Two-Person Zero Sum Games

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We take our inspiration from the book *Game Theory and Strategy* by P. D. Straffin, MAA, 1993 (5th printing in 2004).

The following is a simplified example taken from a real life sociological investigation (chapter 4 in the book). In a Jamaican fishing village (in the 1950's) a quandary faces all fishermen. The fishing grounds are divided into inside and outside banks. The latter are further away and very susceptible to strong underwater currents. The occurrence of those currents is unpredictable, but when they are present they make fishing very hard. The inside bank is protected from the currents, but the fish is of lower quality.

The investigators formulated a pay-off matrix. The matrix considers three possible strategies for the fishermen:
- To fish inside
- To fish outside
- Combine the two by doing both on each trip (so that you will spend less time on each location),

and two strategies for the current: it can
- run
- or it can not run.

The matrix lists the pay-off to the fishermen (in UK pounds per month). Here is the matrix:

$$\begin{pmatrix} & \textit{run} & \textit{not run} \\ \textit{inside} & 17.3 & 11.5 \\ \textit{outside} & -4.4 & 20.6 \\ \textit{combine} & 5.2 & 17.0 \end{pmatrix}$$

Still according to the book, the original researcher did the following (wrong) analysis. He assumed that the a) what the fishers earned, the current somehow paid, and b) that the current could think to choose a strategy. Under those hypotheses the analysis becomes the one you are asked to do in Problem 2.

1) We first look at a different problem. We change one entry in the matrix, so that it becomes:

$$\begin{pmatrix} & \textit{run} & \textit{not run} \\ \textit{inside} & 17.3 & 18.0 \\ \textit{outside} & -4.4 & 20.6 \\ \textit{combine} & 5.2 & 17.0 \end{pmatrix}$$

Now reason that the fishermen (Fisher) will choose their strategy so that whatever the current (Current) does, they maximize the outcome of the column. So if Current runs, the fishermen will go inside. If it doesn't then Fisher will go outside. However Current would 'like' to choose a strategy that minimizes the pay-off to Fisher. What happens in this case if both follow a rational strategy? This outcome is called a *saddle-point*.
Assume that the rational strategy is followed by both. Show that if one of the two deviates from it, he will pay more.

2) a)Now we go back to the original pay-off matrix. Verify there is no saddle-point.
This means that if Fisher chooses *inside* then Current will want to move to *not run*. But if Current does *that*, then Fisher will want to choose *outside*. If Fisher does that, then ... and so on. The solution proposed by game theorists is a so-called *mixed strategy*. Both Fisher and Current will change their strategies randomly. The only choice they make

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is how often each strategy occurs.
b) Given, say, that Fisher follows the same strategy, Current decides to mix up his (its?) choices. A chance p that it runs and $1 - p$ that it doesn't. In the case of each strategy for Fisher, write the expressions for the expected pay-off for Fisher.
c) Draw these functions (for $p \in [0, 1]$) in one graph.
d) Argue that for given p, Fisher wants to choose the highest value of the pay-off. Therefore the stable strategy is to choose that value of p for which the *upper envelope**) in the figure made in 2c) is minimal. In other words: Find the value of $p \in [0, 1]$ for which the maximal pay-off is minimized. Give an estimate of the correct value and pay-off graphically and verify this answer by explicit computation.
e) (MORE DIFFICULT) The above tells us what the strategy is that Current should follow. In that case, what is the strategy Fisher should follow? Make sure that this strategy yields the same pay-off.

*) *An envelope of a collection of graphs in $\mathbb{R}^2$ in this context means a curve forming the boundary of the region "occupied" by the graphs.*

3) It took apparently about 10 years for the error to be pointed out in the literature. Currents don't choose strategies. In particular they don't react to fishermen's strategies. They are a force of nature. The only thing we know is how often (on average) the current runs. That turned out to be 25% of the time. Knowing this Fisher could optimize his strategy.

If Fisher always follows the inside strategy and assuming that Current is randomly distributed with a 25% chance on running and 75% on not running, what is the expected value for the pay-off for Fisher? How about the other strategies? What should Fisher do?

4) It turns out that the fisherman in the original Jamaican village actually DID follow the strategy outlined in problem 2. What was wrong? Did we have to do with a rational current after all? Or were the fishers irrational? Another few years later (after the correction mentioned in Problem 3) an answer was published.
a) The currents are extremely irregular. There may be years where it runs half the time or 50%. Calculate what the payoffs for Fisher would be if he maintains the strategy outlined in Problem 3?
b) Why in fact could it be a good idea for Fisher to follow the strategy of Problem 2e, given that strong fluctuation in the current may occur from year to year?