Numbers from all Angles

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Preface for Instructors and Students

Overview. This work contains a one year (three terms or two semesters), first course in number theory at the advanced undergraduate to advanced graduate level. No prior knowledge of number theory is assumed. We do require knowledge and familiarity with the notions of basic linear algebra (matrices, determinants), calculus (integrals), and proof-writing. But beyond that, this text aims to be completely self-contained[1].

The book is set up as follows. There are three parts, each with five chapters. Part 1 covers what students might learn in an advanced undergraduate first course on number theory. The remainder is pitched at a graduate level. Part 2 discusses the basics of the different branches of number theory (algebraic, analytic, and ergodic or probabilistic). Part 3 is more advanced and contains the proofs of the prime number theorem, the Birkhoff ergodic theorem, and the unsolvability of the general quintic, among other things.

My intent is that each chapter should take about 2 weeks (or 6 to 8 class room hours), thus allowing for a 30-week course, as is typical at many US universities. In my classes, for each of these 2-week periods, students work in groups of 3 to 6 to collaborate on the problems in the chapter at hand. The first week, the "bare bones" material (the main text) is presented by the

[1] With few exceptions: for instance, we do not prove the transcendentality of π.

instructor, and in the second week students present their group work consisting of all the exercises in the given chapter. In my view, it is absolutely *crucial* that equal importance is given to lectures and exercises by students as well as teachers!

Motivation

First of all, the writing of this book was "curiosity-driven". The major theorems in this book are much more than just a piece of quaint history! The pursuit of the proof of the prime number theorem has been a driving force of number theory for hundreds if not thousands of years. The ergodic theorem shaped our perception of typical behavior in both mathematics and physics. The unsolvability of the quintic also counts as one of the most spectacular achievements in mathematics. Students still come into our (mathematics) departments today with the same curiosity for these topics as people in the centuries before them. Yet few introductory books get around to explaining them. I wanted to write about some of the big ideas that have motivated (and still do) people to think about number theory and mathematics in general. I wanted to write the course that I wish I had had the opportunity to take as a student.

Specialization has been a modern trend, also in mathematics text books. In my view this results too often in such detailed treatments that the reader or student is left wondering what the point is. In this work, I disregard boundaries and emphasize the big picture and the connections with different subdisciplines. For example, too often the proof of the prime number theorem is given with only a nodding reference to its complex analytic part. Instead, here I delve fairly deep into the theory behind Cauchy integral formula to explain that part in all its details. The reader will find many other examples of 'horizontal' connections not commonly found in number theory books. Our approach, then, is to try to uncover number theory by any means, regardless of where that takes us.

Another one of my aims is that the book be an efficient resource for anyone interested in learning about the main currents in number theory. I believe that modest-sized books that emphasize the larger picture tend to maintain their value as reference works for generations. The backbone of of the current version of this textbook for a yearlong course are the texts (without the exercises) of the 15 chapters. This covers a lot of material in a modest number of pages. I have taken great care to index all relevant concepts and definitions. If a word is indexed, then its first appearance in the text is almost always underlined[2].

Finally, I tried to adopt an inclusive approach. That is: I tried to make the text accessible to mathematically mature students with a background other than traditional pure mathematics. The reason for this is that the potential audience interested in learning basic, foundational number theory is much larger than the traditional pure mathematics students. Number theory comes up in applied math, physics, biology, computer science, dynamical systems, and so forth. On top of that, even in regular mathematics departments, not all interested students check all the boxes for the prerequisites. This is the background, I think, in which the teaching of this course should take place: an audience much wider than traditional pure mathematics students.

Method

You cannot really learn mathematics without doing it. But on the other hand, you cannot possibly invent everything from scratch either. So a good way to design a course is to give the lay of the land in each chapter, and let students figure out some of the details and corollaries through exercises at the end of the chapter. This has the additional advantage that the basic and most important material is concise and contained in few pages, and therefore can be consulted with great ease to find important results. Thus each chapter consists of a "bare bones" outline of a piece of theory followed by about 25 problem sets. These exercises are meant to achieve two goals. The first is to get the student used to the mechanical or computational aspects of the theory[3]. The second goal of the exercises is to extend the bare bones theory, and fill in some details. All exercises, including the ones of the latter type, should be doable by graduate students. Nonetheless, my recommendation is to have the students work out all the exercises in groups to lessen even more the chance of getting stuck[4].

[2] Check the index for the word "underlined".

[3] We emphasize that even if the algorithm is "more or less" clear or familiar, a wise student will carefully do all the computational problems in order for it to become "thoroughly" familiar.

[4] In my experience, nothing discourages self-study more than being stuck on some exercise. If students tend to get stuck on a particular exercise, I hope I will be notified, so I can add more hints in the next edition.

In order to encourage different learning strategies, both individual and group work is encouraged. Attending the blackboard sessions and reading the main text is an individual activity. On the other hand, the exercises were designed — in terms of quantity and level of difficulty — to be done in groups of 3 to 6 students. Accordingly, the texts of the 15 chapters plus appendices and the over 400 exercises should be considered as equally important parts of the book; the student should aim to expend roughly equal effort on both the text and the problems done in groups. The groups can divvy up the problems. Then each group does their own set of problems and explains their results to the other groups. In turn, they listen to the other groups until they absorb those results as well. This is also the reason why the problem sets are fairly extensive: not every students does every problem individually.

However, since the book is curiosity driven, it should also lend itself well for individual learning. For example, the mathematical physics students who wants to know more about the (Birkhoff) ergodic theorem and its applications. I recommend these students to dive in wherever their interests take them (even if it is an appendix), and follow the cross-references whenever something is unclear. I have made a great effort to make the book as self-contained and as cross-referenced as possible.

Finally, geometric, intuitive insight is also crucially important, and so I added illustrations wherever I could (about 150 figures, and many of these have two parts).

Other Lesson Plans

Because our approach aims to integrate different disciplines within mathematics, parts of the book can easily be used in other courses. For example, a graduate course on dynamical systems or ergodic theory could contain an 8 week segment consisting of Chapters 6, 9, 10, and 11. Chapters 2, 4, 12, 13, and 14 could easily constitute a short analytic number theory course. Chapters 4, 5, 7, and 8 could start off a course in algebraic number theory.

This work *can* be used at the intermediate undergraduate level, where students still need to train in writing proofs. In that case, I recommend a modest program consisting roughly of the following content. Start with Chapters 1 and 2, followed by Sections 3.1 through 3.4 and 3.6, 4.1 and 4.2, and finally 5.1 and 5.4. If this is the course taken, then the instructor would probably want to gather some extra material on how to write proofs, starting with mathematical induction. Some of the exercises may be too advanced for students at this level.

The Appendices

The appendices contain related material that would have led me too far astray from the main story of the book: the three pillars consisting of the prime number theorem, the ergodic theorem, and the insolvability of the quintic. They are not necessarily part of a one year course, as I see it, though they can be used as extra material if needed. My criteria to write these appendices were: it is fun to write, it is not easily found in standard textbooks, and it gives an "aha" experience.

A Request

We ask the reader — whether student or teacher — not to post the answers to problems. Confronting problems without being able to easily check the answer is essential for the learning process. Therefore, posting answers would make that process less efficient. Readers that have comments or corrections should feel free to get in touch with me by email.

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J. J. P. Veerman, Portland, Oregon, August, 2025

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