbation-dependent diophantine condition [22, p. 344]. This condition removes from the averaging set the rotation numbers

that are too closely approximable by rationals, thereby avoiding large fluctuations of the ergodic sum and improving convergence.

Thus, it seems plausible that, for appropriate observables, an ergodic sum such as (2), averaged over KAM tori, will converge in distribution. If so, then what would be the resulting distribution as a function of the perturbation parameter? This question appears worthy of further study.

Acknowledgments

APPENDIX A: THE RESULTS OF TIRNAKLI AND TSALLIS

Let us examine the results of the numerical experiments presented in [1], in view of Theorem 1 of Section II. The authors choose a large set of 2×10^8 initial points uniformly distributed on the 2-torus and compute $T = 2^{22} \approx 4 \times 10^6$ iterates of (1) for each initial point, letting $\langle x \rangle$ be the average of the resulting set of 8×10^{14} data points. For each orbit, the final value S_T of the ergodic sum is stored, and centralized using $\langle x \rangle$. The 2×10^8 values of S_T thus obtained are merged together, their distribution is computed and scaled by the numerical value of $P(0)$, while the latter, together with the normalisation coefficient C_q [6] provide a numerical value for β. Fitting the normalized distribution to their data yields the value $q \approx 1.935$.

The observable used in [1] differs from that of Theorem 1, since $1/2$ is replaced by $\langle x \rangle$, although such a difference can be absorbed by a centralized sequence U_T. If we apply Theorem 1 to these data, the numerical result for q agrees within only one significant figure. The numerical value of P(0) (after employing $ln(T)$ scaling) is $P(0) \approx 1.5$ [23], which is also problematic when compared with the exact value we have obtained, $P(0) = 4$. In order to investigate these discrepancies further, at this stage, we would need more detailed information about the computations presented in [1].

APPENDIX B: COMPUTATION OF ρ IN THEOREM 1

For $x \in [0, 1)$, let q_n be the denominator of the nth convergent of the continued fractions of x [24, Chapter X]. In [7, p. 66] it is proved that there is a unique $\tau > 0$ such that for almost all $x \in (0, 1)$ the following limit exists

$$\tau = \lim_{n \to \infty} \frac{n}{\ln q_n(x)} = \frac{12 \ln 2}{\pi^2}.$$

The parameter ρ is given by the following formula [12]

$$\rho = \frac{2\pi \ln 2}{\tau I} \tag{B1}$$

where

$$I = \int_0^1 \int_0^1 \left| \sum_{k=1}^{\infty} k^{-2} \sin 2\pi k x \sin 2\pi k y \right| \mathrm{d}y \mathrm{d}x. \tag{B2}$$

The integrand has an 8-fold symmetry, being invariant under reflection with respect to the main diagonal and each of the lines $x = 1/2, y = 1/2$ (because of the absolute value). So it suffices to restrict the integration to the triangle with vertices $(0, 0), (1, 0), (1/2, 1/2)$. The sum is absolutely convergent, and over that domain can be shown (via Fourier analysis) to be equal to $\pi^2 y(1 - 2x)$, which is positive. So, by Fubini's theorem, we have

$$I = 8 \int_0^{1/2} \mathrm{d}x \int_0^x \pi^2 y(1 - 2x) \mathrm{d}y = \frac{\pi^2}{24}$$

and thererefore $\rho = 4\pi$ as desired.

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