

Complete Expanding Interval Maps are Mixing

E.L. O’Neil, F.M. Tangerman, J.J.P. Veerman

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Abstract

We give a new and elementary¹ proof that certain smooth, complete, expanding maps from the interval to itself (Definition 1.1) are mixing.

KEYWORDS: Ergodicity, mixing, bounded nonlinearity.

1 Introduction

The aim of this paper is to give an elementary proof that maps like the Gauss map, $T : x \rightarrow 1/x - \lfloor 1/x \rfloor$, are mixing (Theorem 2.5). It is known that this map is *exponentially* mixing [1, 2, 3, 5], however the proofs are far from elementary. Instead, we give a comparatively simple proof of mixing.

In this introduction we set the scene, while the next section contains the proofs. First we fix some notation. For an interval map T , we write $\partial T(x)$ to indicate the derivative with respect to the argument at x , while $\partial_s T(x)$ will indicate $T'(x) \frac{dx}{ds}$.

Definition 1.1 (CEIM) *Let $T : [0, 1] \rightarrow [0, 1]$ be C^2 . We say T is a complete expanding interval map (CEIM, for short) if*

(i) there are $C > 0$ and $\Delta > 1$ so that for all x ,

$$|\partial T^n(x)| \geq C \Delta^n. \tag{1.1}$$

(ii) The unit interval can be partitioned into at most countably many (and at least two) intervals I_i of positive length such that T restricted to I_i is onto $[0, 1]$.

(iii) The nonlinearity of T is bounded, that is: there is $M > 0$ such that

$$\sup_i \sup_{z, u \in I_i} \left| \frac{\partial^2 T(z)}{\partial T(z) \partial T(u)} \right| \leq M,$$

where the supremum is over the intervals defined in (ii).

¹That is: without functional analysis.

The intervals I_i will be called the first level fundamental intervals. Any subsequent iterate of T will then induce a finer partition of $[0, 1]$. The intervals on which the restriction of T^2 is a bijection onto $[0, 1]$ will be called the second level fundamental intervals, and so forth.

It is useful to recall that for a CEIM T , the nonlinearity of T^n is also bounded: there is $D > 0$ such that for all $n > 0$:

$$\sup_i \sup_{z, u \in I_i} \left| \frac{\partial^2 T^n(z)}{\partial T^n(z) \partial T^n(u)} \right| \leq D, \quad (1.2)$$

where now I_i ranges over the n th level fundamental intervals ([4], Proposition 10.16). In addition, for all n and x in an n th level fundamental interval I_i :

$$D^{-1} \leq |\partial T^n(x)| |I_i| \leq D. \quad (1.3)$$

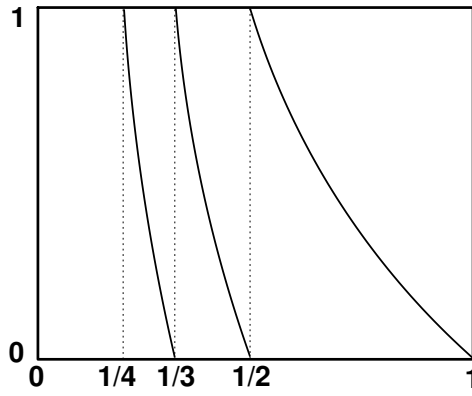


Figure 1.1: A sketch of three branches of the Gauss map.

Clearly, for any integer $n > 1$, the map $x \rightarrow nx \bmod 1$ satisfies all requirements (its second derivative being zero). It is an easy, but tedious, exercise to show that for $d \geq 1$, the map $x \rightarrow x^{-d} \bmod 1$ also satisfies all requirements. We leave this to the reader. Because of its prominent role in number theory, the most important of these examples is of course the Gauss map of which a few branches are sketched in Figure 1.1.

It is well-known that a CEIM T is ergodic with respect to an invariant measure m that is absolutely continuous with respect to the Lebesgue measure. We lift the following from [4], appendix E, where a proof can also be found.

Theorem 1.2 *Let T be a complete expanding interval map. Then there is a unique invariant probability measure m that has the following properties:*

- (i) *m has a continuous density μ ,*
- (ii) *there is a $K > 1$ such that for all $x \in [0, 1]$, $K^{-1} \leq \mu(x) \leq K$,*
- (iii) *T is ergodic with respect to m .*

Consider the following class of functions.

Definition 1.3 *A family of functions $\{f_n\}_{n=0}^\infty = \{j_n + h_n\}_{n=0}^\infty$ from $[0, 1]$ to $\mathbb{R}$ is called piecewise equicontinuous if:*

- (i) The functions j_n are constant except for jump discontinuities²; j_n has bounded variation J_n and $\lim J_n = 0$.
- (ii) The functions h_n are uniformly equicontinuous, that is: for all $\epsilon > 0$, there is $\delta > 0$ such that for all h_n , $|y - x| < \delta$ implies $|h_n(y) - h_n(x)| < \epsilon$.

Our main result is Theorem 2.5. It states that CEIM's are mixing. Here is the definition.

Definition 1.4 Let T be a transformation with an invariant measure m . Then T is mixing with respect to m if for any two intervals A and B

$$\lim_{n \rightarrow \infty} |m(T^{-n}(A) \cap B) - m(A) \cdot m(B)| = 0.$$

2 Proofs

Proposition 2.1 Let $\{f_n\}_{n=0}^\infty = \{j_n + h_n\}_{n=0}^\infty$ a piecewise equicontinuous family. Then the sequence $\{f_n\}$ has a subsequence that converges uniformly to a continuous function g .

Proof. The proof follows largely that of the Arzelà-Ascoli theorem. Choose $\epsilon > 0$ and let δ be such that the family h_n is uniformly ϵ - δ equicontinuous. Furthermore, let X be a dense set from which we can choose finitely many x_i a distance less than δ apart.

Since the f_n are uniformly bounded, there is a subsequence $f_{1,n}$ of f_n such that $f_{1,n}(x_1)$ converges to y_1 . Then we can choose a subsequence $f_{2,n}$ of $f_{1,n}$ such that $f_{2,n}(x_2)$ converges to y_2 . And so forth. Now rename $g_n := f_{n,n} = f_{k_n}$ for some $k_n \geq n$. Clearly,

$$\lim_{n \rightarrow \infty} g_n(x_k) = y_k.$$

Now for any x , take an x_i less than δ away from x . Since $g_n = f_{k_n}$, the jumps in g_n have variation at most $J_{k_n} < J_n$. Thus $|g_n(x_i) - g_n(x)| < \epsilon + J_n$. Also, $\{g_n(x_i)\}_{n=1}^\infty$ is Cauchy, and thus for $\ell > n$ and n large $|g_\ell(x_i) - g_n(x_i)| < \epsilon$. Therefore,

$$\begin{aligned} |g_\ell(x) - g_n(x)| &\leq |g_\ell(x) - g_\ell(x_i)| + |g_\ell(x_i) - g_n(x_i)| + |g_n(x_i) - g_n(x)| \\ &\leq (\epsilon + J_n) + \epsilon + (\epsilon + J_n) < 4\epsilon, \end{aligned}$$

for n large enough. Thus $g_n(x)$ is Cauchy and uniformly converges to some function $g(x)$.

In fact, g is uniformly continuous, because if $|y - x| < \delta$, then

$$\begin{aligned} |g(y) - g(x)| &\leq |g(y) - g_n(y)| + |g_n(y) - g_n(x)| + |g_n(x) - g(x)| \\ &\leq 4\epsilon + (\epsilon + J_n) + 4\epsilon < 10\epsilon, \end{aligned}$$

for n large enough. ■

²A jump discontinuity of j at x means that the right and left limit of j at x exist and are finite, but not equal.

Before we prove the next results, we need more notation. The pushforward (or Ruelle-Perron-Frobenius operator) of a density ϕ_0 under T is ϕ_1 :

$$\phi_1(y) = (T_*\phi_0)(y) := \sum_{x \in T^{-1}y} \frac{\phi_0(x)}{|\partial T(x)|},$$

and $\phi_n = T_*\phi_{n-1}$. By Theorem 1.2, T is ergodic with respect to an invariant density μ , which is continuous and bounded from above and below. Define $f_n(x) := \phi_n(x)/\mu(x)$. We have

$$f_n(y) = \sum_{x \in T^{-1}(y)} f_{n-1}(x) \frac{\mu(x)}{\mu(T(x)) |\partial T(x)|} = \sum_{x \in T^{-n}(y)} f_0(x) \frac{\mu(x)}{\mu(T^n(x)) |\partial T^n(x)|}.$$

By invariance, $1 = \frac{1}{\mu(y)} \sum \frac{\mu(x)}{|\partial T^n(x)|}$, and thus these operations are *weighted averages*. The weights associated with T^n (for $n \in \mathbb{N}$) are positive and will be denoted by $w_n(x)$. Thus

$$f_n(y) = \sum_{x \in T^{-n}(y)} f_0(x) w_n(x). \quad (2.1)$$

The following estimate will be necessary to obtain a modulus of continuity on f_n independent of n .

Lemma 2.2 *There is an increasing real-valued function ϵ_w with $\lim_{t \rightarrow 0} \epsilon_w(t) = 0$ such that for all y and y' , $n > 0$, and x_i, x'_i in the same n th level fundamental interval with $T^n(x_i) = y$ and $T^n(x'_i) = y'$,*

$$\sum_i |w_n(x_i) - w_n(x'_i)| \leq \epsilon_w(|y - y'|).$$

(That is: ϵ_w is a modulus of continuity for w_n for all n .)

Proof. First, for notational convenience we define $a := 1/\mu(y) = 1/\mu(T^n(x_i))$, $b := \mu(x_i)$, and $c := 1/|\partial T^n(x_i)|$, and similarly, $a' := 1/\mu(y') = 1/\mu(T^n(x'_i))$, et cetera. Taking the sum over all n th-level fundamental intervals, we have

$$\sum_i |abc - a'b'c'| \leq \sum_i |(a - a')bc| + \sum_i |(b - b')a'c| + \sum_i |(c - c')a'b'|.$$

Now, a and b are uniformly bounded (recall that μ is bounded away from zero) and the sum $\sum_i |c|$ is bounded by $\sum_i D|I_i| = D$. In the first term on the right hand side $|a - a'|$ has a modulus of continuity inherited from $\mu(y)$. In the second term, we have a weighted sum of moduli $\epsilon_b(|x_i - x'_i|)$ of b . Since $|x_i - x'_i| \leq |y - y'|$, we assert that $\epsilon_b(|y - y'|)$ serves as modulus for this term.

We need to find a modulus for the third term. Rewrite it as $\sum_i |(c/c' - 1)a'b'c'|$. It now is a weighted average of $|c/c' - 1|$ which, in turn, we estimate using (1.2) and (1.3).

$$\left| \ln \frac{|\partial T^n(x)|}{|\partial T^n(x')|} \right| = |\ln |\partial T^n(x)| - \ln |\partial T^n(x')|| = \left| \int_{x'}^x \frac{\partial^2 T^n(z)}{(\partial T^n(z))^2} \cdot \partial T^n(z) dz \right| \leq D|y - y'|.$$

$$\Rightarrow e^{-D|y-y'|} \leq \frac{|\partial T^n(x)|}{|\partial T^n(x')|} \leq e^{D|y-y'|}.$$

And this imparts a modulus of continuity $\epsilon_C(|y - y'|)$ on the third term. ■

In what follows, we will decompose f_n into $j_n + h_n$, jumps plus a continuous function. To facilitate the proof of the next lemma, it is helpful to preface it with an exercise in bookkeeping. Consider the situation if in Figure 2.1. We calculate the continuous part of the difference $f(x)w(x) - f(x')w(x')$, where $w(x)$ is continuous in the interval $[x, x']$ but f has two jump discontinuities:

$$[f(x)w(x) - f(x^{1-})w(x^{1-})] + [f(x^{1+})w(x^{1+}) - f(x^{2-})w(x^{2-})] + [f(x^{2+})w(x^{2+}) - f(x')w(x')].$$

Now take a limit as x^{j+} and x^{j-} tend to x^j , the locus of the discontinuity, and denote the jump in f at x^j by J^j . We use continuity of $w(x)$ to get

$$\begin{aligned} \dots &= f(x)w(x) + \sum_j J^j w(x^j) - f(x')w(x') \\ &= \left[f(x) - (f(x') - \sum_j J^j) \right] w(x) + f(x')(w(x) - w(x')) + \sum_j J^j (w(x^j) - w(x)), \end{aligned}$$

Finally, we note that $f(x) - (f(x') - \sum_j J^j)$ can be replaced by $h(x) - h(x')$ if h is the continuous part of f .

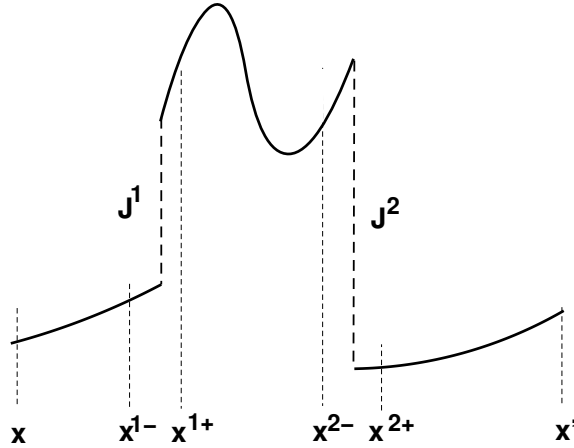


Figure 2.1: Sketch of a function f that is the sum of a continuous function h and two jumps J^j .

Proposition 2.3 Let j_0 be piecewise constant with finitely many finite jumps and h_0 continuous. Then h_n defined in (2.1) has a modulus of continuity which is independent of n .

Proof. By Equation (2.1), we have

$$|f_n(y) - f_n(y')| = \left| \sum_i f_0(x_i)w_n(x_i) - f_0(x'_i)w_n(x'_i) \right|,$$

where $\sum_i$ indicates, as before, that $x'_i \in T^{-n}(y')$ and $x_i \in T^{-n}(y)$ are in the same n th level fundamental interval I_i . We wish to obtain a modulus of continuity for the continuous part h_n of f_n that is independent of n . Denote the jumps of f_0 in the interval $[x_i, x'_i]$ by J_i^j and the loci of the jumps by x_i^j . By the observations prior to this lemma, we obtain

$$h_n(y) - h_n(y') = \sum_i (h_0(x_i) - h_0(x'_i))w_n(x_i) + \sum_i f_0(x'_i)(w_n(x_i) - w_n(x'_i)) + \sum_i \sum_j J_i^j (w_n(x_i^j) - w_n(x_i)).$$

We now give a modulus for each of the terms on the right hand side. The first term consists of a weighted sum of moduli $\epsilon_0(|x_i - x'_i|)$ of h_0 . Thus $\epsilon_0(|y - y'|)$ serves as modulus for this term. The second term has $\epsilon(t)$ of Lemma 2.2 multiplied by $\|f_0\|$ as a modulus. The absolute value of the third term is bounded by the variation of j_0 times the modulus computed in Lemma 2.2. ■

Proposition 2.4 *If ϕ is the sum of a continuous function and a finite jump function and $\int \phi = 0$, then $\lim_{n \rightarrow \infty} \|T_*^n \phi\| = 0$, where $\|\cdot\|$ denotes the supnorm.*

Proof. Denote the operator introduced in (2.1) by $\tilde{T}_*^n$. Since it is a weighted average, $\|f_{n+1}\| = \|\tilde{T}_*^n f_n\| \leq \|f_n\|$. Thus $\|f_n\|$ is positive and decreasing and so $\lim_{n \rightarrow \infty} \|f_n\|$ exists. Call that limit L . We assume $L > 0$ and derive a contradiction.

Consider (2.1) and decompose f_n as before into $j_n + h_n$. The derivatives $|\partial T^n(x)|$ increase exponentially, and therefore the jumps in j_n decrease exponentially. The h_n are uniformly equicontinuous by Proposition 2.3. Thus by Proposition 2.1, the f_n have a subsequence f_{n_i} uniformly convergent to a continuous function g .

By assumption $\|g\| = L > 0$. But since g has average zero, $g(x)$ must alternate sign, and therefore by continuity it must assume the value zero at some point. Again by continuity, there is an interval around that point where $|g(x)| < L/2$. Some m th iterate of $\tilde{T}_*$ must sample g in precisely that interval. But then there is a $\eta > 0$ so that

$$\|\tilde{T}_*^m g\| \leq L - 2\eta.$$

Now by linearity,

$$\|\tilde{T}_*^m f_{n_i}\| = \|\tilde{T}_*^m g + \tilde{T}_*^m (f_{n_i} - g)\| \leq \|\tilde{T}_*^m g\| + \|\tilde{T}_*^m (f_{n_i} - g)\| \leq L - 2\eta + \|\tilde{T}_*^m (f_{n_i} - g)\|.$$

But we can take n large enough so that $\|f_{n_i} - g\| < \eta$. Since $\tilde{T}_*$ cannot increase the supnorm, we finally obtain $\|\tilde{T}_*^m f_{n_i}\| \leq L - \eta$, contradicting that $\|f_{n_i}\| \geq L > 0$. ■

Theorem 2.5 (CEIM's are mixing) *Let T be a CEIM with invariant measure m , absolutely continuous with respect to Lebesgue. Then T is mixing with respect to m .*

Proof. By Theorem 1.2, T has invariant density μ and invariant measure m . Given an interval A , $\mathbf{1}_A$ is the characteristic function on A with m -length m_A . Then

$$\begin{aligned} |m(T^{-n}(A) \cap B) - m_A m_B| &= \left| \int_{T^{-n}(A) \cap B} \mu(y) dy - \int_B m_A \mu(y) dy \right| \\ &= \left| \int_B T_*^n(\mathbf{1}_A \cdot \mu)(y) dy - \int_B m_A \mu(y) dy \right|. \end{aligned}$$

Since μ is invariant under T_* , we may replace the integrand by $(T_*^n[\mathbf{1}_A\mu - m_A\mu])(y)$. Thus

$$|m(T^{-n}(A) \cap B) - m_A m_B| \leq \int |(T_*^n[\mathbf{1}_A\mu - m_A\mu])(y)| dy.$$

By Theorem 1.2, μ is continuous and thus $\mathbf{1}_A\mu - m_A\mu$ is the sum of continuous plus jumps and therefore its iterates satisfy Definition 1.3. Proposition 2.4 now implies the theorem. ■

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