# Asymptotic Reliability Theory of $k$-out-of-$n$ Systems

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**Abstract**

We formulate a theory that allows us to formulate a simple criterion that ensures that two $k$-out-of-$n$ systems $A$ and $\tilde{A}$ are *not* ordered. If the systems fail the criterion, it does not follow they are ordered. Thus the theory only serves to avoid some a priori useless comparisons: when neither $A$ nor $\tilde{A}$ can be said to be better than the other. The power of the theory lies in its wide potential applicability: the assumptions involve very weak estimates on the asymptotic behavior (as $t \to 0$ and as $t \to \infty$) of the constituent survival probabilities. We include examples.

*Key words*: order statistics, stochastic orderings, $k$-out-of-$n$ systems, heterogeneous distributions.

## 1 Introduction

In reliability theory, a $k$-out-of-$n$ system consists of $n$ components of the same kind with independent and identically distributed lifetimes. All $n$ components start working simultaneously, and the system works, if at least $k$ components function; i.e. the system as a whole fails if $(n-k+1)$ components fail. This kind of order statistics has found applications in many industrial processes and other applied areas. For example, an aircraft with four engines will not crash if at least three of them are functioning. The lifetime of a $k$-out-of-$n$ system is described by the $(n-k+1)^{th}$ order statistic of the random variables $X_1, \ldots, X_n$. In particular, the lifetime of a parallel system, which is a 1-out-of-$n$ system, is the same as the largest order statistic, and analogously, the lifetime of a series system, which is a $n$-out-of-$n$ system, is the same as the smallest order statistic.

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If the random variables $X_1, \dots, X_n$ are arranged in ascending order of magnitude, then the $k^{th}$ smallest of $X_k$'s is denoted by $X_{k:n}$. The ordered quantities

$$X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}\,, \tag{1}$$

are called order statistics (OS), and $X_{k:n}$ is the $k^{th}$ order statistic.

In this note we compare two $k$-out-of-$n$ systems where the lifetimes $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ of their components have independent but not identical distributions. The usual approach is to find conditions for which $X_{n-k+1:n} \leq_{st} Y_{n-k+1:n}$, for $1 \leq k \leq n$. This leads to a wealth of information in numerous interesting special cases. In particular, when $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ are two samples of independent exponential random variables with $X_k$ and $Y_k$ having hazard rates $\lambda_k$ and $\theta_k$, respectively, for $1 \leq k \leq n$, Pledger and Proschan [1] were the first to compare stochastically the order statistics from these two samples. Specifically, they showed that

$$(\theta_1, \dots, \theta_n) \leq^{\mathrm{m}} (\lambda_1, \dots, \lambda_n) \Rightarrow Y_{n-k+1:n} \leq_{st} X_{n-k+1:n}.$$

More recently, Khaledi and Kochar [2] studied the case $k = 1$ and proved that

$$(\theta_1, \dots, \theta_n) \leq^{\mathrm{p}} (\lambda_1, \dots, \lambda_n) \Rightarrow Y_{n:n} \leq_{st} X_{n:n}. \tag{2}$$

For the case in which one of the samples is independent and identically distributed, Bon and Păltănea [7] gave a necessary and sufficient condition on the parameters for the inequality $Y_{n-k+1:n} \leq_{st} X_{n-k+1:n}$, $1 \leq k \leq n$.

It is well known that the exponential distribution is a particular case of different models such as the proportional random variables (PRV) model and the proportional hazard rates (PHR) model (see Section 4 for the definition). Pledger and Proschan [1] studied conditions under which the order statistics from two samples $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ in these models can be stochastically ordered. In particular, if the hazard rate, $h(t)$, of $F$ is decreasing and $F$ is an absolutely continuous distribution, then

$$(\theta_1, \dots, \theta_n) \leq^{\mathrm{m}} (\lambda_1, \dots, \lambda_n) \Rightarrow Y_{n-k+1:n} \leq_{st} X_{n-k+1:n},$$

for $1 \leq k \leq n$. When $k = 1$, Khaledi et al. [5] proved that if $t \cdot r(t)$ is decreasing in $t$, where $r(t)$ is the reversed hazard rate of $F$, then

$$(\theta_1, \dots, \theta_n) \leq^{\mathrm{p}} (\lambda_1, \dots, \lambda_n) \Rightarrow Y_{n:n} \leq_{st} X_{n:n}.$$

Khaledi and Kochar [4] further improved (2) from exponential random variables to PHR model, that is,

$$(\theta_1, \dots, \theta_n) \leq^{\mathrm{p}} (\lambda_1, \dots, \lambda_n) \Rightarrow Y_{n:n} \leq_{st} X_{n:n}.$$

Navarro [12] studied the tail behavior of the hazard rate function (when $t \to \infty$) of order statistics from PHR models. Note that the hazard rate ordering implies the usual stochastic ordering.

The strategy in this note is to give less precise information but in a much more general setting. Given the two $k$-out-of-$n$ systems, we look at the asymptotic behavior (as $t \to 0$ and as $t \to \infty$) of the survival functions associated with the order statistics of $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$. If these are sufficiently different we know that $X_{n-k+1:n}$ and $Y_{n-k+1:n}$ are *not* stochastically ordered, for $1 \le k \le n$. The advantage is that this kind of comparison can be done in enormous generality as we show below.

The rest of this paper proceeds as follows. In Section 2, we briefly outline some stochastic orders and majorization. Section 3 then describes the asymptotic behavior of the survival functions associated with $X_{k:n}$, for $1 \le k \le n$. In Section 4 we present applications of our main results to different common types of distributions. In the last section we look at the special case of the exponential distribution.

## 2 Definitions

In this section, we present a brief review of some notions of stochastic orders and majorization. See Shaked and Shantikhumar [9] for an overview of the different notions of ordering, and Marshall and Olkin [8] and Bon and Păltănea [6] for more details on majorization order and $p$-larger order, respectively.

**Definition 1.** *Let $X$ and $Y$ be univariate random variables with cumulative distribution functions (c.d.f.'s) $F$ and $G$, survival functions $\bar{F}\,(= 1 - F)$ and $\bar{G}\,(= 1 - G)$ $X$ is said to be smaller than $Y$ in the usual stochastic order, denoted by $X \le_{st} Y$, if $\bar{F}(t) \le \bar{G}(t)$ for all $t$.*

We shall also be using the concept of majorization in our discussion. Let $\{x_{(1)}, x_{(2)}, \dots, x_{(n)}\}$ denote the increasing arrangement of the components of the vector $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$.

**Definition 2.** *The vector $\boldsymbol{x}$ is said to be majorized by the vector $\boldsymbol{y}$, denoted by $\boldsymbol{x} \le^m \boldsymbol{y}$, if*

$$\sum_{i=1}^{j} x_{(i)} \ge \sum_{i=1}^{j} y_{(i)}, \quad \text{for } j = 1, \dots, n-1 \quad \text{and} \quad \sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}.$$

**Definition 3.** *The vector $\boldsymbol{x}$ is said to be p-smaller than the vector $\boldsymbol{y}$, denoted by $\boldsymbol{x} \le^{\mathrm{p}} \boldsymbol{y}$, if*

$$\prod_{i=1}^{j} x_{(i)} \ge \prod_{i=1}^{j} y_{(i)}, \quad \text{for } j = 1, \dots, n.$$

It is known that $\boldsymbol{x} \le^{\mathrm{m}} \boldsymbol{y} \Rightarrow \boldsymbol{x} \le^{\mathrm{p}} \boldsymbol{y}$. The converse is, however, not true (c.f. Khaledi and Kochar [3]).

# 3 Asymptotic theory

In this section, we establish results on the asymptotic behavior of the survival functions associated with the order statistics from a sample of independent but not identically distributed positive random variables such that their survival functions are contained in the class $\mathcal{F}_{pq}$ defined as following. But first we need to define a $q$-sub-exponential function.

**Definition 4.** *Let $q$ be a positive number. A $q$-sub-exponential function $z(t)$ in $t$ is a function that satisfies:*

$$\forall \epsilon > 0 \quad \lim_{t\to\infty} z(t)e^{-\epsilon t^q} = 0 \quad \text{and} \quad \liminf_{t\to\infty} z(t) \geq 1 \quad .$$

**Definition 5.** *For $p, q > 0$, we say $\bar{F}$ (or $F \equiv 1 - \bar{F}$) is in $\mathcal{F}_{pq}$ if there are positive constants $\alpha$ and $\omega$ and a positive $q$-sub-exponential function $z(t)$ such that*

$$\bar{F}(t) = 1 - \alpha t^p (1 + \phi(t)), \;\; \text{with} \;\; \lim_{t\to 0} \phi(t) = 0,$$

*and*

$$\bar{F}(t) = z(t)e^{-\omega t^q}.$$

*The constants $\alpha$ and $\omega$ are called the (initial and final) asymptotic constants.*

Note that this is the *only* requirement on the distribution $F$. In particular continuity is not required.

The point of this somewhat detailed description of the asymptotic behavior is that it is satisfied very generally by many common distributions. Examples are the generalized Gamma and the exponentiated Weibull distributions. We will look at these and other examples in the next section.

**Definition 6.** *For $0 \leq m \leq n$, let $\mathcal{I}_m$ denote the collection of all indicator functions $I : \{1, 2, ..., n\} \to \{0, 1\}$ such that card$(I^{-1}(1)) = m$. ($\mathcal{I}_m$ consists of $\frac{n!}{m!(n-m)!}$ such functions.)*

Let us denote by $P_{A,k}(t)$ the probability that *exactly* $k$ components of system $A$ (resp., $\tilde{A}$) remain functional after time $t$. Then, it is easily to check that

$$P_{A,k}(t) = \sum_{I\in\mathcal{I}_k} \left( \prod_{i\in I^{-1}(0)} F_i(t) \prod_{i\in I^{-1}(1)} \bar{F}_i(t) \right), \tag{3}$$

where $F_i(t)$ and $\bar{F}_i(t)$ are the distribution and the survival function, respectively, of $X_i$ which is the lifetime of the $i^{th}$ component, for $i = 1, \ldots, n$.

**Proposition 1.** *Let $X_1, \ldots, X_n$ be independent random variables with $X_i$ having survival function $\bar{F}_i \in \mathcal{F}_{pq}$, with the asymptotic constants $\alpha_i$ and $\omega_i$, for $i = 1, \ldots, n$. Then*

$$P_{A,k}(t) = t^{(n-k)p} \left( \sum_{I\in\mathcal{I}_k} \prod_{i\in I^{-1}(0)} \alpha_i \right) (1 + \rho(t)) \;, \;\; \text{with} \;\; \lim_{t\to 0} \rho(t) = 0 \;,$$

*(with the convention that* $\prod_{i\in I^{-1}(0)}\alpha_i = 1$ *if* $I^{-1}(0)=\emptyset$*) and*

$$P_{A,k} = u(t)e^{-\left(\sum_{i=1}^{k}\omega_{(i)}\right)t^q},$$

*where* $u(t)$ *is* $q$*-sub-exponential and* $\omega_{(1)} \le \omega_{(2)} \le \cdots \le \omega_{(n)}$ *is the increasing arrangement of the numbers* $\omega_1, \omega_2, \cdots, \omega_n$ *(with the convention that* $\sum_{i=1}^{0}\cdots = 0$*).*

*Proof.* First we consider small $t$. Definition 5 implies that $\bar{F}_i(t) = 1 + \epsilon_i(t)$ where the $\epsilon_i(t)$ tend to zero for small $t$. Both products in (3) are finite. So we obtain that:

$$\begin{aligned} P_{A,k}(t) &= \sum_{I\in\mathcal{I}_k}\left(\prod_{i\in I^{-1}(0)}\alpha_i t^p(1+\phi_i(t))\prod_{i\in I^{-1}(1)}[1+\epsilon_i(t)]\right) \\ &= t^{(n-k)p}\sum_{I\in\mathcal{I}_k}\left((1+\rho_I(t))\prod_{i\in I^{-1}(0)}\alpha_i\right), \end{aligned}$$

where the $\rho_I(t)$ tend to zero for small $t$. In the last expression set $q_I$ equal to $\prod_{i\in I^{-1}(0)}\alpha_i$. Then note that (the $q_I$'s are positive)

$$\sum_{I\in\mathcal{I}_k} q_I(1+\rho_I) = \left(\sum_I q_I\right)\left(1+\frac{\sum q_I\rho_I}{\sum q_I}\right).$$

The first conclusion follows with $\rho(t) = \frac{\sum q_I\rho_I}{\sum q_I}$.

Now we consider $t$ large. This time we set $F_i(t) = 1+\eta_i(t)$ in (3) where the $\eta_i(t)$ tend to zero for large $t$. Via the same reasoning as before we get:

$$\begin{aligned} P_{A,k}(t) &= \sum_{I\in\mathcal{I}_k}\left(\prod_{i\in I^{-1}(0)}(1+\eta_i(t))\prod_{i\in I^{-1}(1)}z_i(t)\,e^{-\omega_i t^q}\right) \\ &= \sum_{I\in\mathcal{I}_k}\left((1+r_I(t))\quad e^{-\left(\sum_{i\in I^{-1}(1)}\omega_i\right)t^q}\prod_{i\in I^{-1}(1)}z_i(t)\right), \end{aligned}$$

where the $r_I(t)$ tend to zero for large $t$. In the last expression set $s_I$ equal to $\sum_i \omega_i$ and $u_I(t)$ to $\prod_i z_i(t)$. Then of course

$$P_{A,k}(t) = \sum_{I\in\mathcal{I}_k}(1+r_I(t))\;\; u_I(t)\;\; e^{-s_I t^q}.$$

As an anonymous referee noted, if $z$ is $q$-sub-exponential and $\lim_{t\to\infty}\phi(t)=0$, then so is $z(1+\phi)$.

By Definition 4 the behavior for large $t$ is entirely determined by the exponentials. Because of the way the engines are indexed, one of the leading terms among those is $s_0 \equiv \sum_{i=1}^k \omega_i$. Suppose for the moment it is the only leading term. Writing out the terms and separating $s_0$ we get:

$$P_{A,k}(t) = u_0(t)\, e^{-s_0 t^q} \left(1 + r_0(t) + \sum_{I \neq 0} \frac{u_I(t)}{u_0(t)}\, e^{-(s_I - s_0)t^q}\, (1 + r_I(t))\right) .$$

The term in parentheses times $u_0$ is easily seen to be a $q$-sub-exponential function. Now set that function equal to $u(t)$ and then the second statement follows. If there are various $s_I$'s that are minimal this proof can easily be adapted. □

**Definition 7.** *The sign function is defined as follows:*

$$\operatorname{sign}(x) = \begin{cases} -1 & \textit{if } x < 0 \\ 0 & \textit{if } x = 0 \\ 1 & \textit{if } x > 0 \end{cases} .$$

Throughout this article, we suppose without loss of generality that the $\omega_i$'s ($\tilde{\omega}_i$'s) are in increasing order.

**Theorem 1.** *Suppose $X_1, \ldots, X_n$ are independent random variables with $X_i$ having survival function $\bar{F}_i \in \mathcal{F}_{pq}$ and asymptotic coefficients $\alpha_i$ and $\omega_i$, $i = 1, \ldots, n$, and let $Y_1, \ldots, Y_n$ be another set of independent random variables with $Y_i$ having survival function $\bar{G}_i \in \mathcal{F}_{pq}$ and asymptotic constants $\tilde{\alpha}_i$ and $\tilde{\omega}_i$, $i = 1, \ldots, n$. Then there is an $\epsilon > 0$ so that*

$$\operatorname{sign}\left(P_{A,k}(t) - P_{\tilde{A},k}(t)\right) = \begin{cases} \operatorname{sign}\left(\sum_{I \in \mathcal{I}_k} \left(\prod_{i \in I^{-1}(0)} \alpha_i - \prod_{i \in I^{-1}(0)} \tilde{\alpha}_i\right)\right) , & \forall\, t < \epsilon \\ \operatorname{sign}\left(\sum_{i=1}^k (-\omega_i + \tilde{\omega}_i)\right) , & \forall t > 1/\epsilon \end{cases}$$

*assuming that both of the right hand side expressions are non-zero.*

*Proof.* By Proposition 1, we have

$$P_{A,k}(t) - P_{\tilde{A},k}(t) = t^{(n-k)p} \left[\left(\sum_{I \in \mathcal{I}_k} \prod_{i \in I^{-1}(0)} \alpha_i\right)(1 + \rho(t)) - \left(\prod_{i \in I^{-1}(0)} \tilde{\alpha}_i\right)(1 + \tilde{\rho}(t))\right]$$

where $\rho$ and $\tilde{\rho}$ tend to zero as $t \to 0$. Thus for $t$ small enough, $\operatorname{sign}\left(P_{A,k}(t) - P_{\tilde{A},k}(t)\right)$ is the same as the sign of the right hand side of the last equation (unless that is equal to zero).

Similarly Proposition 1 implies that

$$\begin{aligned} P_{A,k}(t) - P_{\tilde{A},k}(t) &= u(t) e^{-(\sum_{i=1}^k \omega_i) t^q} - \tilde{u}(t) e^{-(\sum_{i=1}^k \tilde{\omega}_i) t^q} \\ &= u(t) e^{-(\sum_{i=1}^k \omega_i) t^q} \left(1 - \frac{\tilde{u}(t)}{u(t)}\, e^{-(\sum_{i=1}^k -\omega_i + \tilde{\omega}_i) t^q}\right) \end{aligned}$$

Thus if the sign of $\sum_{i=1}^{k} -\omega_i + \tilde{\omega}_i$ is not equal to zero, it determines the sign of $P_{A,k}(t) - P_{\tilde{A},k}(t)$ for $t$ large. $\square$

**Proposition 2.** *Let $X_1, \ldots, X_n$ be independent random variables with $X_i$ having survival function $\bar{F}_i \in \mathcal{F}_{pq}$, for $i = 1, \ldots, n$. Then, the survival function of the $(n-k+1)^{th}$ order statistic is*

$$\bar{F}_{n-k+1:n}(t) = 1 - t^{(n-k+1)p} \left( \sum_{I \in \mathcal{I}_{k-1}} \prod_{i \in I^{-1}(0)} \alpha_i \right) (1 + \sigma(t)) \ , \ \textit{with} \ \lim_{t \to 0} \sigma(t) = 0 \ ,$$

*and*

$$\bar{F}_{n-k+1:n}(t) = u(t) \, e^{-\sum_{i=1}^{k} \omega_i t^q} \ ,$$

*where $u(t)$ is $q$-sub-exponential. (The same conventions as in Proposition 1 apply.)*

*Proof.* First we consider small $t$. The first part of Proposition 1 immediately imply:

$$\bar{F}_{n-k+1:n}(t) = 1 - \sum_{l=0}^{k-1} P_{A,l}(t) = 1 - \sum_{l=0}^{k-1} \left[ t^{(n-l)p} \left( \sum_{I \in \mathcal{I}_l} \prod_{i \in I^{-1}(0)} \alpha_i \right) (1 + \rho_l(t)) \right] .$$

Notice that

$$\sum_{l=0}^{k-1} t^{(n-l)p} \left( \sum_{I \in \mathcal{I}_l} \prod_{i \in I^{-1}(0)} \alpha_i \right) = t^{(n-k+1)p} \left( \sum_{I \in \mathcal{I}_{k-1}} \prod_{i \in I^{-1}(0)} \alpha_i \right) (1 + \xi(t))$$

where the limit of $\xi$ tends to zero as $t \to 0$. Substituting this expression into that of $\bar{F}_{n-k+1:n}(t)$ and and collecting all these small terms into a single term $\sigma$ proves the first part.

For large $t$, we use the second part of Proposition 1. We obtain:

$$\bar{F}_{n-k+1:n}(t) = \sum_{l=k}^{n} P_{A,l}(t) = \sum_{l=k}^{n} u_l(t) \, e^{-\sum_{i=1}^{l} \omega_i t^q} \ .$$

The leading term is now the one with $l = k$. Then the second statement follows. $\square$

**Theorem 2.** *Under the same assumptions as those in Theorem 1, then there is an $\epsilon > 0$ so that* $\text{sign}\left(\bar{F}_{n-k+1:n}(t) - \bar{G}_{n-k+1:n}(t)\right) =$

$$\begin{cases} \text{sign}\left( \sum_{I \in \mathcal{I}_{k-1}} \left( - \prod_{i \in I^{-1}(0)} \alpha_i + \prod_{i \in I^{-1}(0)} \tilde{\alpha}_i \right) \right) \ , & \forall\, t < \epsilon \\ \text{sign}\left( \sum_{i=1}^{k} (-\omega_i + \tilde{\omega}_i) \right) \ , & \forall t > 1/\epsilon \end{cases}$$

*assuming that both of the right hand side expressions are non-zero.*

*Proof.* This follows immediately from Proposition 2. □

Before we continue we need to define three indices:

$$I_0 \equiv \operatorname{sign}\left(\sum_{I\in\mathcal{I}_{k-1}}\left(-\prod_{i\in I^{-1}(0)}\alpha_i+\prod_{i\in I^{-1}(0)}\tilde{\alpha}_i\right)\right), \tag{4}$$

$$I_\infty \equiv \operatorname{sign}\left(\sum_{i=1}^{k}(-\omega_i+\tilde{\omega}_i)\right), \tag{5}$$

$$I_Q \equiv I_0 \cdot I_\infty . \tag{6}$$

With the aid of the above Theorem, we establish the following result.

**Corollary 1.**

$$\begin{aligned}
I_0 = -1 \quad &\Rightarrow \quad X_{n-k+1:n} \not\geq_{st} Y_{n-k+1:n}\,.\\
I_\infty = -1 \quad &\Rightarrow \quad X_{n-k+1:n} \not\geq_{st} Y_{n-k+1:n}\,.\\
I_Q = -1 \quad &\Rightarrow \quad X_{n-k+1:n} \not\leq_{st} Y_{n-k+1:n} \quad \textit{and} \quad X_{n-k+1:n} \not\geq_{st} Y_{n-k+1:n}\,.
\end{aligned}$$

*Proof.* The proof is straightforward. From Theorem 2, it is easily seen that the sign of $I_0$ (resp. $I_\infty$) gives the sign of $\bar{F}_{n-k+1:n}(t) - \bar{G}_{n-k+1:n}(t)$ as $t$ tends to zero (resp., tends to $\infty$). In particular if $I_Q = -1$ that difference must change sign, i.e., $X_{n-k+1:n}$ and $Y_{n-k+1:n}$ are not ordered according to the usual stochastic order. If any of the signs equal zero there is no conclusion. □

The contrapositive of the above Corollary gives a necessary condition for a stochastic ordering, namely: if $X_{n-k+1:n} \geq^{st} Y_{n-k+1:n}$ then $I_Q = 1$. The same holds for the other statements.

# 4 Examples

In order to illustrate the performance of our main results established in Section 3, we present here some interesting examples.

## 4.1 Generalized Gamma Distributions

A random variable $X$ is said to have a generalized gamma distribution, denoted by $X \sim GG(a,p,q)$, if it admits the following survival function:

$$\bar{F}_{a,p,q}(t) = \frac{a^{p/q}\,p}{\Gamma(p/q)}\int_t^\infty s^{p-1}\,e^{-as^q}\,ds\,,$$

where the parameters $p$, $q$, and $a$ are henceforth understood to be positive. This distribution includes, as special cases, exponential ($p = q = 1$), Weibull ($p = q$)

and gamma ($q = 1$) distributions. We have the following asymptotic estimates:

$$\int_0^t s^{q-1}\, e^{-s}\, ds = \frac{t^q}{q}\, (1+\epsilon_1(t))\ , \qquad \text{where } \lim_{t\to 0}\ \epsilon_1(t) = 0\ ,$$

$$\int_t^\infty s^{q-1}\, e^{-s}\, ds = t^{q-1}\, e^{-t}\, (1+\epsilon_2(t))\ , \quad \text{where } \lim_{t\to \infty}\ \epsilon_2(t) = 0\ .$$

From this one can easily deduce the following observation (cited without proof):

$$\bar{F}_{a,p,q}(t) = \begin{cases} 1 - \dfrac{a^{p/q}\ t^p}{\Gamma(p/q+1)}\ (1+\phi(t)), & \text{where } \lim_{t\to 0}\ \phi(t) = 0\ , \\ \dfrac{a^{p/q-1}\ t^{p-q}\ e^{-at^q}}{\Gamma(p/q)}\ . & \end{cases}$$

Then, the generalized gamma distributions satisfy the Definition 5 with

$$\begin{cases} \alpha = & \dfrac{a^{p/q}}{\Gamma(p/q+1)}\ , \\ \omega = & a\ , \\ z(t) = & \dfrac{a^{p/q-1}\ t^{p-q}}{\Gamma(p/q)}\ . \end{cases} \tag{7}$$

Suppose that the random variables $X_1 \cdots, X_n$ all have generalized gamma distributions with parameters $a_i$, $p_i$, and $q_i$. Then Theorem 2 says that if all $p_i$'s are equal and all $q_i$'s are equal (namely to $p$ and $q$ resp.) then the asymptotic behavior of $\bar{F}_{a,p,q}(t)$ can be calculated from the $a_i$. In particular (see also Corollary 1) if we have another set of random variables $Y_1 \cdots, Y_n$ also with generalized gamma distributions $\bar{F}_{b,p,q}(t)$, then the asymptotic behavior of those systems can be compared. Those results give conditions on the $a_i$ and $b_i$ for which those system are not stochastically ordered. Specifically, if $k = 1$, it is easy to check, from (4), (5) and (7), that

$$I_0 \equiv \text{sign}\left(-\prod_{i=1}^n a_i + \prod_{i=1}^n b_i\right) \quad \text{and} \quad I_\infty \equiv \text{sign}\,(-a_1+b_1)\,.$$

Therefore, if $a_1 > b_1$ and $\prod_{i=1}^n a_i < \prod_{i=1}^n b_i$ or $a_1 < b_1$ and $\prod_{i=1}^n a_i > \prod_{i=1}^n b_i$, then from Corollary 1 we have that $X_{n:n} \not\leq_{st} Y_{n:n}$ and $X_{n:n} \not\geq_{st} Y_{n:n}$ . When $k = n$, we get

$$I_0 \equiv \text{sign}\left(-\sum_{i=1}^n a_i^{p/q} + \sum_{i=1}^n b_i^{p/q}\right) \quad \text{and} \quad I_\infty \equiv \text{sign}\left(-\sum_{i=1}^n a_i + \sum_{i=1}^n b_i\right).$$

It is immediate that, if $\boldsymbol{a} = (1, 1, 5)$, $\boldsymbol{b} = (1, 2, 3)$, $p = 2$ and $q = 10$, then $I_0 = -1$ and $I_\infty = +1$. Hence $X_{n:n}$ and $Y_{n:n}$ are not ordered according with the usual stochastic ordering.

### 4.2 PRV and PHR models

As we pointed out in the introduction, the exponential distribution is a special case of the PRV and the PHR models. Here, we give their definitions formally.

**Definition 8.** *Let $\bar{F}$ be a survival function of some non-negative random variable $X$. Then the independent random variables $X_1 \dots, X_n$ follow the proportional random variables (PRV) model if there exists $\lambda_1 > 0, \dots, \lambda_n > 0$ such that,*

$$\bar{F}_k(t) = \bar{F}(\lambda_k t),$$

*for $k = 1, \dots, n$.*

**Definition 9.** *Let $\bar{F}$ be a survival function of some non-negative random variable $X$. Then, the independent random variables $X_1 \dots, X_n$ follow the proportional hazard rates (PHR) model (or scale model) if there exists $\lambda_1 > 0, \dots, \lambda_n > 0$ such that,*

$$\bar{F}_k(t) = \left(\bar{F}(t)\right)^{\lambda_k},$$

*for $k = 1, \dots, n$.*

The following lemma is simple and hence the proof is omitted.

**Lemma 1.** *Let $X_1, \dots, X_n$ be a sequence of independent random variables that follow the PRV or the PHR model with a base-line distribution $F(t)$. If $\bar{F} \in \mathcal{F}_{pq}$, then $\bar{F}_k \in \mathcal{F}_{pq}$.*

Let $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ be sets of random variables whose distribution functions are given by $F(\lambda_i t)$ and $F(\theta_i t)$ where $F$ is an arbitrary members of $\mathcal{F}_{pq}$. From Definition 5, we have that $\alpha_i = \alpha\lambda_i^p$ ($\tilde{\alpha}_i = \alpha\theta_i^p$ ) and $w_i = w\lambda_i^q$ ($\tilde{w}_i = w\theta_i^q$), then, if $k = 1$, we get

$$I_0 \equiv \operatorname{sign}\left(-\prod_{i=1}^{n}\lambda_i + \prod_{i=1}^{n}\theta_i\right), \quad \text{and} \quad I_\infty \equiv \operatorname{sign}\left(-\lambda_1 + \theta_1\right).$$

Therefore, if $\theta_1 > \lambda_1$ and $\prod_{i=1}^n \theta_i < \prod_{i=1}^n \lambda_i$ or $\theta_1 < \lambda_1$ and $\prod_{i=1}^n \theta_i > \prod_{i=1}^n \lambda_i$, then from Corollary 1 we have that $X_{n:n}$ and $Y_{n:n}$ are not ordered according to the usual stochastic ordering. Note that, in this case, for the PHR model the indices $I_0$ and $I_\infty$ are the same.

When $X_1 \dots, X_n$ follow a PHR model with $\bar{F}_k(t) = \left(\bar{F}(t)\right)^{\lambda_k}$ and $Y_1 \dots, Y_n$ follow another PHR model with $\bar{G}_k(t) = \left(\bar{G}(t)\right)^{\theta_k}$, Navarro [12] proved that, if $\lambda_{(1)} + \dots + \lambda_{(k)} < \theta_{(1)} + \dots + \theta_{(k)}$ and $\limsup_{t\to\infty} h_F(t)/h_G(t) \le 1$, then $X_{n-k+1:n} \ge_{a-hr} Y_{n-k+1:n}$. Recall that a univariate random variable $X$ is said to be less or equal than another univariate random variables $Y$ asymptotically in the hazard rate order (denoted by $X \le_{a-hr} Y$) if there exists $a > 0$ such that the corresponding hazard rate functions satisfy $h_X(t) \ge h_Y(t)$ for (almost) all $t > a$. Clearly, $X \le_{a-hr} Y$ implies $\bar{F}(t) \le \bar{G}(t)$ when $t \to \infty$. From Lemma 1 and Theorem 2, we get that $\bar{F}_{n-k+1:n}(t) \ge \bar{G}_{n-k+1:n}(t)$ when $t$ tends to $\infty$ if $w\sum_{i=1}^k \lambda_{(i)} < \tilde{w}\sum_{i=1}^k \theta_{(i)}$. Note that our conditions are weaker than those in Navarro [12].

# 5 The Exponential Distribution

The exponential distribution is a special case of each of the examples given in the last section. If $X_1, \ldots, X_n$ $(Y_1, \ldots, Y_n)$ are heterogeneous exponential random variables with hazard rates $(\lambda_1, \ldots, \lambda_n)$ and $(\theta_1, \ldots, \theta_n)$ respectively, then $\alpha_i = \omega_i = \lambda_i$ $(\tilde{\alpha}_i = \tilde{\omega}_i = \theta_i)$, for $i = 1, \ldots, n$. The indices $I_0$ and $I_\infty$ become

$$I_0 \equiv \text{sign} \sum_{I \in \mathcal{I}_{k-1}} \left( - \prod_{i \in I^{-1}(0)} \lambda_i + \prod_{i \in I^{-1}(0)} \theta_i \right), \tag{8}$$

$$I_\infty \equiv \text{sign} \sum_{i=1}^{k} (-\lambda_i + \theta_i). \tag{9}$$

## 5.1 Results

Theorem 2 yields an easy condition ($I_0 \cdot I_\infty = -1$) that guarantees that the order statistics $X_{n-k+1:n}$ and $Y_{n-k+1:n}$ are not stochastically ordered. In the case in which one of the samples is independent and identically distributed we have:

$$I_0 \equiv \text{sign} \left( - \sum_{I \in \mathcal{I}_{k-1}} \prod_{i \in I^{-1}(0)} \lambda_i + \binom{n}{n-k+1} \theta^{n-k+1} \right), \tag{10}$$

$$I_\infty \equiv \text{sign} \left( - \sum_{i=1}^{k} \lambda_i + k\theta \right). \tag{11}$$

Recall that Theorem 2 only implies that if $I_0 = 1$ then $\overline{G}_{n-k+1:n}(t) \leq \overline{F}_{n-k+1:n}(t)$ holds for small enough time $t$. However we have the following:

**Theorem 3** (Bon and Păltănea [7])**.** *Let $X_1, \ldots, X_n$ be a sequence of independent exponential random variables with respective hazard rates $\lambda_1, \ldots, \lambda_n$, and let $Y_1, \ldots, Y_n$ another sequence of independent exponential random variables with the common parameter $\theta > 0$. Then, for any $k$,*

$$I_0 = 1 \quad \Leftrightarrow \quad Y_{n-k+1:n} \leq_{st} X_{n-k+1:n} \ ,$$

**Remark 1.** *As Bon and Păltănea [7] observed, the sign of $I_0$ equals the sign of $\theta - m^n_{n-k+1}(\{\lambda_i\}^n_{i=1}$, where*

$$m^{(n)}_{n-k+1}(\{\lambda_i\}^n_{i=1}) \equiv \left( \binom{n}{n-k+1}^{-1} \sum_{I \in \mathcal{I}_{k-1}} \prod_{i \in I^{-1}(0)} \lambda_i \right)^{\frac{1}{n-k+1}},$$

*also known as the $n-k+1^{st}$ symmetric mean of the $\lambda_i$. These quantities were studied by McLaurin and satisfy the inequality that $m^{(n)}_{i+1} \leq m^{(n)}_i$. Note* *that $m_1^{(n)}$ is the usual arithmetic average (see [10]). As a further curiosity we observe that Theorem 2 and Theorem 3 together imply that if $I_0 = 1$ then $I_\infty = 1$. This implies that for all (ordered sets of $n$) positive reals $\lambda_i$:*

$$m_{n-k+1}^{(n)}(\{\lambda_i\}_{i=1}^n) \geq m_1^{(n)}(\{\lambda_i\}_{i=1}^k)\,.$$

*The last quantity is the arithmetic mean of the first $k$ of the $\lambda_i$'s.*

The next result improves on a result by Navarro and Lai (see Figure 1, [11])

**Proposition 3.** *Let $X_1$ and $X_2$ be two independent exponential random variables with respective hazard rates $\lambda_1$ and $\lambda_2$, and let $Y_1$ and $Y_2$ be another two independent exponential random variables with respective hazard rates $\theta_1$ and $\theta_n$. Then (see Figure 1),*
*i) $(\lambda_1, \lambda_2) \leq^{\mathrm{p}} (\theta_1, \theta_2) \Rightarrow Y_{2:2} \leq_{st} X_{2:2}$*
*ii) $(\theta_1, \theta_2) \leq^{\mathrm{p}} (\lambda_1, \lambda_2) \Rightarrow X_{2:2} \leq_{st} Y_{2:2}$*
*iii) Neither $(\lambda_1, \lambda_2) \leq^{\mathrm{p}} (\theta_1, \theta_2)$ nor $(\theta_1, \theta_2) \leq^{\mathrm{p}} (\lambda_1, \lambda_2) \Rightarrow Y_{2:2}$ and $X_{2:2}$ are not stochastically ordered.*

*Proof.* Statements $(i)$ and $(ii)$ follow from Khaledi and Kochar [2]. If neither $(i)$ nor $(ii)$ holds, then $\theta_1 > \lambda_1$ and $\theta_1\theta_2 < \lambda_1\lambda_2$, or $\theta_1 < \lambda_1$ and $\theta_1\theta_2 > \lambda_1\lambda_2$. In both cases, $I_Q = I_0 \cdot I_\infty$ equals $-1$, and so by Corollary 1 $X_{2:2}$ and $Y_{2:2}$ are not stochastically ordered. Hence, the required result follows. □

**Remark**: In the case that $\theta_1 = \theta_2$, using Theorem 3 implies statement $(i)$.

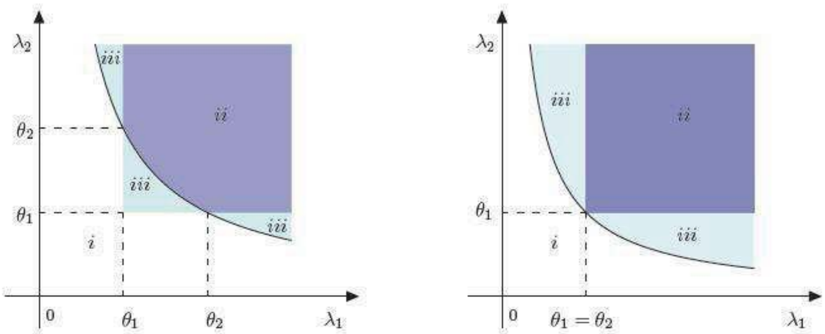


Figure 1: (color online)*Stochastic ordering according to Proposition 3. In the first figure $\theta_1 < \theta_2$ and in the second $\theta_1 = \theta_2$. The regions are labeled as in the statement of Proposition 3*

## 5.2 Counterexamples

Proposition 3 says that if $k = 1$ and $n = 2$ then $I_Q = 1$ implies that $X_{n-k+1:n}$ and $Y_{n-k+1:n}$ are stochastically ordered. Equation 2 says that if $k = 1$ and $n > 1$

then $\lambda \leq^{\mathrm{p}} \theta$ implies that $X_{n:n} \geq_{st} Y_{n:n}$. Thus the question arises (especially in view of Theorem 3) whether partial converses are also true. In particular, for $k = 1$ and $n > 1$ is it true that

i: If $X_{n:n}$ and $Y_{n:n}$ do not admit a stochastic ordering, is $I_Q = -1$?

ii: Does $X_{n:n} \geq_{st} Y_{n:n}$ imply that $\lambda \leq^{\mathrm{p}} \theta$?

Both questions can be answered in the negative as the following 1-out-of-3 examples clearly show. Let

$$\begin{aligned} \lambda_1 &\in \{0.73, 0.732, 0.74, 0.80, 0, 90\} & & \theta_1 = 1 \\ \lambda_2 &= 2 & \text{and} \quad & \theta_2 = 1 \\ \lambda_3 &= 2 & & \theta_3 = 4 \end{aligned}$$

One easily verifies that for each of the 5 values of $\lambda$ and for $\theta$ we have that $\lambda \not\leq^{\mathrm{p}} \theta$ and $I_Q = 1$. Nevertheless as Figure 2 indicates *both* $X_{n:n} \geq_{st} Y_{n:n}$ and $X_{3:3}$ *and* $Y_{3:3}$ *do not admit a stochastic ordering* occur.

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## References

[1] P. Pledger and F. Proschan (1971), *Comparisons of order statistics and of spacings from heterogeneous distributions*, In Optimizing Methods in Statistics, ed. Rustagi, J. S., New York: Academic Press, pp. 89–113.

[2] B. Khaledi and S.C. Kochar (2000), *Some new results on stochastic comparisons of parallel systems*, Journal of Applied Probability, 37, 1123-1128.

[3] B. Khaledi and S.C. Kochar (2002), *Dispersive ordering among linear combinations of uniform random variables*, J. Statist. Plann. Inference 100, 13-21.

[4] B. Khaledi and S.C. Kochar (2006), *Weibull distribution: some stochastic comparisons results*, J. Statist. Plann. Inference 136, 3121-3129.

[5] B. Khaledi, S. Farsinezhad and S.C. Kochar (2011), *Stochastic comparisons of order statistics in the scale model*, J. Statist. Plann. Inference 141, 276-286.

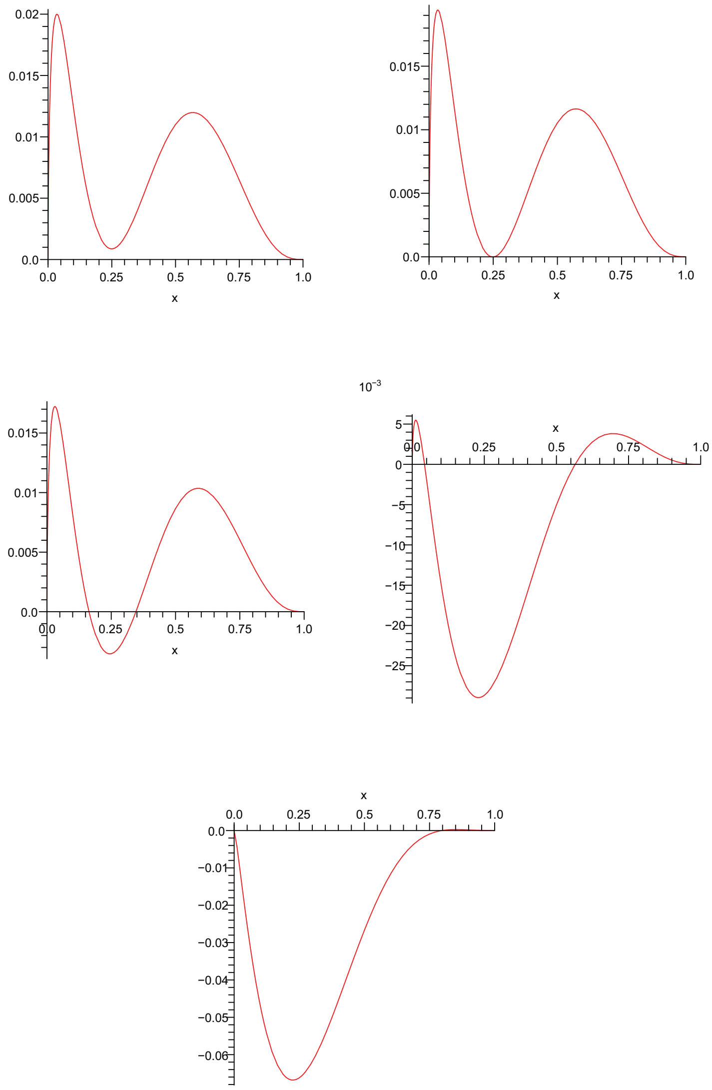


Figure 2: *The values of the parameters are as given above. The parameter* $\lambda_1$ *increases (see text) in value from upper-left to lower right. The horizontal axis is* $x = e^{-t}$. *We have drawn the difference of the survival distributions when* $k = 1$, *that is,* $\bar{F}_{3:3}(t) - \bar{G}_{3:3}(t)$.

[6] J.-L. Bon and E. Păltănea (1999), *Ordering properties of convolutions of exponential random variables*, Lifetime Data Analysis 5, 185–192.

[7] J.-L. Bon and E. Păltănea (2006), *Comparisons of order statistics in a random sequence to the same statistics with iid variables*, ESAIM: Probability and Statistics 10, 1-10.

[8] A.W. Marshall and I. Olkin, Inequalities : Theory of Majorization and Its Applications, Academic Press, New York, 1979.

[9] M. Shaked, J.G. Shanthikumar, Stochastic Orders and their applications, Springer, New York, 2007.

[10] G. H, Hardy, J. E. Littlewood, G. Polya, *Inequalities*, Cambridge University Press, 1934.

[11] J. Navarro, C-D. Lai, (2007), *Ordering Properties of Systems with Two Dependent Components*, J. Communications in Statistics - Theory and Methods, 36, 645-655.

[12] J. Navarro, (2007), *Tail hazard rate ordering properties of order statistics and coherent systems*, Naval Research Logistics, 54, 820-828.