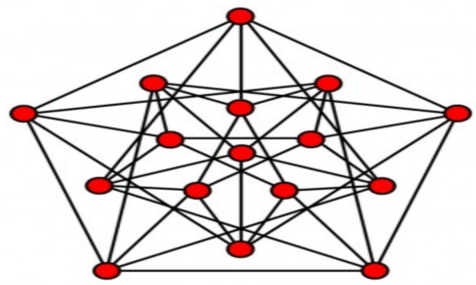
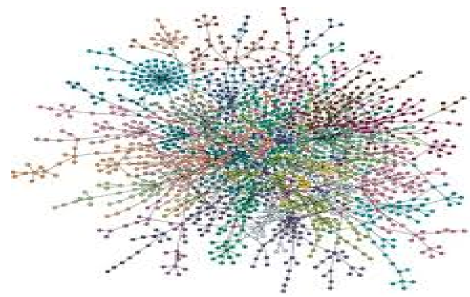
Como, Italy, December 2022





DIGRAPHS I

Mathematical Background: Perron-Frobenius, Spectral Theorem, Jordan Normal Form, Cauchy-Binet, Jacobi's Formula

Based on various sources.

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SUMMARY:

* This is a review of four theorems from linear algebra that are important for the development of the algebraic theory of directed graphs. These theorems are the Perron-Frobenius theorem, the Cauchy-Binet formula, the Jordan Normal Form, and Jacobi's Formula.

OUTLINE:

The headings of this talk are color-coded as follows:

Graph Theory Definitions

Perron-Frobenius

The Spectral Theorem

Jordan Normal Form

Cauchy-Binet

Jacobi's Formula

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ELEMENTARY GRAPH THEORY

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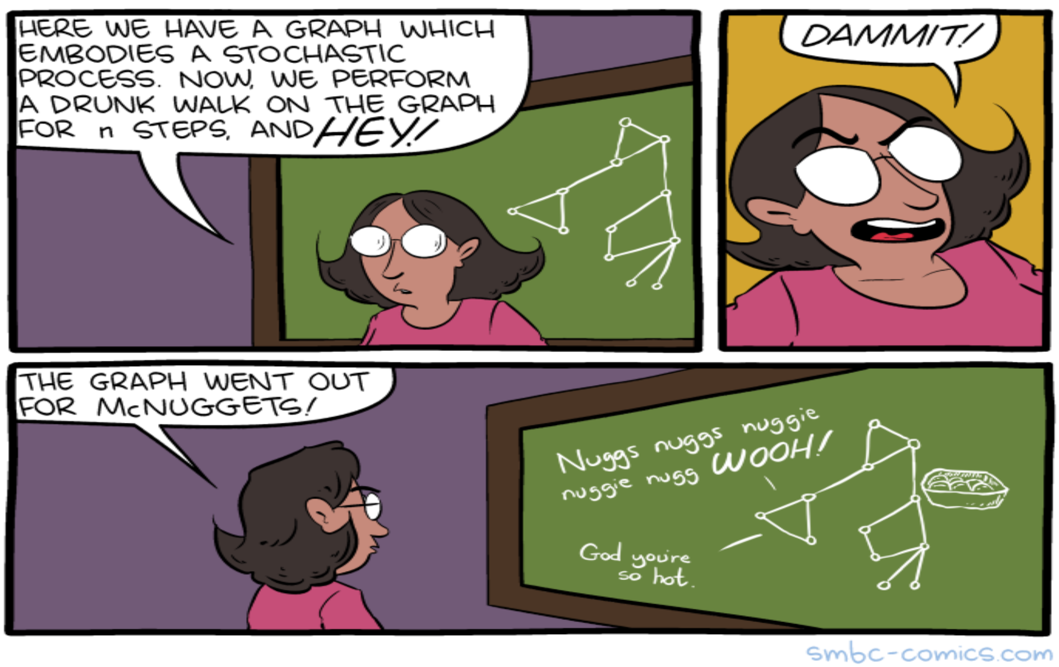
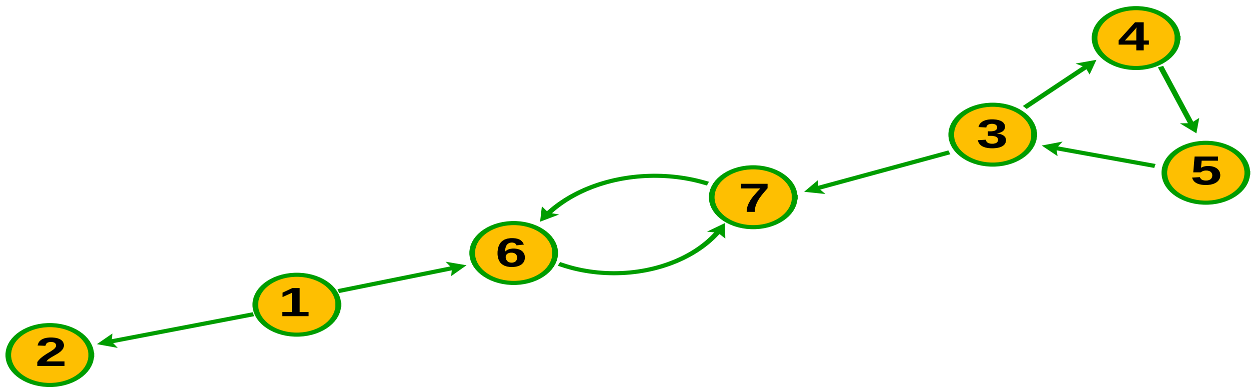


Definitions: Digraphs

Definition: A directed graph (or **digraph**) is a set $V = \{1, \cdots n\}$ of **vertices** together with set of ordered pairs $E \subseteq V \times V$ (the **edges**).



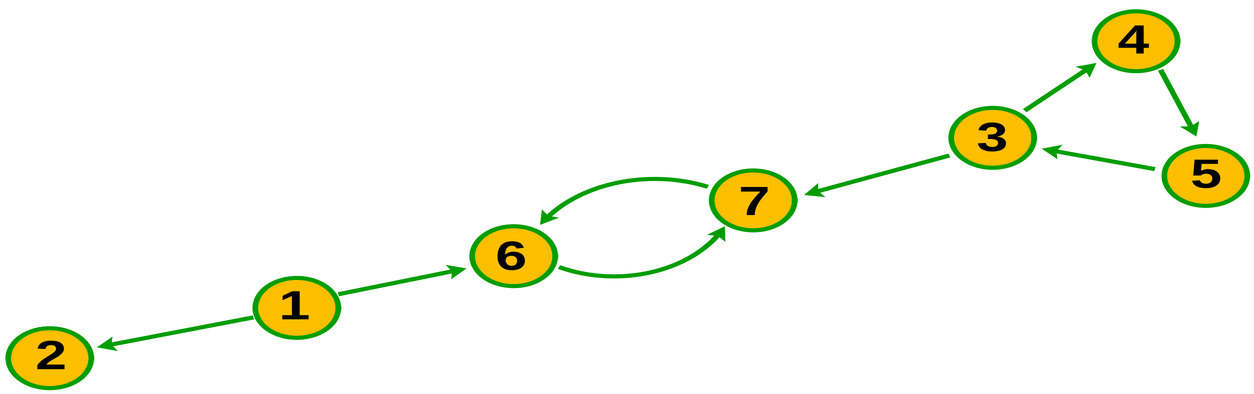
A directed edge $j \to i$, also written as ji.
A directed path from j to i is written as $j \rightsquigarrow i$.

Digraphs are everywhere: models of the internet [7], social networks [8], food webs [12], epidemics [11], chemical reaction networks [13], databases [6], communication networks [5], and networks of autonomous agents in control theory [9], to name but a few.

A BIG topic: Much of mathematics can be translated into graph theory (discretization, triangulation, etc). In addition, many topics in graph theory that do not translate back to *continuous* mathematics.

Definitions: Connectedness of digraphs

Undirected graphs are connected or not. But...



Definition: A digraph G is

* **strongly connected or SC** if for every ordered pair of vertices (i, j), there is a path $i \rightsquigarrow j$.

* **unilaterally connected** if for every ordered pair of vertices (i, j), there is a path $i \rightsquigarrow j$ or a path $j \rightsquigarrow i$.

* **weakly connected** if the **underlying UNdirected graph** is connected.

* **not connected** if it is not weakly connected.

* **componentwise strongly connected or CSC** if each weak component is strongly connected.

* **Multilaterally connected** **weakly connected** but not **unilaterally connected**.

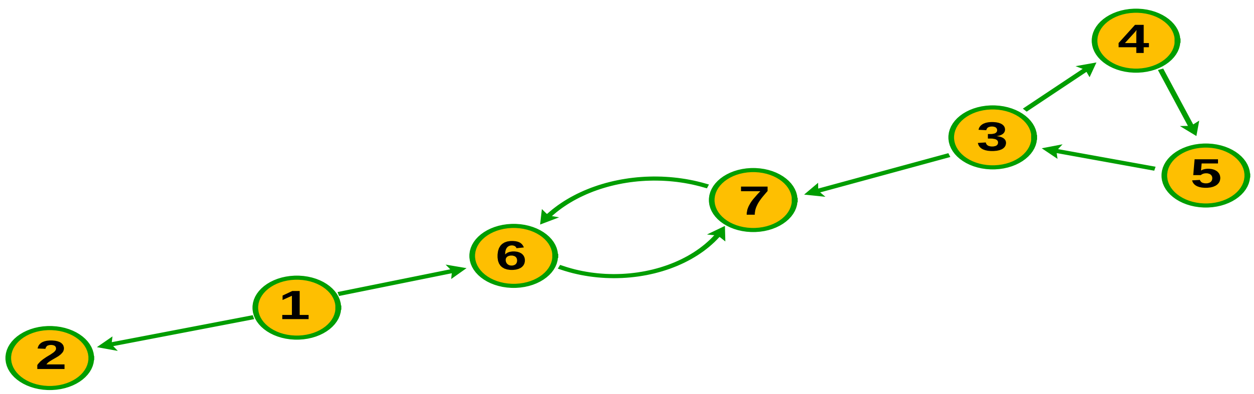
Exercise: Find a graph of each type.

Exercise: Find examples of digraphs and classify conn.ness. Try: the graph of the Figure and the largest strict subset relation of subsets of $\{1, 2, 3\}$ ($A \to B$ if A is a maximal strict subset of B). The latter gives a 3-dimensional cube.

The Adjacency Matrix

Definition: The **combinatorial adjacency matrix** Q of the graph G is the matrix whose entry $Q_{ij} = 1$ if there is an edge ji and equals 0 otherwise.

Interpretation: We think of $Q_{ij} = 1$ as information going from j to i. Or: i "sees" j. In the graph below, both 2 and 6 "see" 1. So $Q_{21} = Q_{61} = 1$.



$$Q = \left(\begin{array}{c|c|ccc|cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right)$$

Exercise: Find the combinatorial adjacency matrices of examples of previous page.

THE PERRON FROBENIUS THEOREM

Non-Negative Matrices

Definition: A non-negative matrix Q is **irreducible** if for every i, j, there is a k such that $(Q^k)_{ij} > 0$.

Exercise: Q is **irreducible** if for all i, j, there is path from j to i: $j \rightsquigarrow i$. (Hint: $(Q^2)_{ij} > 0$ iff there is k such that $Q_{ik} > 0$ and $Q_{kj} > 0$.)

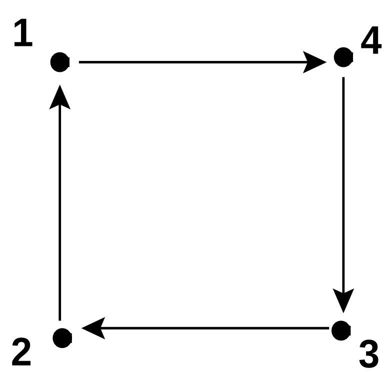
So: Q adjacency matrix of graph G: Q irreducible iff G is SC.

Definition: A non-negative matrix Q is **primitive** if there is a k such that for every i, j, we have $(Q^k)_{ij} > 0$.

Exercise: Q is **primitive** if $\exists$ k such that for all i, j, there is $j \rightsquigarrow i$ of length k.

Irreducible but **not primitive**: any cyclic permutation.

$$Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$



Perron-Frobenius

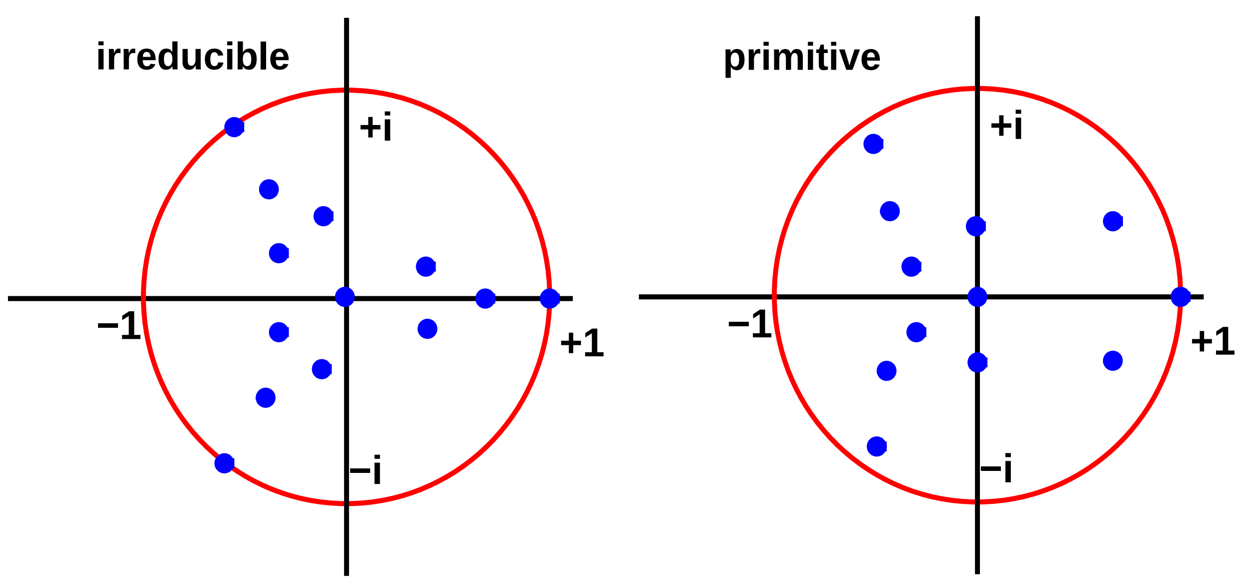
The single most important theorem in algebraic graph theory!!
Gives leading eigenpair of many important matrices.
1st order description of dynamical processes on graphs.
More details in [1] and [14].

Theorem 1A: Let $A \geq 0$ be irreducible. Then:
(a) Its spectral radius $\rho(A)$ is a simple eval of A.
(b) Its associated evec is the only strictly positive evec.

Thus its largest eval is simple, real, and positive. But there may be other evals of the same modulus.

Theorem 1B: Let $A \geq 0$ be primitive. Then also:
All other evals have modulus strictly smaller than $\rho(A)$.

(Note 3-fold rotational symmetry in irreducible case.)



Irreducible Has Period p

In the irreducible case, the matrix A has a **period $p \geq 1$**. That is: after permutation of vertices, A is **block cyclic**.
Example: $p = 3$:

$$A = \begin{pmatrix} 0 & A_1 & 0 \\ 0 & 0 & A_2 \\ A_3 & 0 & 0 \end{pmatrix}$$

In this **cyclic block form**, the A_i are **rectangular**!

Exercise: Show that

$$A^3 = \begin{pmatrix} A_1A_2A_3 & 0 & 0 \\ 0 & A_2A_3A_1 & \\ 0 & 0 & A_3A_1A_2 \end{pmatrix}$$

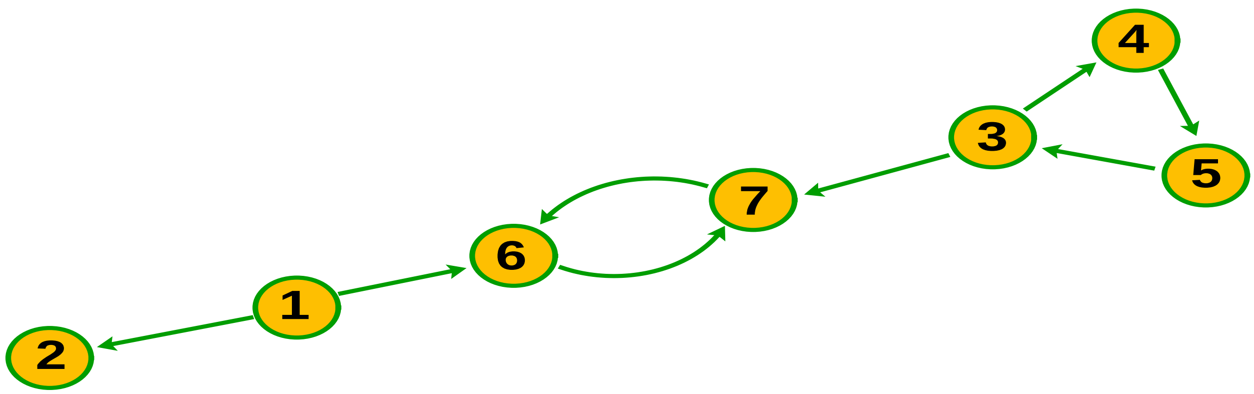
Now, the diagonal blocks are primitive.

By **Cauchy-Binet** (later):
each diagonal block D of A^3 has same non-zero spectrum.
Suppose non-zero spectrum D is: $\{\lambda_i\}_{i=1}^s$.

The non-zero spectrum of A consists of **all 3rd roots** of these.

Earlier Example

To check irreducibility, need check paths of length at most 6. Then must repeat.



$$\sum_{i=1}^{6} Q^i = \left(\begin{array}{c|c|ccc|cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 & 0 & 0 \\ \hline 3 & 0 & 4 & 2 & 3 & 3 & 3 \\ 3 & 0 & 5 & 3 & 4 & 3 & 3 \end{array}\right)$$

So, Q is block-triangular and thus *not* irreducible. But:
The two non-trivial ($\dim > 1$) diagonal blocks are **irreducible but *not* primitive**. Notice the grouping of the evals.

The spectrum of Q is $\{0, 0, \quad 1, e^{2\pi i/3}, e^{-2\pi i/3}, \quad 1, -1\}$.

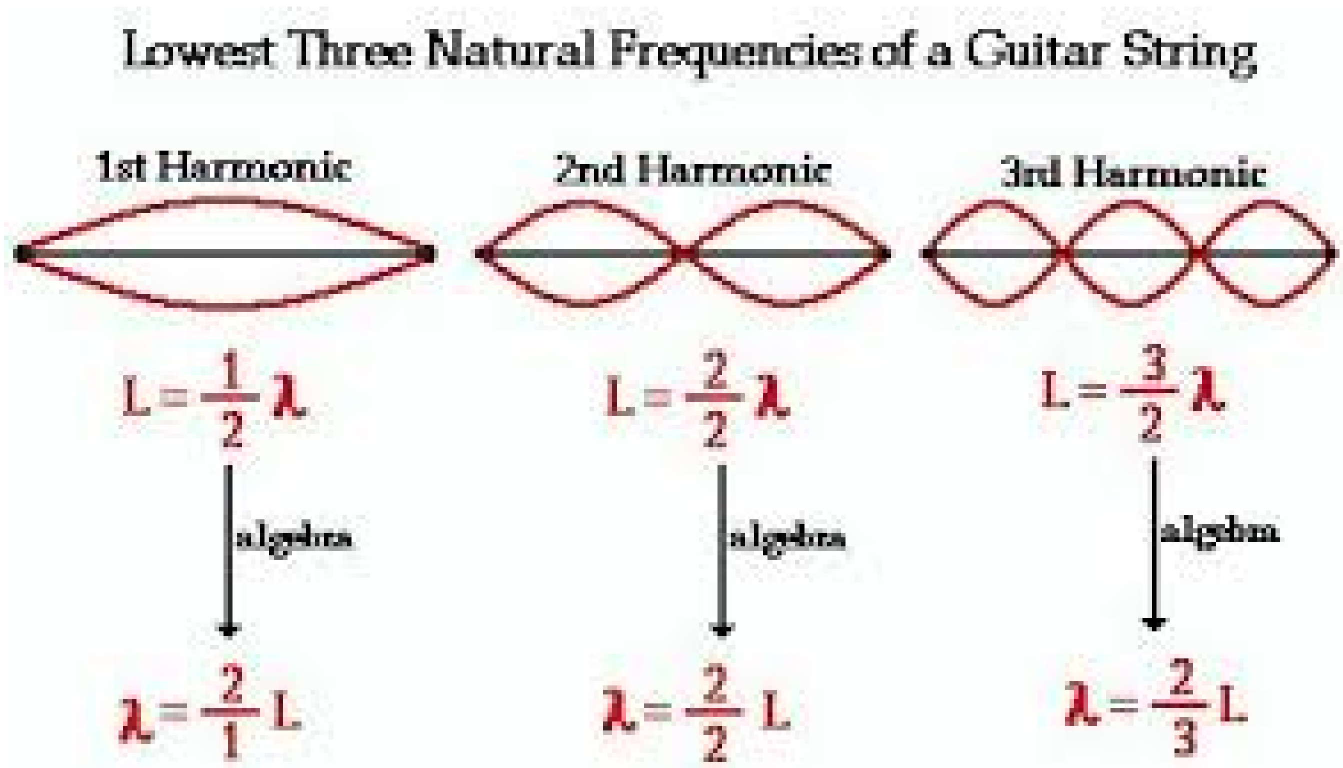
Exercise: Prove all statements (use [1], [14], or others). Find examples.

Other Eigenvectors

Theorem 1C: Let A be irreducible. Any other evec but the leading <u>cannot</u> be real and non-negative.

This is clear if the eigenvalue is non-real. So only needs proof for real evecs.

This is the beginning of the study of **Nodal Domains**.
A classical problem in analysis (since Courant): count the number of nodal domains of e.fns to the Laplace operator. See Figure.



For undirected graphs there are many results. But for digraphs very little is known. (After all, evecs may not be real!)

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THE SPECTRAL THEOREM

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Spectral Theorem

From now: A is $n \times n$ matrix with real or complex coeff's:
real symmetric $\subset$ self-adjoint $\subset$ normal.
(A is normal if $A^*A = AA^*$. A^* is conjugate transpose $\overline{A}^T$.)

Theorem 2 (Spectral Theorem): A has orthonormal basis of evecs $\{v_i\}_{i=1}^n$ iff A normal.

These evals are <u>real</u>, if A is self-adjoint.

Definition: Standard (Hermitian) inner product on $\mathbb{C}^n$ is

$$(x, y) = x_1\overline{y}_1 + \cdots + x_n\overline{y}_n\,,$$

$\overline{z}$ indicates complex conjugate of z.

Normal is common in physics and engineering. Makes life easy, because computations simplify:

Let A a (normal) matrix with e.pairs $\{\lambda_i, v_i\}$.
Suppose $\dot{x} = Ax$ with initial condition $x(0) = x_0$. Then:

$$\boldsymbol{x(t) = \sum_i \frac{(x_0, v_i)}{(v_i, v_i)} e^{\lambda_i t} v_i}$$

where $(.,.)$ is real or Hermitian inner product and $|v| = \sqrt{(v, v)}$.
$(x_0, v_i)v_i/|v_i|^2$ is the orthogonal projection of x_0 onto v_i.

Orthogonal Basis of Evecs Implies Normal

A is an $n \times n$ matrix. ASSUME $\{v_i\}_{i=1}^n$ orthonormal basis of e.vecs. Set

$$H := (v_1, v_2, \cdots, v_n),$$

so that its columns are the e.vecs of A.

Exercise: Show that $A = HDH^{-1}$, where D is diagonal.

Exercise: Show that the ith row of H^* equals $\overline{v}_i$, or

$$H^* = \begin{pmatrix} \overline{v}_1^T \\ \overline{v}_2^T \\ \vdots \\ \overline{v}_n^T \end{pmatrix}$$

Exercise: Show that $H^*H = I$, and so $H^* = H^{-1}$.

Exercise: Show that $A^* = H\overline{D}H^{-1}$.

Exercise: Show that $A^*A = AA^*$.

Observe that these exercises prove one direction of the spectral theorem!

Normal Implies Orthogonal Basis of Evecs

A is an $n \times n$ matrix. ASSUME A is normal.

Exercise: Show that $Av = \lambda v$ iff $A^* v = \overline{\lambda} v$.
Hint: *Show that* $(A - \lambda I)(A^* - \overline{\lambda} I) = (A^* - \overline{\lambda} I)(A - \lambda I)$. *Then use normality to show that*

$$((A - \lambda I)v, (A - \lambda I)v) = \big((A^* - \overline{\lambda} I)v, (A^* - \overline{\lambda} I)v\big)$$

where $(,)$ *is (Hermitian) inner product. So, if one is zero, then the other is too.*

Exercise: If A has two e.pairs (λ, v) and (μ, w) and $\lambda \neq \mu$, then $(v, w) = 0$.
Hint: $(\lambda - \mu)(v, w) = (Av, w) - (v, A^* w)$ *by the previous and defition of Hermitian inner product.*

Exercise: Show that A has no non-trivial (dim > 1) Jordan blocks.
Hint: *If* A *has Jordan block, then there is* λ *and* v *such that*

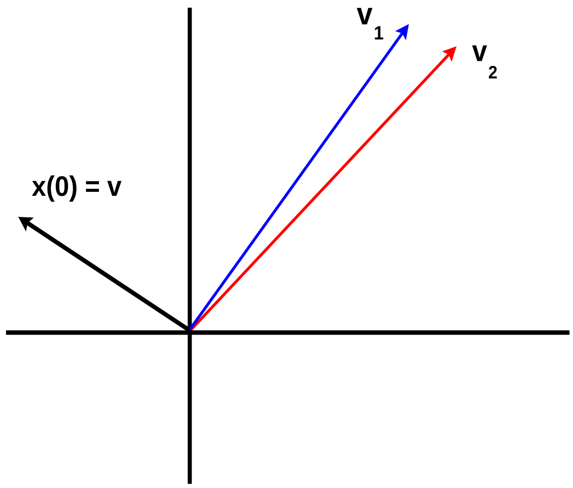
$$(A - \lambda I)v \neq 0 \qquad \text{and} \qquad (A - \lambda I)^2 v = 0\,.$$

But then by normality and first exercise

$$0 \neq ((A - \lambda I)v, (A - \lambda I)v) = \big((A^* - \overline{\lambda} I)(A - \lambda I)v, v\big) = 0$$

Observe that this proves the other direction of the spectral theorem!

Life in a Non-normal Universe

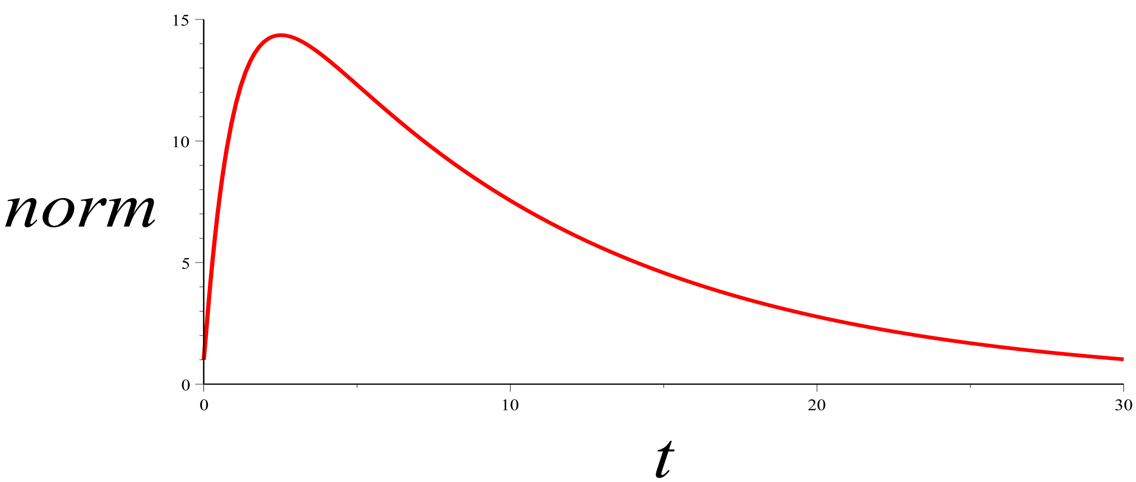


Let $\dot{x} = Ax$. Sps evecs v_1 and v_2 nearly parallel.

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$

Example: $\lambda_i = \{-0.1, -1.0\}$ and init. condn $x(0)$ as indicated.

Large **transient**! Stable system may initially "look" unstable. Below we plot $|x(t)|$.



Exercise: Define a 2-dim. system of ODE plus initial condition that exhibits this type of behavior.

Another Convenience of Normality

Exercise: Matrix norm $\|A\| \equiv \sup_x\{|Ax| \; : \; |x| = 1\}$ equals norm of its largest eval if A is normal.

Hint: WLOG $|v_i| = 1$, $|x| = 1$.
a) Show $\sum (v_i, x)^2 = 1$;
b) Show that $Ax = \sum \lambda_i (v_i, x) v_i$;
c) Showthat (Ax, Ax) is a weighted mean of λ_i^2.

This may fail in particular for matrices that have a non-trivial Jordan block.

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THE JORDAN NORMAL FORM

$$\begin{pmatrix} \boxed{\begin{matrix} \lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1 \end{matrix}} & & & & \\ & \boxed{\begin{matrix} \lambda_2 & 1 \\ & \lambda_2 \end{matrix}} & & & \\ & & \boxed{\lambda_3} & & \\ & & & \ddots & \\ & & & & \boxed{\begin{matrix} \lambda_n & 1 \\ & \lambda_n \end{matrix}} \end{pmatrix}$$

Case I: n Lin. Indep. Eigenvectors

Let A be $n \times n$ matrix, but not necessarily normal!
In general, it may have real and/or complex e.pairs.

Evals are the solutions $\{\lambda_i\}_{i=1}^{k}$ (with $k \leq n$) of

$$\det(A - \lambda I) = 0$$

Case I: n **linearly independent** evecs $\{v_i\}_{i=1}^{n}$.
Given λ_i, then $\{v_i\}$ is a solution of

$$(A - \lambda_i I)v = 0$$

Let H the matrix whose ith column equals v_i. Then A is **diagonalizable**, or:

$$D = H^{-1} A H$$

with D diagonal with $D_{ii} = \lambda_i$ (real if A is self-adjoint).

Application: Suppose $\dot{x} = Ax$ with init. cond. x_0. Then:

$$x(t) = \sum_i \alpha_i e^{-\lambda_i t} v_i$$

But the α_i are less simple to calculate. Set $t = 0$, you get:

$$H\alpha = \alpha_1 v_1 + \cdots + \alpha_n v_n = x_0$$

Exercise: Check the statements on this page.

Case II: Less than n LI Eigenvectors

Let A be $n \times n$ matrix.

Case II: less than n **linearly independent** evecs $\{v_i\}_{i=1}^n$.

This happens when for some i, λ_i is a root of order $\underline{k > 1}$ of

$$\det(A - \lambda I) = 0$$

but

$$(A - \lambda_i I)v = 0$$

has less than k linearly independent solutions for v.

Definition: The **algebraic multiplicity** of an eigenvalue λ_i of A is the **order** of the root λ_i of $\det(A - \lambda I)$.
The **geometric multiplicity** of λ_i is the **number** of linearly independent evecs associated with λ_i.

In this case A is not diagonalizable but **block diagonalizable**. There is matrix H so that

$$J = H^{-1}AH$$

Exercise: J has diagonal **Jordan blocks** (or JB), all of the form:

$$B_i = \begin{pmatrix} \lambda_i & 1 & 0 & .. \\ 0 & \lambda_i & 1 & .. \\ .. & .. & .. & 1 \\ .. & .. & 0 & \lambda_i \end{pmatrix}$$

Case II: Not Enough LI Eigenvectors

Find all evals λ satisfying

$$\det(A - \lambda I) = 0$$

For each eval λ_i, find its evecs:

$$(A - \lambda_i I)v = 0$$

These vectors span the **eigenspace** of λ_i.
For simplicity: assume there is only one: v_i.

If geom mult$(\lambda_i) <$ alg mult(λ_i), do this:
Start with evec v_i.
Find vector w_{i1} such that $\{w_{i1}, v_1\}$ LI and

$$(A - \lambda_i I)w_{i1} = v_i and$$

Find w_{i2} such that $\{w_{i2}, w_{i1}, v_1\}$ LI and

$$(A - \lambda_i I)w_{i2} = w_{i1}$$

Etc. The v_i together with w_{ij} are **generalized eigenvectors**. They span the **generalized eigenspace** of λ_i.

Thus there are exactly n **linearly independent** generalized eigenvectors.

Exercise: Check the statements on this page.

Case II: Construction of the Matrix H

H is the matrix whose columns are:

$$\{\mathbf{v_1}, \mathbf{w_{11}}, \cdot\cdot\ \mathbf{w_{1n_1}}, \mathbf{v_2}, \mathbf{w_{21}}, \cdot\cdot\ \mathbf{w_{2n_2}}, \cdot\cdot, \mathbf{v_k}, \mathbf{w_{k1}}, \cdot\cdot\ \mathbf{w_{kn_k}}\}$$

equals v_i. Then

$$\boldsymbol{J = H^{-1}AH}$$

and J consists of non-trivial Jordan blocks.

Example: If 1st block has dim ≥ 3 (or $n_1 \geq 2$):

$$\lambda_1 e_1 \xleftarrow{H^{-1}} \lambda_1 v_1 \xleftarrow{A} v_1 \xleftarrow{H} e_1$$

$$\lambda_1 e_2 + e_1 \xleftarrow{H^{-1}} \lambda_1 w_{11} + v_1 \xleftarrow{A} w_{11} \xleftarrow{H} e_2$$

$$\lambda_1 e_3 + e_2 \xleftarrow{H^{-1}} \lambda_1 w_{12} + w_{11} \xleftarrow{A} w_{12} \xleftarrow{H} e_3$$

Definition: Thus the first diagonal block of J becomes:

$$\begin{pmatrix} \lambda_1 & 1 & 0 & \cdots & \cdots \\ 0 & \lambda_1 & 1 & \cdots & \cdots \\ 0 & 0 & \lambda_1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

This is called **Jordan normal form**.

Exercise: Check the statements on this page.

$\dot{x} = Ax$, General Case

Exercise: Let I be the identity and

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad J = \lambda I + N = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

a) Compute e^{Jt} via the usual expansion.
(*Hint:* $e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$.)
b) Use a) to give solutions of $\dot{x} = Jx$, where $x(0) = (a_1, a_2)^T$.
(*Hint:* $e^{\lambda t} \begin{pmatrix} a_1 + a_2 t \\ a_2 \end{pmatrix}$.)

The expansion of e^{Jt} in the exercise

$$e^{Jt} = I + Jt + \frac{J^2 t^2}{2} + \frac{J^3 t^3}{3!} + \cdots$$

simplifies because $J = \lambda I + N$ and $N^2 = 0$.

Exercise: Solve general problem $\dot{x} = Ax$, $x(0) = x_0$.
Step 1: Write init. cond as sum of **gener. evecs.**

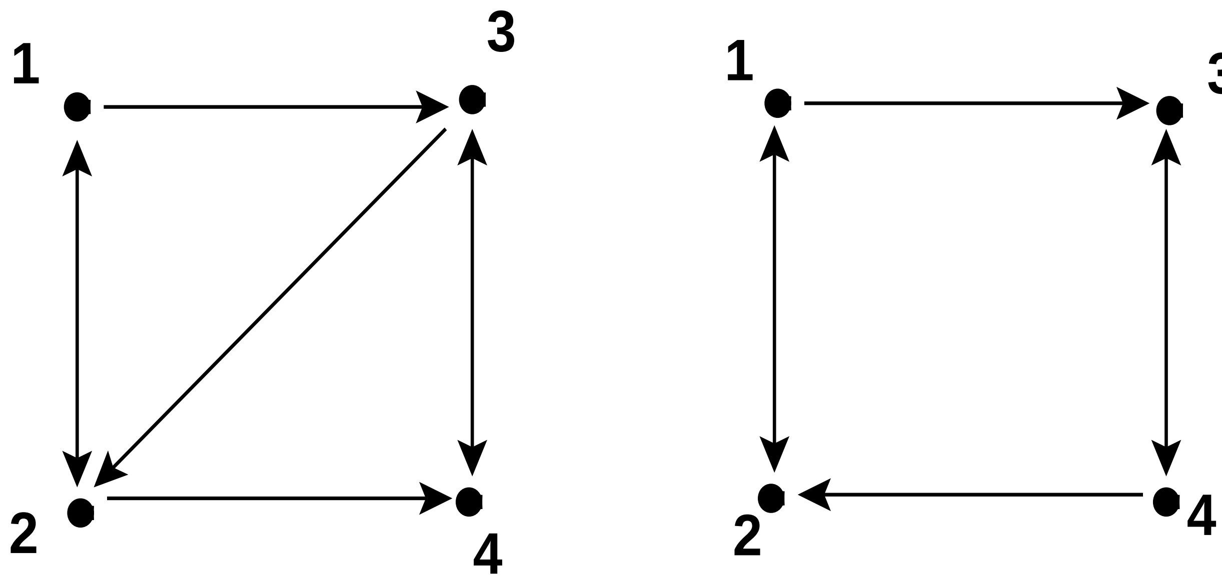
$$x_0 = \sum \alpha_i v_i \qquad \text{where} \qquad H\alpha = x_0$$

Step 2: Suppose $x_0 = \alpha_{12} w_{12}$. Then

$$x(t) = \alpha_{12}\, e^{\lambda t} \left(\frac{t^2}{2} v_1 + t\, w_{11} + w_{12} \right)$$

Step 3: Sum those contributions.

Two Examples



Exercise: Check that the first graph has adjacency matrix

$$Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

with spectrum $\{1.68, -1.03 \pm 0.74i, 0.37\}$ (approximately).

Exercise: The second graph has adjacency matrix

$$Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

with spectrum $\{0^{(2)}, \pm\sqrt{2}\}$. The eigenvalue 0 has an associated 2-dimensional Jordan block.

Additional Exercises

Exercise: Show that the matrix

$$\begin{pmatrix} a-b & c \\ -cd & a+b \end{pmatrix}$$

has a non-trivial Jordan block (JB) if $b^2 = c^2 d$ and $c \neq 0$ and $d \neq 0$.

Exercise: So you may think JB's are rare (co-dimension one). But symmetries can change that. Show that
a) Newton's equation $\ddot{x} = 0$ gives rise to a JB.
b) That JB explains why two bodies without forcing separate linearly in time (Newton's first law).

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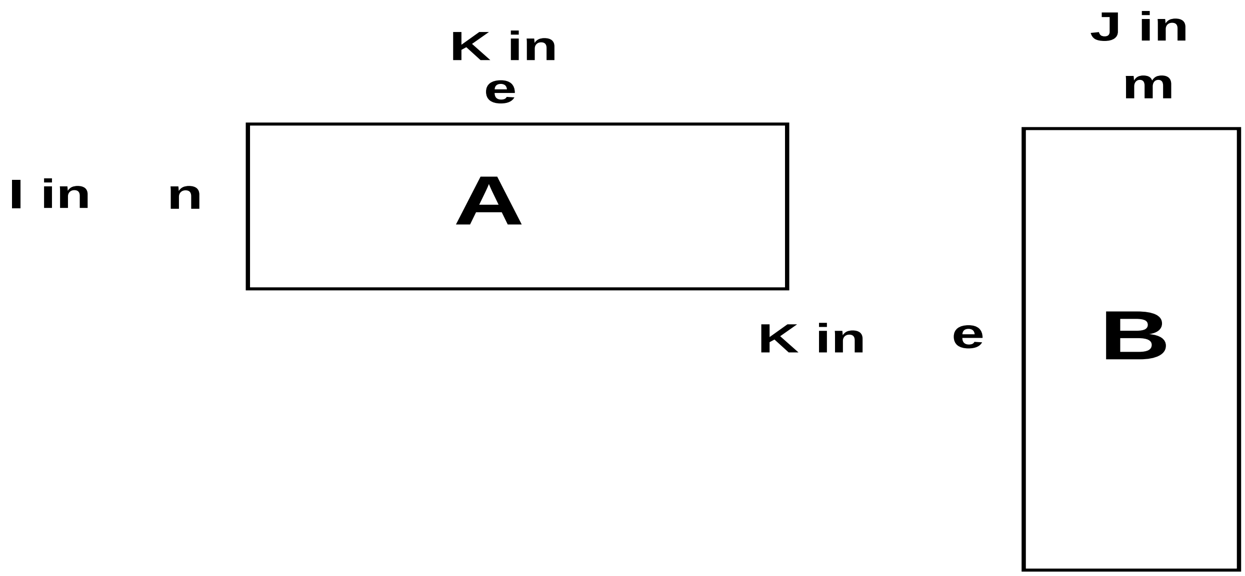
THE CAUCHY - BINET FORMULA

Generalized Cauchy-Binet

A is a $n \times e$ matrix and B is a $e \times m$ matrix.



Notation: $k \leq n, m \leq e$. (See figure). Let $I \subseteq \{1, \cdots n\}$, $J \subseteq \{1, \cdots m\}$, and $K \subseteq \{1, \cdots e\}$. All subsets have the same cardinality k.

Definition: The matrix consisting of the entries of A in $I \times K$ is called a **minor** of A. **Principal minor** if $I = K$. It is denoted by $A[I, K]$.

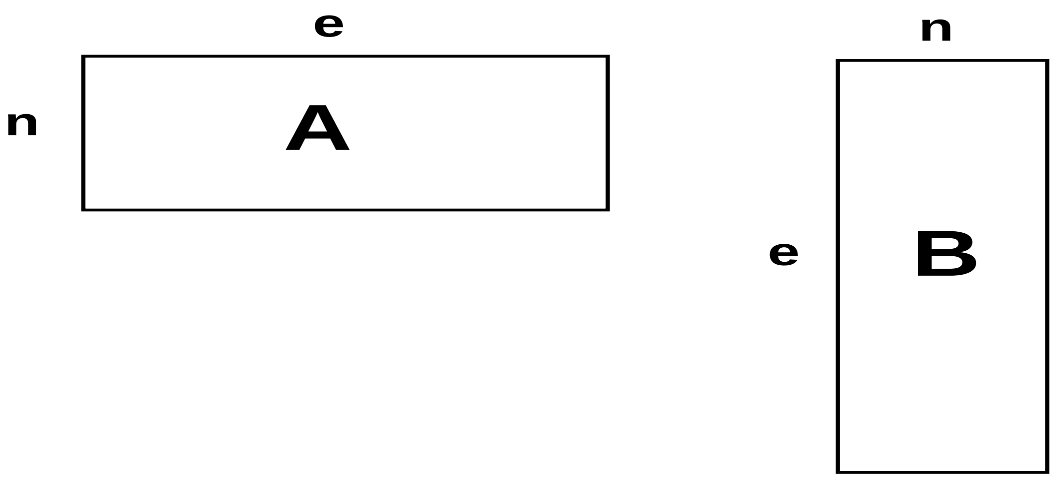
Theorem 3 (generalized Cauchy-Binet):

$$\det\left((AB)[I, J]\right) = \sum_{K} \det(A[I, K]) \det(B[K, J])$$

where the sum is over all $K \subseteq \{1, \cdots e\}$ with $|K| = k$.

Corollaries

A and B as depicted, where $n \leq e$. Now $I = J = \{1, \cdots n\}$



Corollary (Cauchy-Binet): We have

$$\mathbf{det}\,(\boldsymbol{AB}) = \sum \mathbf{det}(\boldsymbol{A}[\boldsymbol{J}, \boldsymbol{K}])\, \mathbf{det}(\boldsymbol{B}[\boldsymbol{K}, \boldsymbol{J}])$$

where the sum is over all $K \subseteq \{1, \cdots e\}$ with $|K| = n$.

If X is $n \times n$, by standard matrix computation

$$\det(X + z\, Id) = \cdots + z^{n-k} \sum_{|K|=k} \det X[K, K] + \cdots$$

By generalized C-B, we also have for $k \leq n$:

$$\sum_{|K|=k} \sum_{|L|=k} \det A[K, L] \det B[L, K]$$

equals $\sum_{|K|=k} \det(AB)[K, K]$ **and** $\sum_{|L|=k} \det(BA)[L, L]$**.**

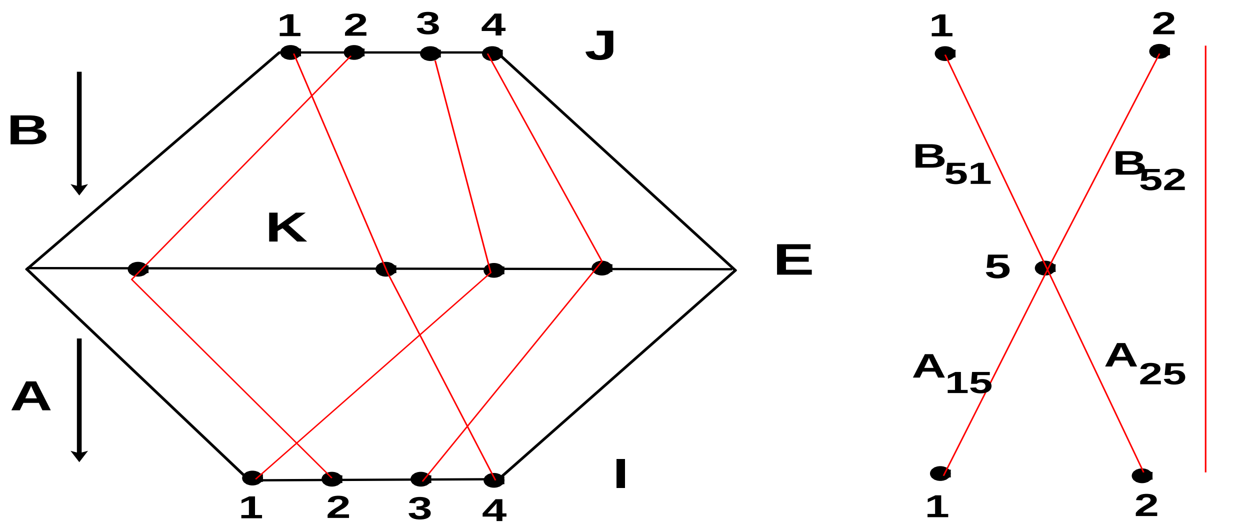
Corollary: We have

$$\mathbf{det}(\boldsymbol{BA} + \boldsymbol{z}\, \boldsymbol{Id}) = \boldsymbol{z}^{\boldsymbol{e}-\boldsymbol{n}}\, \mathbf{det}(\boldsymbol{AB} + \boldsymbol{z}\, \boldsymbol{Id})$$

Exercise: Prove this. (Write both determinants (green). Use blue equality.)

Sketch of Proof of Cauchy-Binet

Inspired by Gessel-Viennot [10]. ($n = 4$ in this example.)



$I = J = \{1, \cdots n\}$ and $E = \{1, \cdots e\}$ with $n \leq e$.

$$\det AB = \sum_\sigma \text{sgn}\sigma \prod_{i\in I} (AB)_{i\sigma(i)}$$
$$= \sum_\sigma \text{sgn}\sigma \prod_{i\in I} \sum_{\ell\in E} A_{i\ell}\, B_{\ell\sigma(i)}$$

Fix σ. What is the meaning of $\prod_{i\in I} \sum_{\ell\in E} A_{i\ell}\, B_{\ell\sigma(i)}$?

For all $i \in I$, fix endpoints of paths $i \rightsquigarrow \sigma(i)$. For any $(\ell_1, \cdots \ell_n) \in E^n$ form the product of the paths (left figure).

$$\prod_{i\in I} A_{i\ell_i}\, B_{\ell_i\sigma(i)}$$

and sum over all possible $\overrightarrow{\ell} = (\ell_1, \cdots \ell_n)$ (see also pg 33):

$$\sum_{\overrightarrow{\ell}\in E^n} \prod_{i\in I} A_{i\ell_i}\, B_{\ell_i\sigma(i)}$$

This includes $\overrightarrow{\ell}$'s with "crossing" paths, e.g. $\overrightarrow{\ell} = (5, 5, \cdots)$.

Sketch of Proof Continued

But crossing paths give canceling contributions.
For the crossing as pictured (right figure), contributions are:

$$A_{15}B_{5\mathbf{1}}A_{25}B_{5\mathbf{2}}A_{3\ldots}\cdots \quad \text{and} \quad A_{15}B_{5\mathbf{2}}A_{25}B_{5\mathbf{1}}A_{3\ldots}\cdots$$

BUT with <u>opposite</u> sign: σ changes by 1 transpos.: $1 \leftrightarrow 2$.
The next expression avoids crossing paths:

$$\det AB = \sum_{\sigma} \operatorname{sgn}\sigma \sum_{K,|K|=n} \sum_{\vec{\ell} \in \operatorname{bij}(I,K)} \prod_{i \in I} A_{i\ell_i} B_{\ell_i \sigma(i)}$$

where $\operatorname{bij}(I, K)$ is the set of bijections from I to K.

Re-introduce (canceling) crossing paths within K.

$$\det AB = \sum_{\sigma} \mathbf{sgn}\sigma \sum_{K,|K|=n} \prod_{i \in I} \sum_{\ell \in K} A_{i\ell} B_{\ell\sigma(i)}$$

Swap two summations, so that we get:

$$\det AB = \sum_{K,|K|=n} \sum_{\sigma} \mathbf{sgn}\sigma \prod_{i \in I} \sum_{\ell \in K} A_{i\ell} B_{\ell\sigma(i)}$$

For fixed K, $\sum_{\sigma} \mathbf{sgn}\sigma \prod_{i \in I} \sum_{\ell \in K} A_{i\ell} B_{\ell\sigma(i)}$ is the determinant of product of square matrices. This equals product of the determinants. So

$$\det \boldsymbol{AB} = \sum_{|\boldsymbol{K}|=\boldsymbol{n}} \det(\boldsymbol{A}[\boldsymbol{I}, \boldsymbol{K}]) \det(\boldsymbol{B}[\boldsymbol{K}, \boldsymbol{J}])$$

Exercises

Helpful Exercise: To understand the exchange of $\sum$ and $\prod$ better, show that

$$\prod_{i\in I}\sum_{\ell\in J} x_{i\ell} = \sum_{\vec{\ell}\in J^m}\prod_{i\in I} x_{i\ell_i}\,,$$

where $|I| = m$ and $|J| = n$.

Example:

$$(x_{11}+x_{12}+x_{13})(x_{21}+x_{22}+x_{23})(x_{31}+x_{32}+x_{33})$$

is equal to

$$x_{1\mathbf{1}}x_{2\mathbf{1}}x_{3\mathbf{1}} + x_{1\mathbf{1}}x_{2\mathbf{1}}x_{3\mathbf{2}} + x_{1\mathbf{1}}x_{2\mathbf{1}}x_{3\mathbf{3}} + x_{1\mathbf{1}}x_{2\mathbf{2}}x_{3\mathbf{1}} + \cdots$$

The red indices range over all of J^m.

Exercise: Explicitly verify all steps of the previous if B is a 3×2 matrix and A is 2×3.

Exercise: Discuss application to Perron-Frobenius, page 11.

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JACOBI'S FORMULA

The Formula and Its Corollaries

A a square matrix, **adj**(A) its **adjugate**: adj(A) is the transpose of the cofactor matrix and satisfies

$$A \operatorname{adj}(A) = \operatorname{adj}(A)\, A = \det(A)\, I$$

Suppose A depends (differentiably) on a parameter t.

Theorem 4: $\displaystyle \frac{d}{dt}\det(A) = \operatorname{Tr}\left(\operatorname{adj}(A)\,\frac{dA}{dt}\right)$.

We give some common corollaries as easy exercises.

Replace $\dfrac{dA}{dt}$ by B whose only non-zero entry is $B_{k\ell} = 1$:

Exercise: Show that $\displaystyle \frac{d}{dA_{k\ell}}\det(A) = (\operatorname{adj}(A))_{\ell k}$.

Instead, replace A by e^{Bt} and so $\operatorname{adj}(A)$ by $e^{-Bt}\det\left(e^{Bt}\right)$:

Exercise: Show that $\displaystyle \frac{d}{dt}\det(e^{tB}) = \operatorname{Tr}(B)\,\det(e^{tB})$.

The latter gives an ODE. Solve it:

Exercise: Show that the latter implies: $\det(e^{tB}) = e^{\operatorname{Tr}(Bt)}$.

Sketch of Proof

B has evals λ_i (with mult.). Then $I + \epsilon B$ has evals $1 + \epsilon\lambda_i$:

$$\det(I + \epsilon B) = \prod_i (1 + \epsilon\lambda_i)$$

Thus

$$\lim_{\epsilon\to 0} \frac{\det(I + \epsilon B) - \det(I)}{\epsilon} = \sum_i \lambda_i = \mathrm{Tr}(B)$$

For an invertible A:

$$\lim_{\epsilon\to 0} \frac{\det(A + \epsilon B) - \det(A)}{\epsilon} =$$

$$\lim_{\epsilon\to 0} \frac{\det(A)\left[\det(I + \epsilon A^{-1}B) - \det(I)\right]}{\epsilon} =$$

$$\det(A)\mathrm{Tr}(A^{-1}B) = \mathrm{Tr}(\det(A)A^{-1}B)$$

Extend to non-invertible: replace $\det(A)\, A^{-1}$ by $\mathrm{adj}(A)$:

$$\cdots = \mathrm{Tr}\left(\mathrm{adj}(A)\, B\right)$$

... And replace B by $\frac{dA}{dt}$:

$$\cdots = \mathrm{Tr}\left(\mathrm{adj}(A)\, \frac{dA}{dt}\right)$$

Exercise: Set $\partial_B A := \lim_{\epsilon\to 0} \dfrac{\det(A + \epsilon B) - \det(A)}{\epsilon}$. Prove

$$\partial_B \det A = \mathrm{Tr}\left(\mathrm{adj}(A)\, B\right)$$

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