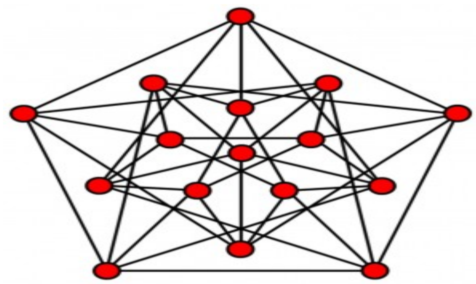
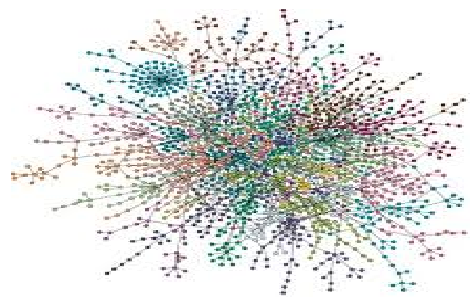
Como, Italy, October 2022

DIGRAPHS II
Diffusion and Consensus on Digraphs

Based on:

[1]: J. S. Caughman[1], J. J. P. Veerman[1],
***Kernels of Directed Graph Laplacians*,**
Electronic J. of Combinatorics, 13, No 1, 2006.
[2]: J. J. P. Veerman[1], E. Kummel[1],
***Diffusion and Consensus on Weakly Connected Directed Graphs*,**
Linear Alg. and Its Appl., 578, 184-206, 2019.
[3]: J. J. P. Veerman[1], R. Lyons[1],
***A Primer on Laplacian Dynamics in Directed Graphs*,**
Nonl. Phenom. in Compl. Syst., 23(2), 196-206, 2020.

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SUMMARY:

* This is a review of two basic dynamical processes on a weakly connected, directed graph G: consensus and diffusion, as well their discrete time analogues. We will omit proofs in this lecture. A self-contained exposition of this lecture with proofs included can be found in [1, 2].

* We consider them as dual processes defined on G by:
$\dot{x} = -\mathcal{L}x$ for consensus and $\dot{p} = -p\mathcal{L}$ for diffusion.

* We give a complete characterization of the asymptotic behavior of both diffusion and consensus — discrete and continuous — in terms of the null space of the Laplacian (defined below).

* Many of the ideas presented here can be found scattered in the literature, though mostly outside mainstream mathematics and not always with complete proofs.

OUTLINE:

The headings of this talk are color-coded as follows:

Definitions

Peculiarities of Directed Graphs

Consensus and Diffusion

Left and Right Kernels of $\mathcal{L}$

Asymptotics

Continuous and Discrete Processes

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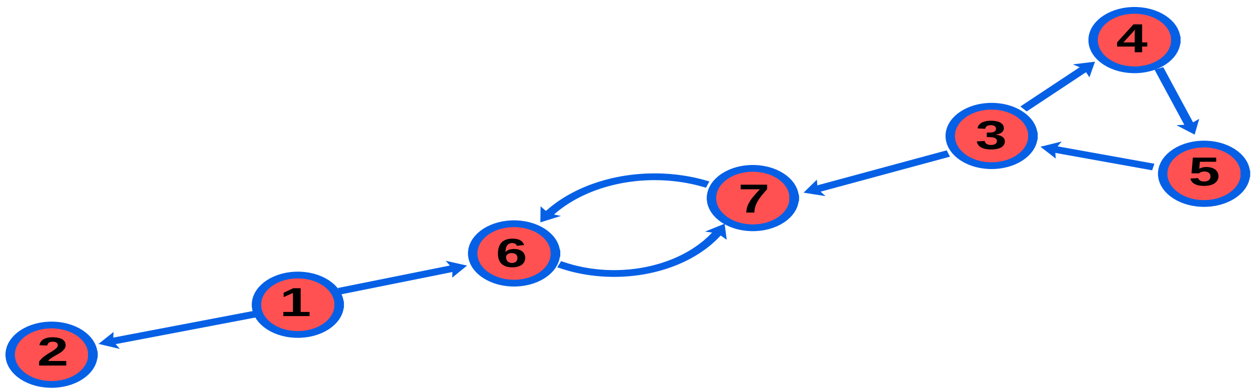
D E F I N I T I O N S

Definitions: Digraphs

Definition: A directed graph (or **digraph**) is a set $V = \{1, \cdots n\}$ of **vertices** together with set of ordered pairs $E \subseteq V \times V$ (the **edges**).



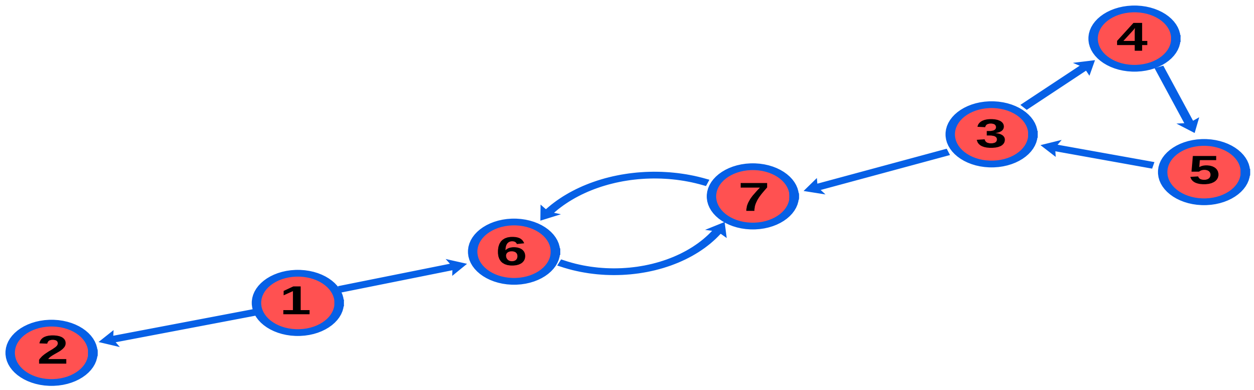
A directed edge $j \to i$, also written as ji.
A directed path from j to i is written as $j \rightsquigarrow i$.

Digraphs are everywhere: models of the internet [6], social networks [7], food webs [10], epidemics [9], chemical reaction networks [13], databases [5], communication networks [4], and networks of autonomous agents in control theory [8], to name but a few.

A BIG topic: Much of mathematics can be translated into graph theory (discretization, triangulation, etc). In addition, many topics in graph theory that do not translate back to *continuous* mathematics.

Definitions: Connectedness of digraphs

Undirected graphs are connected or not. But...

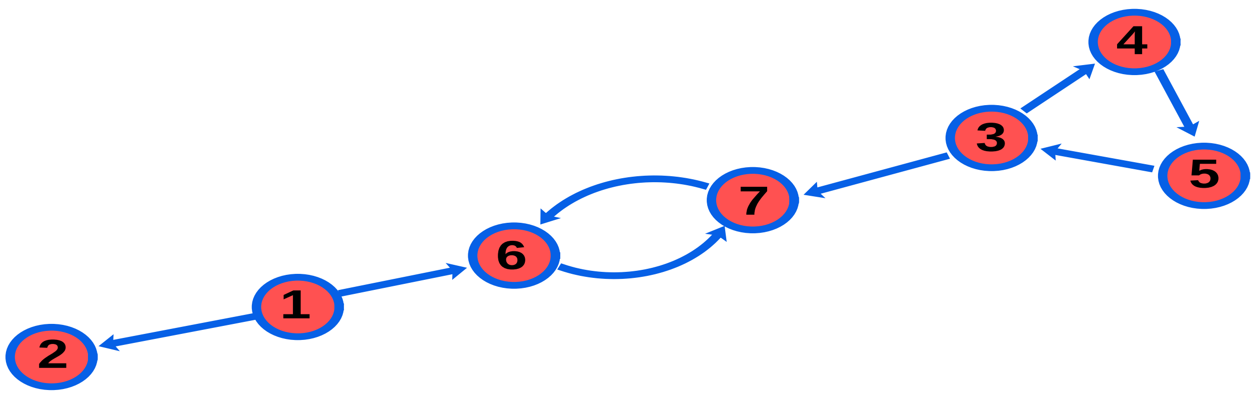


Definition:
* A directed edge from i to j is indicated as $i \to j$ or ij.
* A digraph G is **strongly connected** if for every ordered pair of vertices (i, j), there is a path $i \leadsto j$.
* A digraph G is **unilaterally connected** if for every ordered pair of vertices (i, j), there is a path $i \leadsto j$ or a path $j \leadsto i$.
* A digraph G is **weakly connected** if the **underlying UNdirected graph** is connected.
* A digraph G is **not connected:** if it is not weakly connected.

Definition: **Multilaterally connected**: **weakly connected** but not **unilaterally connected**.

Note: Maximal Strongly Connected Component: **SC** component, or **SCC**.

Definitions: Graph Structure

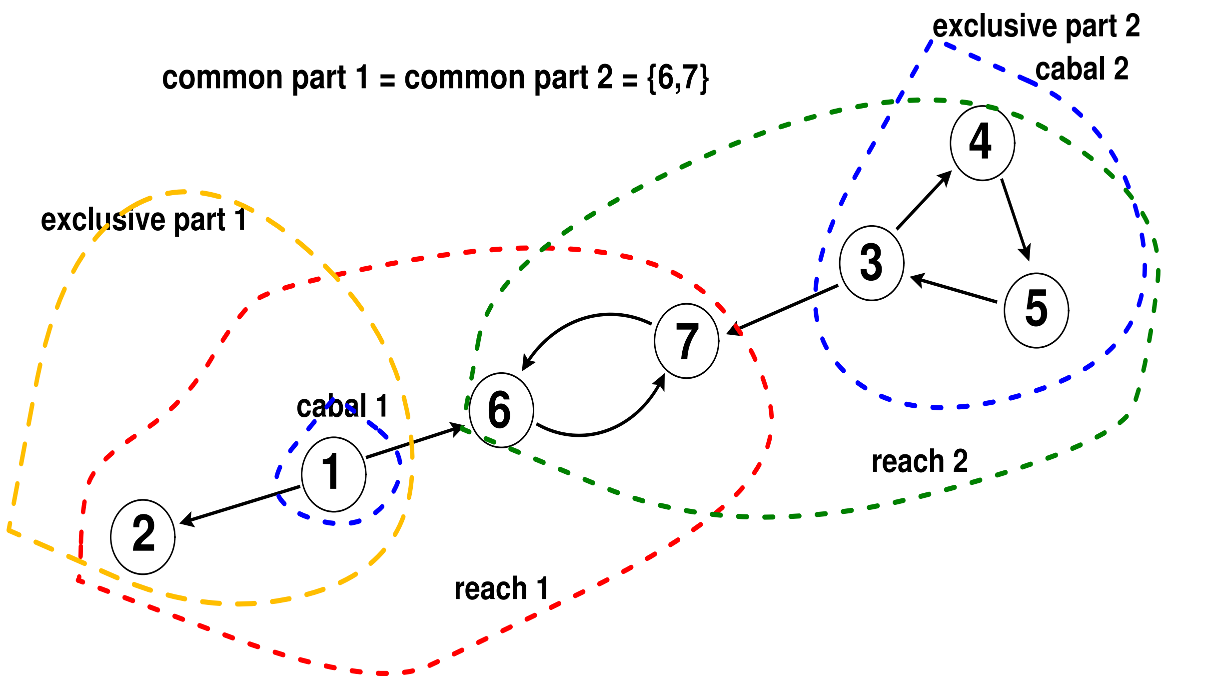


Definition: Only the blue definitions are used downstream.
* **Reachable Set** $R(i) \subseteq V$**:** $j \in R(i)$ if $i \rightsquigarrow j$.
* **Reach** $R \subseteq V$**:** A maximal reachable set. Or: a maximal unilaterally connected set.
* **Exclusive part** $H \subseteq R$**:** vertices in R that do not "see" vertices from other reaches. If not in cabal, called **minions**.
* **Common part** $C \subseteq R$**:** vertices in R that also "see" vertices from other reaches.
* **Cabal** $B \subseteq H$**:** set of vertices from which the entire reach R is reachable. If single, called **leader**.
* **Gaggle** $Z \subseteq R$**:** an SCC with no outgoing edges. If single, called **goose**.

Exercise: So gaggles and cabals are SCC's.

Exercise: If we reverse edge orientation, then gaggles turn into cabals, and so on. SCC's remain SCC's. Reaches are not preserved.

Definitions: Reaches



cabal = SCC w. no incoming edges

gaggle = SCC w. no outgoing edges **{2} and {6,7}**

{2} = goose = minion

{1} = leader

Fun exercise: Invert orientation and do the taxonomy again.

Surprising exercise: The number of reaches may change if orientation is reversed! (Thus the spectrum is not invariant.) Example: $o \longleftarrow o \longrightarrow o$

Definitions: Laplacian

Definition: The **combinatorial adjacency matrix** Q of the graph G is defined as:
$Q_{ij} = 1$ if there is an edge ji (if **"i sees j"**) and 0 otherwise.
If vertex i has no incoming edges, set $Q_{ii} = 1$ (create a loop).

Remark: Instead of creating a loop, sometimes all elements of the ith row are given the value $1/n$. This is called Teleporting! The matrix is denoted by Q_t. (n is no. vertices.)

Definition: The **in-degree matrix** D is a diagonal matrix whose ith diagonal entry equals the number of (directed, incoming) edges xi, $x \in V$.

Definition: Matrices $S \equiv D^{-1}Q$ and $S_t \equiv D^{-1}Q_t$ are the **normalized adjacency matrices**. By construction, they are **row-stochastic** (non-negative, every row adds to 1).

Definition: (In-degree) Laplacians describe **decentralized** or **relative** observation. Common cases:
The **combinatorial Laplacian**: $L \equiv D - Q$.
The **random walk (rw) Laplacian**: $\mathcal{L} \equiv I - D^{-1}Q$.
The **rw Laplacian with teleporting**: $\mathcal{L} \equiv I - D^{-1}Q_t$.

Definition: In general, a matrix is called Laplacian if
(a) row-sum zero, (b) diag elmts ≥ 0, and (c) non-diag elmts ≤ 0. **Equivalently:** L is Laplacian if $L = D - DS$, where D non-neg diag and S row-stoch [1].

Definitions: the "Usual" Laplacian

Crude discretization of 2nd deriv. of function $f : \mathbb{R} \to \mathbb{R}$:

$$f''(j) \approx (f(j+1) - f(j)) - (f(j) - f(j-1)) \text{ or}$$

$$f''(j) \approx f(j-1) - 2f(j) + f(j+1)$$

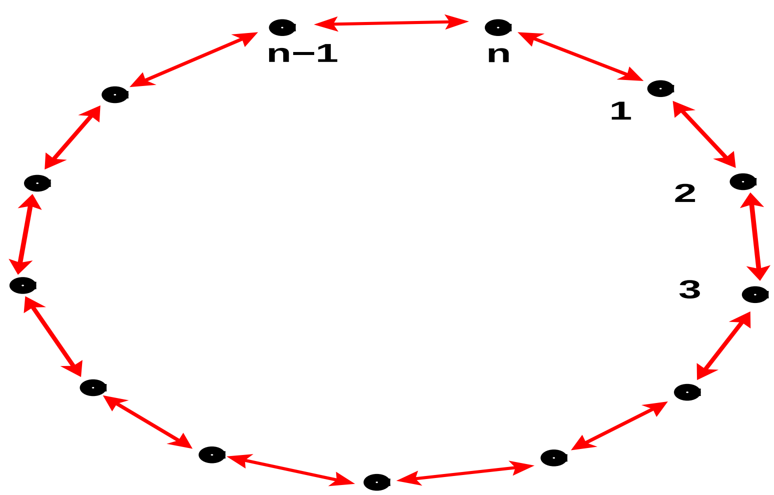
Suppose has period n (large). Get (combinatorial) Laplacian

$$L = \begin{pmatrix} -2 & 1 & 0 & \cdots & 1 \\ 1 & -2 & 1 & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & \cdots & 1 & -2 \end{pmatrix}$$

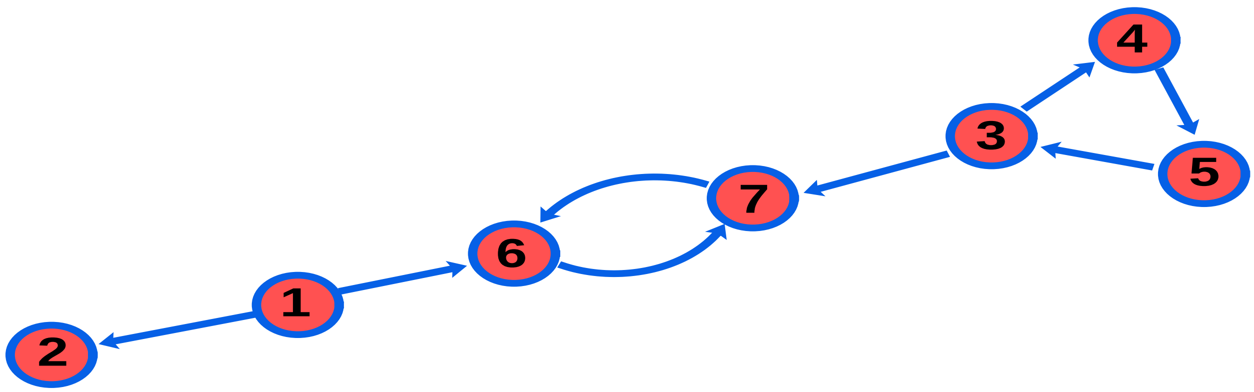
Graph theorists add a "-" to get eigenvalues ≥ 0.

Random walk Laplacian: Divide by 2 (and multiply by -1).

The corresponding graph G:



Definitions: rw Laplacian



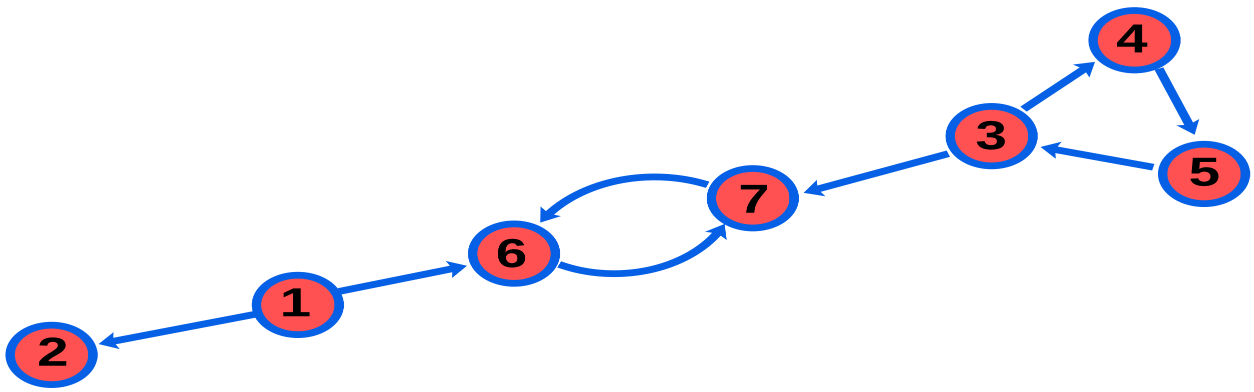
$$Q = \left(\begin{array}{c|c|ccc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right) \qquad D = \mathrm{diag}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}$$

So

$$\mathcal{L} \equiv I - D^{-1}Q = \left(\begin{array}{c|c|ccc|cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ \hline -1/2 & 0 & 0 & 0 & 0 & 1 & -1/2 \\ 0 & 0 & -1/2 & 0 & 0 & -1/2 & 1 \end{array}\right)$$

Spectrum: $\left\{0, 0, \frac{1}{2}, 1, \frac{3}{2}, \frac{3}{2} + i\,\frac{\sqrt{3}}{2}, \frac{3}{2} - i\,\frac{\sqrt{3}}{2}\right\}$.

Definitions: Combinatorial Laplacian



$$Q = \left(\begin{array}{c|c|ccc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right) \qquad D = \mathrm{diag}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}$$

So

$$L \equiv D - Q = \left(\begin{array}{c|c|ccc|cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ \hline -1 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 0 & 0 & -1 & 2 \end{array}\right)$$

Spectrum: $\left\{0, 0, 1, 1, 3, \frac{3}{2} + i\frac{\sqrt{3}}{2}, \frac{3}{2} - i\frac{\sqrt{3}}{2}\right\}$.

Definitions: Generalized Laplacians

$$\mathcal{L} \equiv I - D^{-1}Q = \left(\begin{array}{c|c|ccc|cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ \hline -1/2 & 0 & 0 & 0 & 0 & 1 & -1/2 \\ 0 & 0 & -1/2 & 0 & 0 & -1/2 & 1 \end{array}\right)$$

Definition: A generalized Laplacian is a Laplacian plus a non-negative diagonal matrix D^*. Common cases:

The **generalized combinatorial Laplacian**:
$L^* \equiv D^* + D - Q$.

The **generalized random walk (rw) Laplacian**:
$\mathcal{L}^* \equiv I - (D + D^*)^{-1}Q$.

The **generalized rw Laplacian with teleporting**:
$\mathcal{L}^* \equiv I - (D + D^*)^{-1}Q_t$.

Observation: The charpoly of the Laplacian of a weakly connected graph is the product of the charpolys of generalized Laplacians of its strongly connected components.

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P E C U L I A R I T I E S

O F

D I R E C T E D G R A P H S

Directed and Undirected

In the math community, directed graphs are still much less studied than undirected graphs (especially true for the algebraic aspects). As a consequence, very few good text books.

What are the reasons for this?

Directed graphs are **a lot messier** than undirected graphs:
- Combinatorial Laplacians of undirected graphs are **symmetric**. So: real eigenvalues, orthogonal basis of eigenvectors, no non-trivial Jordan blocks, etc.
- Connectedness of undirected graphs is much simpler.
- No standard convention on how to orient a digraph.

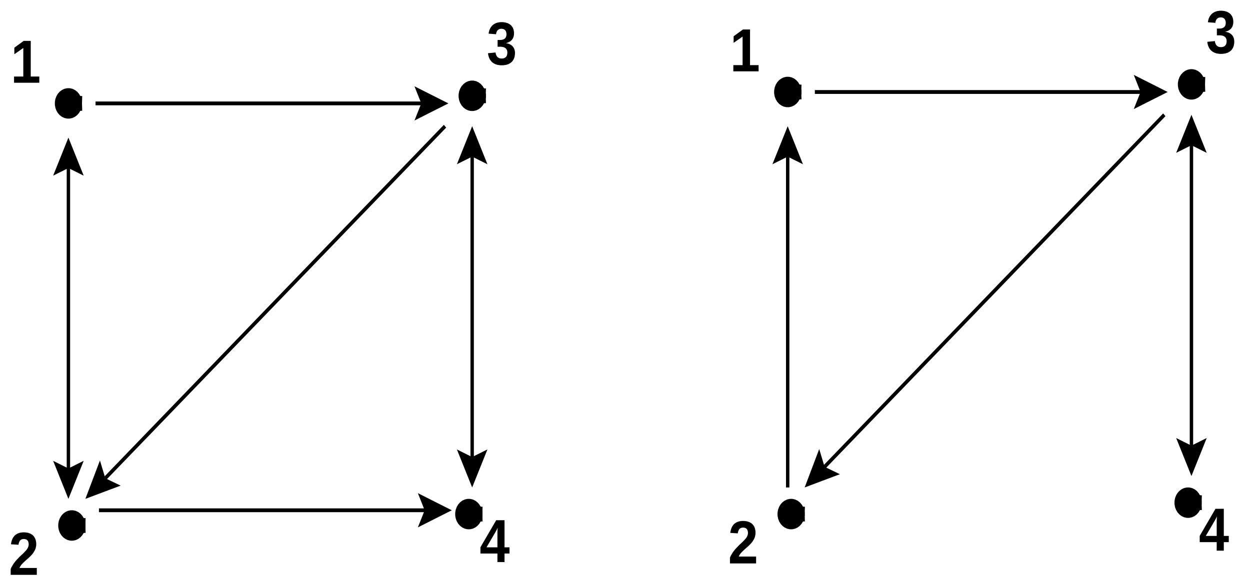
rw Laplacians of undirected graphs are **"almost symmetric"**, because they are conjugate to symmetric matrices.

Exercise: Show that $D^{-1}Q = D^{-\frac{1}{2}} \cdot D^{-\frac{1}{2}}QD^{-\frac{1}{2}} \cdot D^{\frac{1}{2}}$.

Proposition: G undirected. Then the eigenvectors of the rw Laplacian form a complete basis, and the eigenvalues are real.

(**Well-known result:** mathematicians like 'clean', not 'messy'.)

SC but Messy Comb Laplacians 1



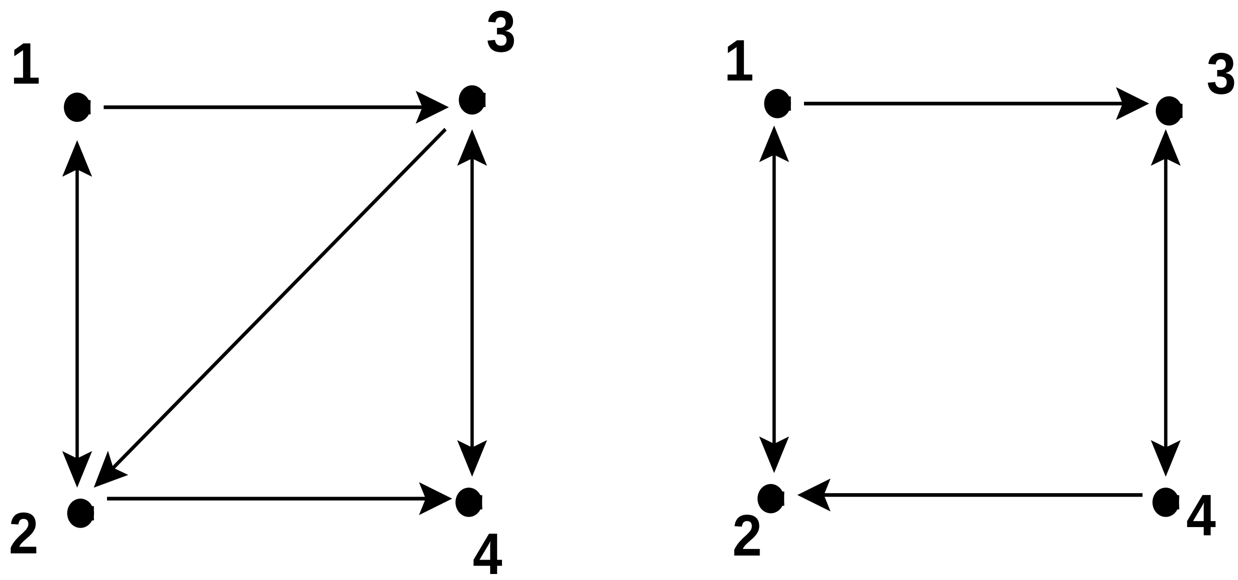
$$L_{\text{left}} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$$

with char. polynomial $x^4 - 7x^3 + 16x^2 - 11x$ and spectrum $\{0, 1.245, 2.877 \pm 0.745i\}$ (approximately).

$$L_{\text{right}} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

with char. polynomial $x^4 - 5x^3 + 8x^2 - 4x$ and spectrum $\{0, 1, 2^{(2)}\}$. The eigenvalue 2 has an associated 2-dimensional Jordan block.

SC but Messy RW Laplacians 2



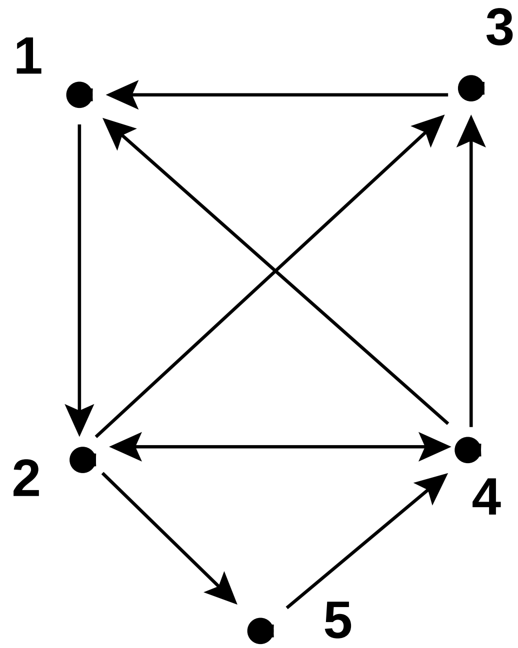
$$\mathcal{L}_{\text{left}} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1/2 & 1 & -1/2 & 0 \\ -1/2 & 0 & 1 & -1/2 \\ 0 & -1/2 & -1/2 & 1 \end{pmatrix}$$

with char. polynomial $\frac{x}{8}(8x^3 - 32x^2 + 42x - 17)x$ and spectrum $\{0, 1.616 \pm 0.396i, 0.77\}$ (approximately).

$$\mathcal{L}_{\text{right}} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1/2 & 1 & 0 & -1/2 \\ -1/2 & 0 & 1 & -1/2 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

with char. polynomial $x(x-2)(x-1)^2$ and spectrum $\{0, 1^{(2)}, 2\}$. The eigenvalue 1 has an associated 2-dimensional Jordan block.

SC but Messy Laplacians 3



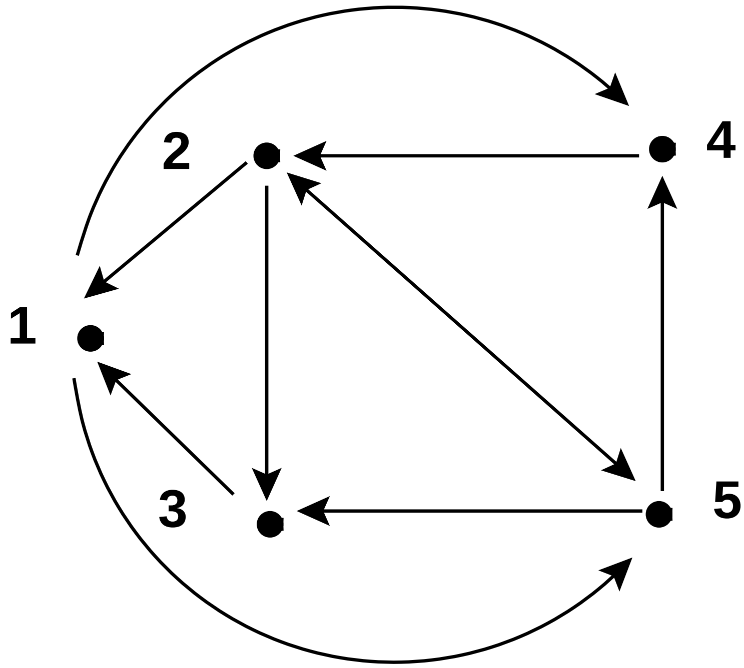
$$L = \begin{pmatrix} 2 & 0 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & -1 & 0 & 2 & -1 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix}$$

with char. polynomial $x(x^2 - 5x + 7)(x - 2)^2$ and spectrum $\{0, \frac{1}{2}(5 \pm i\sqrt{3}), 2^{(2)}\}$ (cmplx eval plus 2-d J block).

$$\mathcal{L} = \begin{pmatrix} 1 & 0 & -1/2 & -1/2 & 0 \\ -1/2 & 1 & 0 & -1/2 & 0 \\ 0 & -1/2 & 1 & -1/2 & 0 \\ 0 & -1/2 & 0 & 1 & -1/2 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix}$$

with char. polynomial $\frac{x}{16}(7 - 10x + 4x^2)^2$ and spectrum $\{0, \frac{1}{4}(5 \pm i\sqrt{3})^{(2)}\}$ (a 4-d complex Jordan block).

SC but Messy Laplacians 4



$$L = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 \\ 0 & 2 & 0 & -1 & -1 \\ 0 & -1 & 2 & 0 & -1 \\ -1 & 0 & 0 & 2 & -1 \\ -1 & -1 & 0 & 0 & 2 \end{pmatrix}$$

with char. polynomial $x(x^2 - 5x + 7)^2$ and spectrum $\{0, \frac{5}{2} \pm \frac{i}{2}\sqrt{3})^{(2)}\}$ (a 4-d complex Jordan block). Jordan evals are 2+cube roots of -1.

This is an example of minimal dimension (must have eval 0).

The associated graph is strongly connected.

The adjacency matrix is *primitive* ($\exists$ 3-cycle and 4-cycle).

The following also have a 4-d complex Jordan block

$$A = 2I - L \qquad \text{and} \qquad \mathcal{L} = \frac{1}{2}L$$

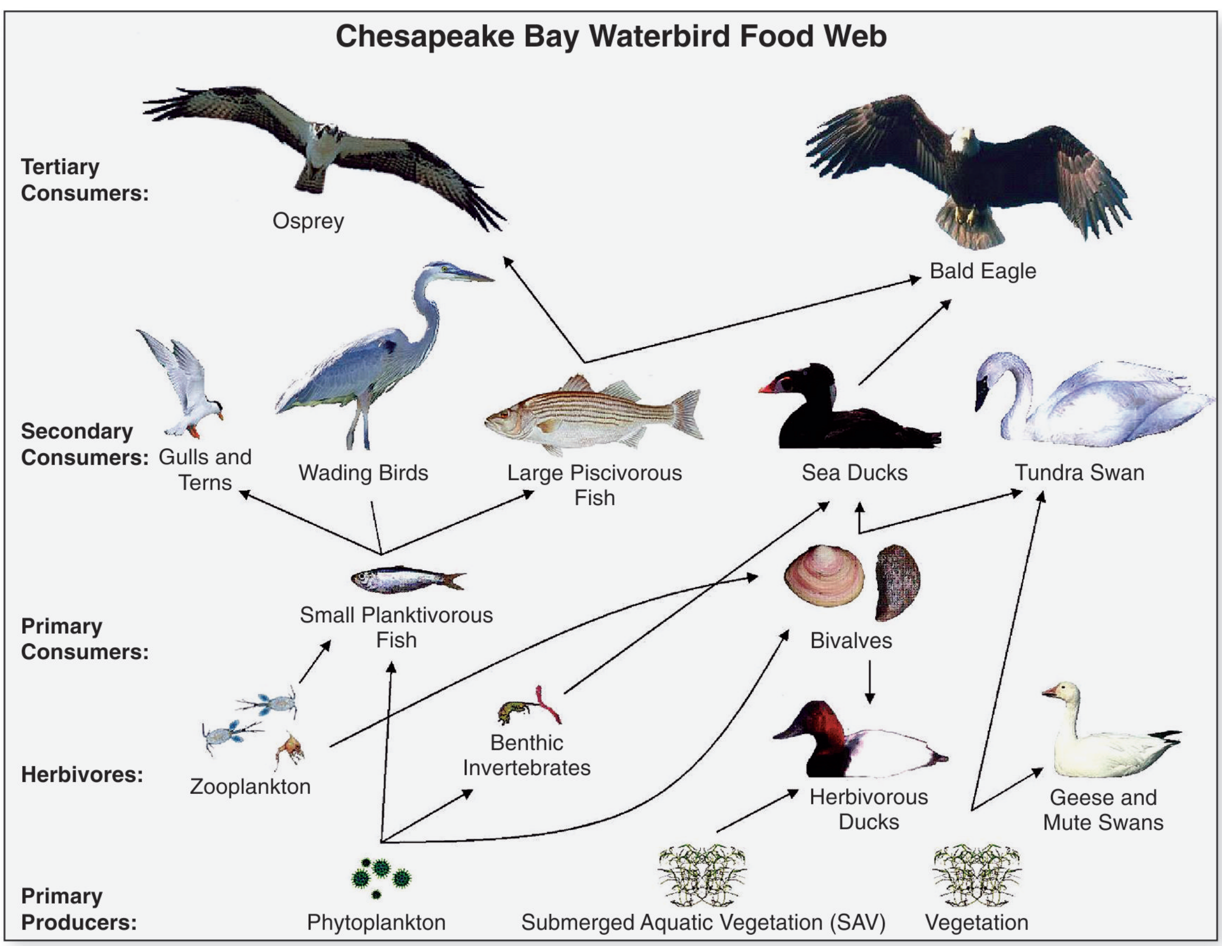
(I am indebted to Ewan Kummel for providing this example.)

Which Direction??

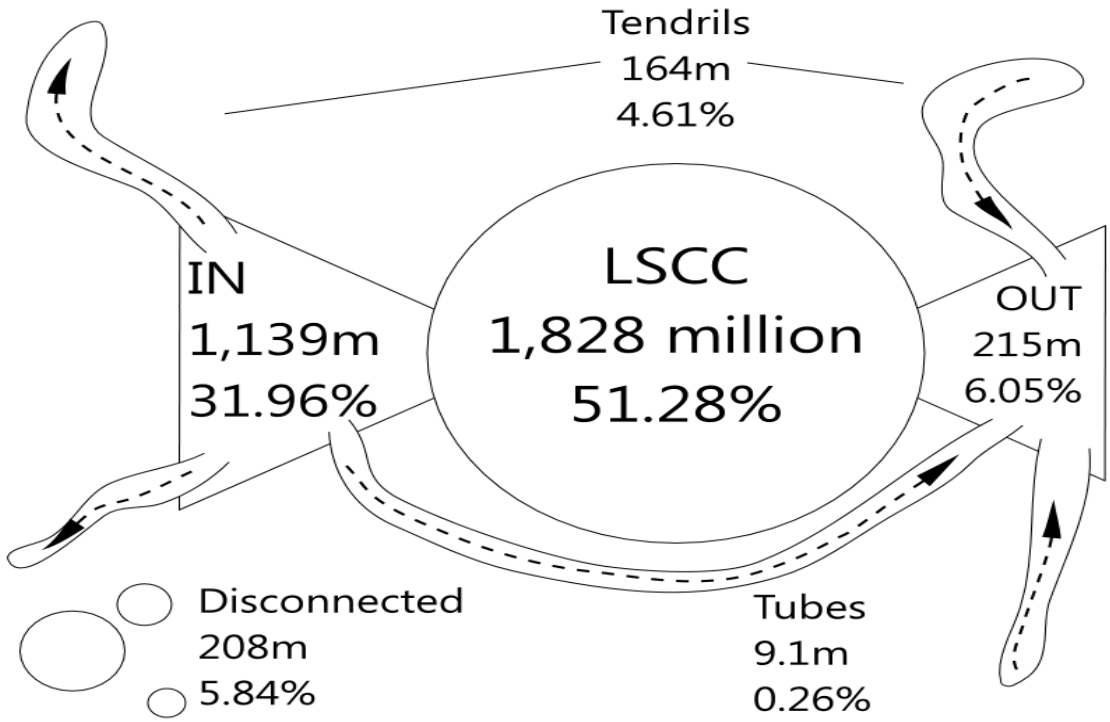
In this review, we are interested in information flow, as opposed to a physical flow (oil, traffic, for example). We propose a new convention:

The direction of the edges should be the same as the direction of the flow of the information.

In many cases, this makes sense. In a food web, the predator needs to locate the prey. Thus arrows go from prey to predator. See this food web. Taken from the US Geological Survey [12].



Bow-tie Structure of Web



(These arrows run *against* the information flow!)

- **LSCC or core**: Large strongly connected component.
- **IN component**: there is directed path *to* core.
- **OUT component**: directed path *from* core;
- **TENDRILS**: pages reachable from IN, or that can reach OUT.
- **TUBES**: paths from IN to OUT.
- **DISCONNECTED**: All other pages.

(Sources: [6] in 2000, and [11] in 2015.)

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DUAL PROCESSES: CONSENSUS AND DIFFUSION

Consensus and Diffusion

$\mathcal{L}$ has form $I - S$ or $I - S_t$ where S and S_t are row-stochastic. From now on x is a column vector and p is a row vector. Assume that edge $k \to i$ has weight $w > 0$.

Consensus: $\dot{x} = -\mathcal{L}x$. (Usual matrix multiplication.)
Properties: The all ones vector $\mathbf{1}$ is a solution.
Edge $k \to i$ contributes $w(x_k - x_i)$ to $\dot{x}_i$.
Exercise: Prove by writing out eqn for $\dot{x}_i$ in ex. pg 11.
Influence of opinion is felt **downstream**!

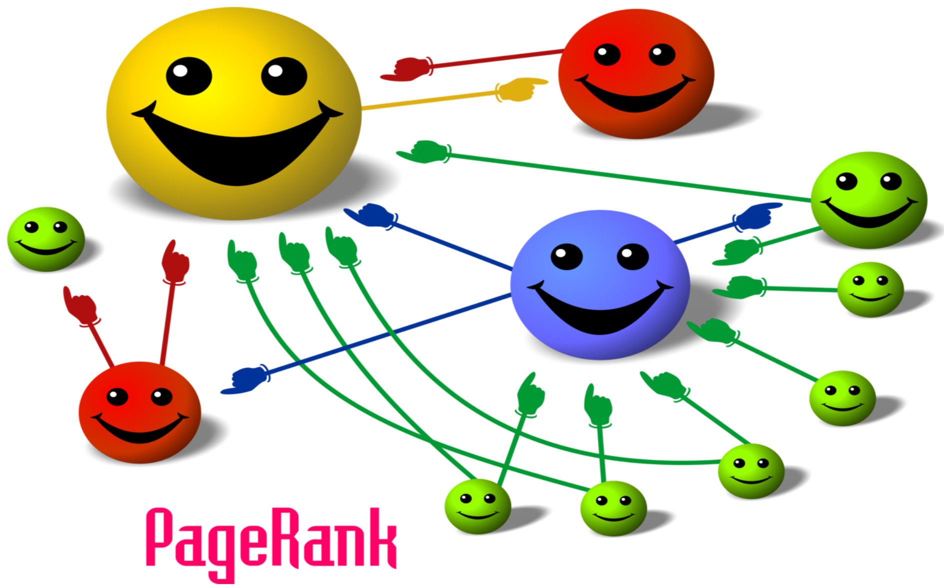
Diffusion: $\dot{p} = -p\mathcal{L}$. (Usual matrix multiplication.)
Properties: $\sum_i \dot{p}_i = 0$ (row-sum $\mathcal{L}$ is zero).
Edge $k \to i$ contributes $+wp_i$ to $\dot{p}_k$ and $-wp_i$ to $\dot{p}_i$.
Exercise: Prove by writing out the eqn for initial condn $p = p_i e_i^T$ in example[1] pg 11.
Diffusion moves **upstream** (against arrows)!

Remark: The physicist's definition of $\mathcal{L}$ would be the negative of the one we use here (cf. "Usual Laplacian"). Graph theorists like eigenvalues of symmetric Laplacians to be non-negative.

Theorem 1: The eigenvalues of S lie within the closed unit disk (Gersgorin). So the non-zero eigenvalues of $\mathcal{L} = I - S$ have positive real part.
Exercise: Prove this.

[1] $p_i e_i^T$ is the column vector whose only non-zero entry is the ith, which equals p_i.

Orientation of the Web



A web page can be **linked** to another one (see picture). This means that there is a reference to data in another page that you can land on by tapping or clicking.

The **pagerank** algorithm employs these links to make random walks following links. The stationary measure determines the expected frequency of visits to pages. The higher the frequency, the more "important" the pages.

Important Remark: The flow of information is **opposite to the direction of the links**. In other words, with our convention the orientation of the edges is reversed.

Important Remark: For rw, S_{ij} is the probability $i \to j$. For discrete consensus, S_{ij}, is the step $x(i)$ makes following a unit step of $x(j)$.

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LEFT AND RIGHT KERNELS OF LAPLACIANS

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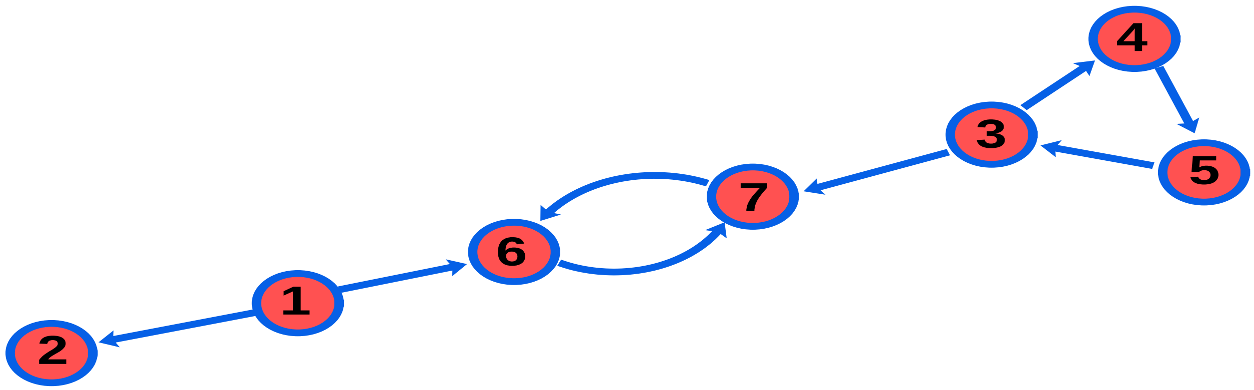
First: Eigenvalue Zero



SCC: $i \sim j$ if i and j are in same SCC. This is an equivalence. Partial order on SCC's: $S_1 < S_2$ if $S_1 \rightsquigarrow S_2$.

Topological sorting: extend partial order to total order.

Theorem 2: S and L are block triangular with SCC's as blocks. The blocks are generalized rw Laplacians.

$$L = \left(\begin{array}{c|c|ccc|cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ \hline -1/2 & 0 & 0 & 0 & 0 & 1 & -1/2 \\ 0 & 0 & -1/2 & 0 & 0 & -1/2 & 1 \end{array}\right)$$

1st and 3rd block both give a zero eigenvalue. To understand how SCC's are connected, we will look at their eigenvectors, i.e.: the kernel of L.

The Right Kernel of L

Recall: reach R_i, excl. part H_i, cabal B_i, common part C_i.

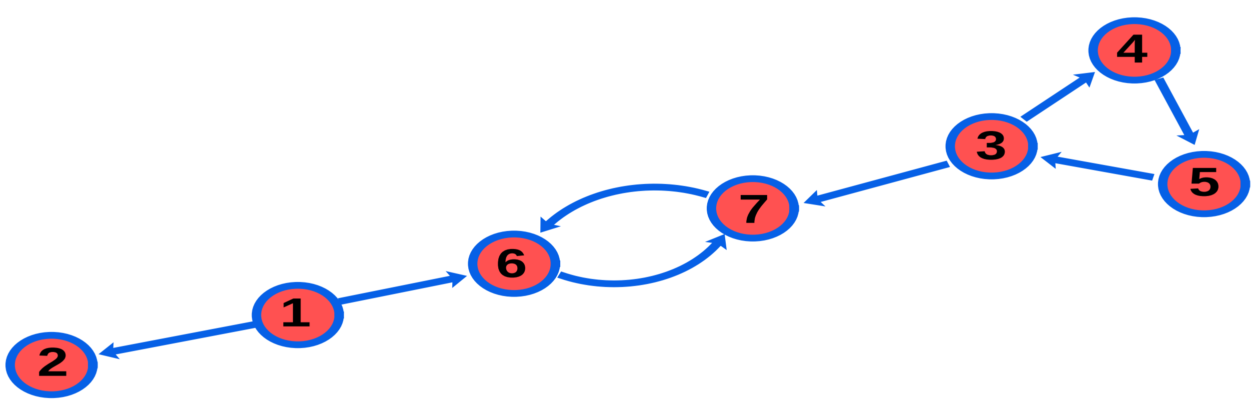
From Now On: (i) There are exactly **$\underline{k}$ reaches** $\{R_i\}_{i=1}^k$.
ii) L is a general Laplacian of the form $L = D - DS$ [1].

Theorem 3 [1]: The algebraic and geometric multiplicity of the eigenvalue 0 of L equals k.

Thus: **no non-trivial Jordan blocks in kernel!**

Theorem 4 [1]: The *right* kernel of L is spanned by the *column* vectors $\{\gamma_1, \cdots, \gamma_k\}$, where:

$$\begin{cases} \gamma_m(j) = 1 & \text{if} \quad j \in H_m \quad \textbf{(excl.)} \\ \gamma_m(j) \in (0,1) & \text{if} \quad j \in C_m \quad \textbf{(common)} \\ \gamma_m(j) = 0 & \text{if} \quad j \notin R_m \quad \textbf{(reach)} \\ \sum_{m=1}^k \gamma_m(j) = \mathbf{1} & \end{cases}$$



$\gamma_1^T = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$ and $\gamma_2^T = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$

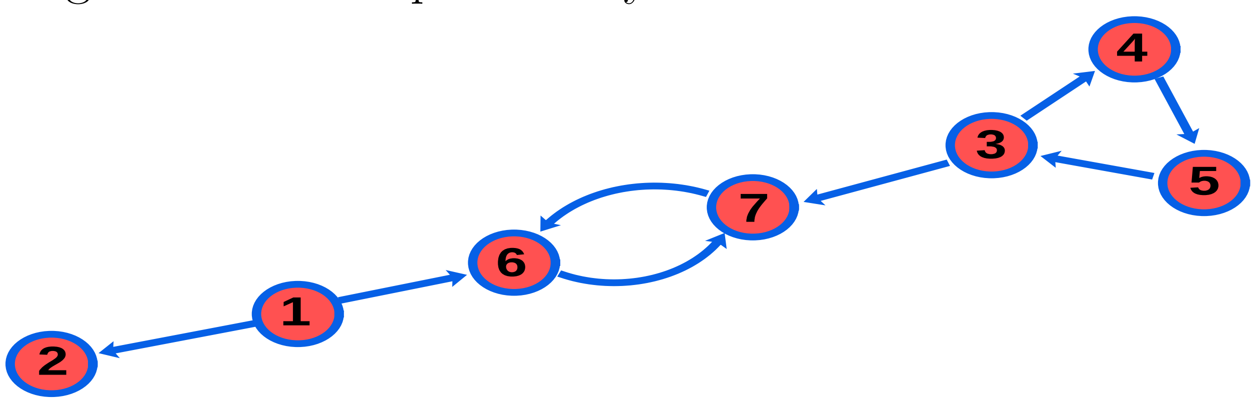
The Left Kernel of L

Theorem 5 [2]: The *left* kernel of L is spanned by the *row* vectors $\{\bar{\gamma}_1, \cdots, \bar{\gamma}_k\}$, where:

$$\begin{cases} \bar{\gamma}_m(j) > 0 & \text{if } \; j \in B_m \; \textbf{(cabal)} \\ \bar{\gamma}_m(j) = 0 & \text{if } \; j \notin B_m \\ \sum_{j=1}^{k} \bar{\gamma}_m(j) = 1 & \\ \{\bar{\gamma}_m\}_{m=1}^{k} \textbf{ are orthogonal} & \end{cases}$$

Mnemonic: the horizontal "bar" on $\bar{\gamma}$ indicates a (horizontal) row vector.

Thus in this case the row vectors $\{\bar{\gamma}_1, \cdots, \bar{\gamma}_k\}$ are a set of orthogonal invariant probability measures.



$$\bar{\gamma}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \bar{\gamma}_2 = \begin{pmatrix} 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \end{pmatrix}$$

Observations about the Kernels

Theorem 6 (folklore, [2]): A random walker starting at vertex j has a chance $\gamma_m(j)$ of ending up in the mth cabal B_m.

In the following G is a (weakly connected) digraph with rw Laplacian $\mathcal{L}$. The union of its cabals is called B. Its complement is denoted as B^c.

Theorem 7 (folklore): If $\tau(i)$ is the expected time for a rw starting at vertex i to reach B, then τ is the unique solution of

$$\mathcal{L}\tau = \mathbf{1}_{B^c} \text{ with } \tau|_B = 0$$

τ is often called the **expected hitting time**.

Sketch of Proof of Thm 7

The **boundary condition** ($\tau|_B = 0$) is clearly correct.

Recall: a) $S_{ij} > 0$ means 'i sees j'.
b) But rw goes <u>against</u> arrows. So:

S_{ij} is probability of $i \to j$, so for $i \in B^c$ (complement of B):

$$\tau(i) = 1 + \sum_j S_{ij}\tau(j)$$

Rewriting gives the **equation of the theorem**.

Existence and uniqueness: Reorder the vertices so that vertices in B appear before vertices in B^c. Then by Theorem 2, $\mathcal{L}$ is lower block triangular. The equation becomes

$$\begin{pmatrix} \mathcal{L}_{BB} & \mathbf{0} \\ \mathcal{L}_{B^cB} & \mathcal{L}_{B^cB^c} \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ \tau_{B^c} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix}$$

The matrix $\mathcal{L}_{B^cB^c}$ is non-singular [1]. So the solution exists and is unique.

Exercise: Prove that $\mathcal{L}_{B^cB^c}$ is non-singular. Hint: suppose $\mathcal{L}_{B^cB^c}x = 0$. Then pad x with zeroes to get a vector in the null space of $\mathcal{L}$. Now use Theorem 4 to see that there is no such vector in the right kernel of $\mathcal{L}$.

Exercise: Prove Theorem 6 using the same method.

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A S Y M P T O T I C

B E H A V I O R

Continuous or Discrete

Recall: for rw Lapl. $\mathcal{L}$ and normalized adjacency matrix S

$$\mathcal{L} = I - S$$

If evals $\mathcal{L}$ are λ_m, then evals S are $1 - \lambda_m$.

Continuous Consensus:

$$\dot{x} = -\mathcal{L}x$$

Discrete Consensus:

$$x^{(t+1)} - x^{(t)} = -\mathcal{L}x^{(t)} \quad \Longrightarrow \quad x^{(t+1)} = Sx^{(t)}$$

Similarly, **Continuous Diffusion:**

$$\dot{p} = -p\mathcal{L}$$

Discrete Diffusion or Random Walk:

$$p^{(t+1)} - p^{(t)} = -p^{(t)}\mathcal{L} \quad \Longrightarrow \quad p^{(t+1)} = p^{(t)}S$$

Definition: a (right) eigenpair (λ_m, η_m) of $\mathcal{L}$ is a pair such that $\mathcal{L}\eta_m = \lambda_m\eta_m$. A left eigenpair $(\lambda_m, \bar{\eta}_m)$ satisfies $\bar{\eta}_m\mathcal{L} = \lambda_m\bar{\eta}_m$.

Definition: G a digraph with n vertices and k reaches, define the $n \times n$ matrix Γ as follows (γ_m and $\bar{\gamma}_m$ as in Thm 4 & 5):

$$\Gamma_{ij} \equiv \sum_{m=1}^{k} \gamma_m(i)\bar{\gamma}_m(j) \quad \text{or} \quad \Gamma = \sum_{m=1}^{k} \gamma_m \otimes \bar{\gamma}_m$$

Asymptotics of Self-Adjoint

Continuous consensus: If $\mathcal{L}$ is any symmetric (or self-adjoint) square matrix with right eigenpairs (λ_m, η_m) and left eigenpairs $(\lambda_m, \bar{\eta}_m)$. Note that $\bar{\eta}_m = \eta_m^T$. Then

$$\dot{x} = -\mathcal{L}x$$

is solved by

$$x^{(t)} = \sum_{m=1}^{n} \frac{(\eta_m, x^{(0)})}{(\eta_m, \eta_m)} e^{-\lambda_m t} \eta_m$$

The terms with $\mathrm{Re}(\lambda_m)$ positive converge to 0.

Notation: x has n components labeled by i. Each of these depends on time (t): $\boldsymbol{x^{(t)}(i)}$.

Discrete diffusion or random walk: Similar notation $\boldsymbol{p^{(t)}(i)}$.

$$p^{(t+1)} = p^{(t)} S$$

gives

$$p^{(t)} = \sum_{m=1}^{n} \frac{(\bar{\eta}_m, p^{(0)})}{(\bar{\eta}_m, \bar{\eta}_m)} (1 - \lambda_m)^t \bar{\eta}_m$$

The terms with $|1 - \lambda_m| < 1$ converge to 0.

Exercise: write solutions for discrete consensus and continuous diffusion.

Asymptotics

But non-orthogonality and Jordan blocks destroy this simple picture! However, for our bases for kernels of $\mathcal{L}$ (theorems 4 and 5) with Γ on pg 32, we still get the following.

Theorem 8 [2]: The soln of the continuous consensus problem satisfies

$$\lim_{t\to\infty} x^{(t)}(i) = \sum_{j=1}^{n}\left(\sum_{m=1}^{k}\gamma_m(i)\bar{\gamma}_m(j)\right)x^{(0)}(j)$$

or

$$\lim_{t\to\infty} x^{(t)} = \Gamma x^{(0)}$$

Theorem 9 [2]: The soln of the (discrete) random walk satisfies

$$\lim_{n\to\infty}\frac{1}{n}\sum_{t=0}^{n-1} p^{(t)} = p^{(0)}\Gamma$$

The p are probability row vectors.
In discrete case: *first* take average, *then* take limit!

Exercise: state similar theorems for discrete consensus and continuous diffusion.

These theorems do not follow immediately from theorems 1, 4, and 5 (see [2]).

Another Interpretation of γ_m

From Thm 8: Displacements in consensus caused by initial displacement x_0:

$$\dot{x} = -\mathcal{L}x \quad \Longrightarrow \quad \lim_{t\to\infty} x^{(t)} = \Gamma x^{(0)}$$

Left multiplying by $\frac{1}{n}\mathbf{1}^T$ has the effect of taking an average of these displacements.

Definition: The <u>**influence**</u> $I(i)$ of the vertex i is **average** of the displacements caused by unit displacement e_i:

$$I(i) \equiv \frac{1}{n}\mathbf{1}^T\,\Gamma\,e_i = \frac{1}{n}\mathbf{1}^T\left(\sum_{m=1}^{k}\gamma_m \otimes \bar{\gamma}_m\right)e_i$$

$\mathbf{1}$ is the *all ones* vector.

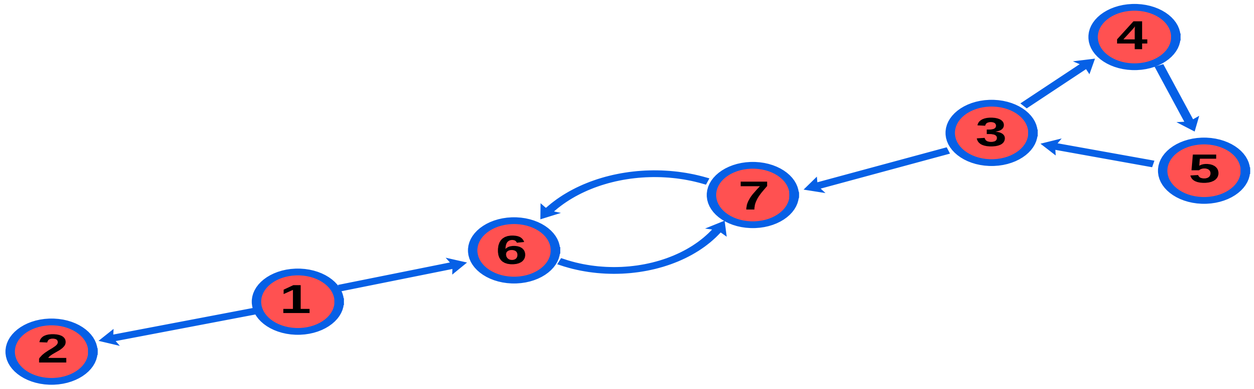
Theorem 10: The influence $I(i)$ of vertex i in the mth cabal is given by

$$I_m(i) = \frac{1}{n}\mathbf{1}^T(\gamma_m \otimes \bar{\gamma}_m)(i) \geq 0$$

If i not in a cabal, then its influence is zero. The sum of all influences equals 1.

Exercise: prove this theorem. *Hint: It is enough to show that Γ is row-stochastic. That can be shown from its definition.*

Asymptotics: Example



$$\gamma_1^T = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \quad \text{and} \quad \gamma_2^T = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

$$\bar{\gamma}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \bar{\gamma}_2 = \begin{pmatrix} 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \end{pmatrix}$$

So

$$\Gamma = \sum_{m=1}^{k} \gamma_m \otimes \bar{\gamma}_m = \frac{1}{9} \begin{pmatrix} 9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 3 & 3 & 0 & 0 \\ 0 & 0 & 3 & 3 & 3 & 0 & 0 \\ 0 & 0 & 3 & 3 & 3 & 0 & 0 \\ 6 & 0 & 1 & 1 & 1 & 0 & 0 \\ 3 & 0 & 2 & 2 & 2 & 0 & 0 \end{pmatrix}$$

Let $x^{(0)}$ and $p^{(0)}$ be concentrated on vertex 7 only. Then

$$\lim_{t\to\infty} x^{(t)} = \mathbf{0} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} p^{(i)} = \frac{1}{9}(3,0,2,2,2,0,0)$$

Exercise: Check that Γ is as given.

Exercise: Find another interesting example.

DISCRETE

AND

CONTINUOUS

Are They Different??

Up to now, we have used the matrices S and $\mathcal{L}$ to model discrete and continuous versions of consensus and diffusion.

We have seen that these models have many aspects in common and some differences.

Now, a different question presents itself.
Given a continuous process, can I find a discrete process that gives the time 1 map of the continuous one.
And vice versa, given a discrete process, can I find a continuous process whose time 1 map gives me back the the discrete one.

Exercise: If $\mathcal{L} = I - S$, is $x^{(t+1)} = Sx^{(t)}$ the time 1 map of $\dot{x} = -\mathcal{L}x$? *Hint: no.*

From Continuous to Discrete

Start with the continuous processes: $\dot{\boldsymbol{x}} = -\boldsymbol{\mathcal{L}x}$ (consensus)
. $\dot{\boldsymbol{p}} = -\boldsymbol{p\mathcal{L}}$ (diffusion)

Soln: $x^{(t)} = e^{-\mathcal{L}t}x^{(0)}$. Time one map: $x^{(t+1)} = e^{-\mathcal{L}}x^{(t)}$.

$$(1) \quad S^{(d)} \equiv e^{-\mathcal{L}} = I - \mathcal{L} + \frac{\mathcal{L}^2}{2} - \cdots$$

$$(2) \quad S^{(d)} \equiv e^{-\mathcal{L}} = e^{S-I} = e^{-1}\left(I + S + \frac{S^2}{2} + \cdots\right)$$

Properties of $e^{-\mathcal{L}}$: (1) row-sum one, (2) off-diagonal elmts non-negative. Thus $S^{(d)}$ is a row-stochastic matrix. So....

Obtain **Discrete Consensus:** $\boldsymbol{x}^{(t+1)} = \boldsymbol{S}^{(d)}\boldsymbol{x}^{(t)}$.

and **Discrete Diffusion:** $\boldsymbol{p}^{(t+1)} = \boldsymbol{p}^{(t)}\boldsymbol{S}^{(d)}$.
(The usual term is random walk.)

Define the discrete Laplacian: $\mathcal{L}^{(d)} = I - S^{(d)}$. From (1):

Theorem 11 [2]: $\mathcal{L}^{(d)}$ and $\mathcal{L}$ have the same kernels.

As before: the leading eigenspace of $S^{(d)}$ is kernel of $\mathcal{L}^{(d)}$.

Corollary: These discrete processes have the same asymptotic behavior as the original continuous ones.

Every Possible Discrete Process??

One more Property of $e^{-\mathcal{L}}$: Recall

$$(2) \quad S^{(d)} = e^{-\mathcal{L}} = e^{S-I} = e^{-1}\left(I + S + \frac{S^2}{2} + \cdots\right)$$

Thus $e^{-\mathcal{L}}$ is **transitively closed**: if there is a path $i \rightsquigarrow j$, then there is an edge ij.

So, the answer to question in the header is: **NO !**

Digraphs like $\boldsymbol{o} \leftrightarrows \boldsymbol{o}$ with $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

cannot occur as time one maps (not transitively closed).

Another obstruction is that $S^{(d)} = e^{-\mathcal{L}}$ *cannot* have 0 as eigenvalue.

The **question** exactly which maps can be considered as a time one map of a Laplacian system is **open**, though several obstructions are known (such as the ones above).

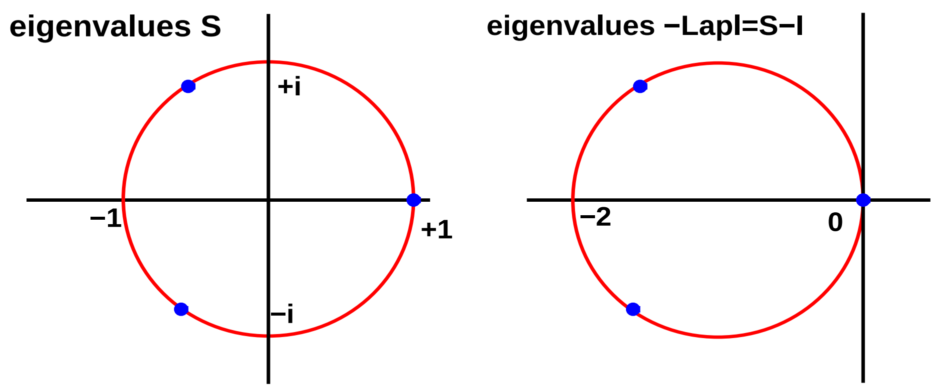
Exercise: Give an example of a discrete process where S has an eigenvalue 0 (or $\mathcal{L} = I - S$ an eval 1).

Periodic Behavior

Possibility of periodic behavior changes asymptotics:
Consider:
Consensus (continuous): $\dot{\boldsymbol{x}} = -\boldsymbol{\mathcal{L}}\boldsymbol{x}$.
Consensus (discrete): $\boldsymbol{x}^{(t+1)} = \boldsymbol{S}\boldsymbol{x}^{(t)}$.

The eigenvalues of S lie within the closed unit disk.

Asymptotic behavior as $\boldsymbol{t} \to \infty$ is determined by
Continuous: null space of $\mathcal{L}$.
Discrete: (i) eigenspace of S assoc. to eigenvalue 1 or
. (ii) eigenspaces of S assoc. to roots of unity.
All else converges to zero.



To get asymptotics
For discrete: must average: $\lim_{n\to\infty} \frac{1}{n} \sum_{t=0}^{n-1} \boldsymbol{x}^{(t)}$.
For continuous, no need: $\lim_{t\to\infty} \boldsymbol{x}^{(t)}$.

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