Professor Stipak

Example of Integer Programming Problem Worked Out Using CMMS Program

(Problem is the knapsack problem from Eck textbook, p. 156-157.)

C O M P U T E R   M O D E L S   F O R   M A N A G E M E N T   S C I E N C E

INTEGER PROGRAMMING

-=*=- INFORMATION ENTERED -=*=-

NUMBER OF VARIABLES	:	5
NUMBER OF <= CONSTRAINTS	:	1
NUMBER OF = CONSTRAINTS	:	0
NUMBER OF >= CONSTRAINTS	:	0

MAX Z = 12 X1 + 40 X2 + 15 X3 + 20 X4 + 10 X5

SUBJECT TO:

.2 X1 + .5 X2 + .2 X3 + .2 X4 + .3 X5 <= .8

-=*=- RESULTS -=*=-

VARIABLE	VALUE	ORIGINAL COEFFICIENT
X1	0	12
X2	1	40
X3	0	15
X4	1	20
X5	0	10

CONSTRAINT NUMBER	ORIGINAL RIGHT-HAND VALUE	SLACK OR SURPLUS
1	.8	.1

OBJECTIVE FUNCTION VALUE: 60

---------- E N D   O F   A N A L Y S I S ----------

OBJECTIVE FUNCTION VALUE: 60

Note: To use the CMMS program to do integer programming, first select the integer programming module and then do the following:

1) choose to enter a problem from the keyboard
2) then run the problem
3) finally, view the problem

Used by Permission:R.D. Eck (1979). Business Applications of Integer programming, Introduction to Quantitative Methods for Business Application, 153-173. South-Western Publishing Co.

7

Synopsis

ESSENTIAL CONCEPTS:
The nature of integer programming models
Knapsack problems in business
Multiple-choice models for business applications
Formulation of if-then and either-or constraints
Business applications of fixed charge models
Modeling vanishing restrictions

APPLICATIONS:

Aircraft component choices	New product development
Auditing team selection	Production planning
Banquet planning	Portfolio analysis
Department store design	Real estate development
Energy management	Route selection
Floating stock issues	Stock market transactions

R.D. Eck, An Introduction to Quantitative Methods for Business Application (Belmont: Wadsworth, 1979)

Business Applications of Integer Programming

Integer programming models are very similar to linear programming models; they differ only to the extent that integer programming models require some or all of the decision variables to have integer (whole number) values. Although the integer requirement is a seemingly modest change from linear programming, it significantly expands our ability to model and solve important business problems. This chapter introduces some integer programming terminology and explores many applications of integer programming in business.

The algorithms that are used to solve integer programming models are more complex than those that have been presented elsewhere in this book. Therefore, algorithms for solving integer programming models will not be discussed. The interested reader is referred to Chapter 10 in *Operations Research for Business* (Wadsworth, 1976) for a relatively easy-to-follow exposition of integer programming solution techniques.

Although the exposition of the logic behind integer programming algorithms is beyond the scope of this book, it should be understood that computer programs are available that perform the computations needed to obtain optimal solutions. Once business people have formulated an appropriate integer programming model, data processing personnel can easily obtain optimal solutions for the models.* This chapter concentrates on how to formulate the models.

An integer programming model is a model that has an algebraic representation that is identical to a linear programming model, with the exception that one or more of the decision variables are required to have only integer values.

The Nature of Integer Programming Models

An *integer programming model* is a model that has an algebraic representation that is identical to a linear programming model, with the exception that one or more of the decision variables are required to have only integer values. The *linear* programming model

Maximize $z = x_1 + 2x_2$

subject to: $2x_1 + 3x_2 \leq 41.3$

would become an *integer* programming model if it were also specified that either x_1 or x_2 or both x_1 and x_2 are to have integer values.

A mixed-integer programming model is an integer programming model in which some but not all of the decision variables are required to have integer values.

Suppose that x_1 is to be integer valued. Then the *integer* programming model would be

Maximize $z = x_1 + 2x_2$

subject to: $2x_1 + 3x_2 \leq 41.3$

(x_1 to be integer valued)

A pure-integer programming model is an integer programming model in which all the decision variables are required to have integer values.

Integer programming models are often classified as being either mixed-integer programming models, pure-integer programming models, or zero-one integer programming models.

A zero-one integer programming model is a special case of the pure-integer programming model in which all decision variables are to be integer valued and are to have values of either zero or one.

A *mixed-integer programming model* is an integer programming model in which some but not all of the decision variables are required to have integer values. A *pure-integer programming model* is an integer programming model in which all the decision variables are required to have integer values. A *zero-one integer programming model* is a special case of the pure-integer programming model in which all decision variables are to be integer valued *and* are to have values of either zero or one.

EXAMPLE 7.1

Types of Integer Programming Models Classify each of the following integer programming models as being either a mixed, a pure, or a zero-one integer programming model:

Model A

Maximize $z = x_1 + 2x_2$

subject to: $2x_1 + x_2 \leq 2$

(x_1 to be integer valued)
(x_2 to be integer valued)

* Researchers are still working on the development of fast and accurate algorithms for solving some large and troublesome integer programming models.

Model B

Maximize $z = x_1 + 2x_2$

subject to: $2x_1 + x_2 \leq 2$

(x_1 to be integer valued)

Model C

Maximize $z = x_1 + 2x_2$

subject to: $2x_1 + x_2 \leq 2$

(x_1 to have value of 0 or 1)
(x_2 to have value of 0 or 1)

SOLUTION

Model A is a pure-integer programming model. Model B is a mixed-integer programming model. Model C is a zero-one integer programming model. ■

To illustrate the ease with which optimal solutions can be obtained when a manager has recourse to computer facilities, imagine that a manager wants to obtain an optimal solution for the following model:

Minimize $z = 2x_1 + 3x_2 - 4x_3 + 7.2x_4$

subject to: $x_1 + x_2 + x_3 + x_4 = 25$

$3x_1 - 4x_2 \geq 7$

(x_3 to be integer valued)
(x_4 to be zero-one integer valued)

A typical set of instructions that could be keypunched for computer processing is shown in Figure 7.1.

```
*USE MIXED INTEGER PROGRAMMING ALGORITHM
X3 INTEGER
X4 ZERO-ONE
MINIMIZE Z = 2X1 + 3X2 - 4X3 + 7.2X4
SUBJECT TO
X1 + X2 + X3 + X4 = 25
3X1 - 4X2 .GREATER OR EQUAL. 7
*EXECUTE
```

FIG. 7.1 Typical computer instructions for solving an integer programming model.

The instructions inform the computer that the model is to be solved by a mixed-integer programming algorithm that is stored in the computer. All the details of the model are provided in a terse language that can be processed by the computer. The final instruction, EXECUTE, instructs the computer to execute the computations that are needed to obtain and report an optimal solution. In a typical computer facility, the computer would report the optimal solution within seconds after the EXECUTE command.

Knapsack Problems in Business

Knapsack problems are based on the following prototypal situation. A hiker has various cans of food of assorted sizes that are to be placed in a knapsack. Unfortunately, the knapsack is not large enough to hold all the cans. Therefore, the hiker must determine which cans are to be included in the knapsack. The hiker would like to make a selection that will optimize something, for example, maximize the total protein content of the food in the knapsack. This problem can be formulated as a zero-one integer programming problem.

To show how a knapsack problem can be formulated as an integer programming model, suppose that a hiker's knapsack will hold no more than 0.8 cubic feet of canned goods. Some food items that are being considered for inclusion in the knapsack are shown in Table 7.1. The hiker wants to maximize the protein content of the items that are selected for inclusion in the knapsack.

Let x_i be a zero-one integer-valued variable. If, in the solution, $x_i = 1$, item i is to be included in the knapsack. If, in the solution, $x_i = 0$, item i will not be placed in the knapsack. The variable x_i can have no values other than 0 or 1.

The knapsack model has the form

Maximize the total protein content of the items in the knapsack
subject to: total cubic feet of items in the knapsack ≤ 0.8

In algebraic form, the model is

$$\text{Maximize } z = 12x_1 + 40x_2 + 15x_3 + 20x_4 + 10x_5$$

$$\text{subject to: } 0.2x_1 + 0.5x_2 + 0.2x_3 + 0.2x_4 + 0.3x_5 \leq 0.8$$

(all variables to have values of 0 or 1)

TABLE 7.1

Items That Might Be Selected for Inclusion in a Knapsack

Item	Volume, cubic feet	Grams of protein
1	0.2	12
2	0.5	40
3	0.2	15
4	0.2	20
5	0.3	10

The optimal solution for the model is

$$x_1 = 0 \quad x_2 = 1 \quad x_3 = 0 \quad x_4 = 1 \quad x_5 = 0$$

which means that only items 2 and 4 are to be included in the knapsack. Substituting this solution in the objective function, it is seen that the total protein content will be

$$\begin{aligned} z &= 12x_1 + 40x_2 + 15x_3 + 20x_4 + 10x_5 \\ &= 12(0) + 40(1) + 15(0) + 20(1) + 10(0) \\ &= 60 \text{ grams} \end{aligned}$$

KEY CONCEPT

A knapsack problem is a problem in which all the decision variables have values of either zero or one, indicating whether or not an item is to be included in a collection of items. The items are chosen to optimize an objective subject to the restriction that knapsack capacities cannot be exceeded.

To confirm that the solution satisfies the restriction on cubic feet, substitute the solution in the right side of the constraint and observe that

$$\begin{aligned} &0.2x_1 + 0.5x_2 + 0.2x_3 + 0.2x_4 + 0.3x_5 \\ &\quad = 0.2(0) + 0.5(1) + 0.2(0) + 0.2(1) + 0.3(0) \\ &\quad = 0.7 \text{ cubic feet} \end{aligned}$$

which satisfies the limit of 0.8 cubic feet.

Some common business problems that are analogous to the knapsack problem are as follows:

1. Select departments to be included in the limited space of a planned department store.
2. Select cargo items to be included in s steamship or cargo plane.
3. Select stocks to be included in a stock portfolio, where limited funds restrict the stocks that can be purchased.

EXAMPLE 7.2

A Retailing Application of the Knapsack Problem Cromwell's Department Store will gain an additional 50,000 square feet of display space when a construction program is completed. Data for some new departments that might be included in the addition are shown in Table 7.2.

TABLE 7.2

Department	Square feet required (in thousands)	Expected annual profit of department (in thousands)
1	35	$10.0
2	28	7.0
3	10	3.0
4	10	2.8
5	8	2.5

REQUIRED

(a) Verbally state a knapsack-type model that can be used to select the specific departments that should be included in the new addition.

(b) Provide an algebraic formulation of the model.
(c) Suppose that departments 2, 3, and 4 are selected for inclusion in the new addition. By substituting this solution in the model, show that the constraint will be satisfied and that an expected annual profit of \$12.8 (thousand) will result.

SOLUTION

(a) Determine the departments that should be included in the new addition so as to

Maximize total expected annual profit

subject to: the total space required by the departments that are included in the addition ≤ 50 (thousand)

(b) Let x_i be a zero-one integer-valued variable, where $x_i = 1$ if department i is included, and $x_i = 0$ if department i is not included.

$$\text{Maximize } z = 10.0x_1 + 7.0x_2 + 3.0x_3 + 2.8x_4 + 2.5x_5$$

$$\text{subject to:} \quad 35x_1 + 28x_2 + 10x_3 + 10x_4 + 8x_5 \leq 50$$

(all variables to be zero-one integer-valued variables)

(c) If departments 2, 3, and 4 are selected for inclusion in the new addition, then

$$x_1 = 0 \quad x_2 = 1 \quad x_3 = 1 \quad x_4 = 1 \quad x_5 = 0$$

Substituting these values in the constraint yields

$$35(0) + 28(1) + 10(1) + 10(1) + 8(0) \leq 50$$

or

$$48 \leq 50$$

which confirms that the constraint is satisfied. For this solution, the objective will have a value of

$$z = 10.0(0) + 7.0(1) + 3.0(1) + 2.8(1) + 2.5(0)$$
$$= \$12.8 \text{ (thousand)}.$$

■

Multiple-Choice Models for Business Applications

Consider a chef who must plan a banquet. Possible menu items and cost data are shown in Table 7.3. The banquet is to consist of *at least* one appetizer; *exactly* one salad; *at least* one, *but no more than* two vegetables; and *exactly* one entree. The chef wants to select items that will minimize the total cost of the banquet.

This problem is called a multiple-choice problem because the chef must pick one or more items from each of several categories. The chef's task is analogous to

TABLE 7.3

Item	Cost per serving
Appetizers:	
Type 1	\$0.25
Type 2	0.30
Type 3	0.35
Salads:	
Type 4	0.25
Type 5	0.45
Vegetables:	
Type 6	0.20
Type 7	0.15
Type 8	0.40
Entrees:	
Type 9	1.25
Type 10	2.50
Type 11	3.00

the task that confronts a student during a multiple-choice examination. Th problem can also be viewed as being a somewhat enriched knapsack problem i which one or more items from each category are to be selected for inclusion in tl "knapsack." As you might suspect, the chef's multiple-choice problem can l modeled as a zero-one integer programming problem.

The chef's multiple-choice problem can be verbally stated as

Minimize the total cost of items selected for inclusion in the banquet
subject to:

(a) Total number of types of appetizers selected for inclusion ≥ 1
(b) Total number of types of salads selected for inclusion = 1
(c) Total number of types of vegetables selected for inclusion ≥ 1
(d) Total number of types of vegetables selected for inclusion ≤ 2
(e) Total number of types of entrees selected for inclusion = 1

To provide an algebraic formulation of the model, let x_i be a zero-one integ valued variable, where $x_i = 1$ if a type i item is included and $x_i = 0$ if a type i it is not included. The multiple-choice model is shown.

$$\text{Minimize } z = 0.25x_1 + 0.30x_2 + 0.35x_3 + 0.25x_4 + 0.45x_5$$
$$+ 0.20x_6 + 0.15x_7 + 0.40x_8 + 1.25x_9 + 2.50x_{10} + 3.00x_{11}$$

subject to:
(a) $x_1 + x_2 + x_3 \geq 1$
(b) $x_4 + x_5 = 1$
(c) $x_6 + x_7 + x_8 \geq 1$
(d) $x_6 + x_7 + x_8 \leq 2$
(e) $x_9 + x_{10} + x_{11} = 1$
(all variables to be zero-one integer-valued variables)

KEY CONCEPT

In a multiple-choice problem, zero-one integer-valued variables are used to identify the choice(s) that is(are) made in each of several categories.

In interpreting the model for the chef's multiple-choice problem, it is essential to remember that the variables are zero-one variables that indicate whether or not a food item is to be included in the banquet. The variables *do not indicate the size of servings*. For example, a feasible (but not optimal) solution is

$x_1 = 1 \quad x_2 = 1 \quad x_3 = 0 \quad x_4 = 1 \quad x_5 = 0 \quad x_6 = 0 \quad x_7 = 1$

$x_8 = 0 \quad x_9 = 1 \quad x_{10} = 0 \quad x_{11} = 0$

This solution represents a banquet consisting of the type 1 and type 2 appetizers, the type 4 salad, the type 7 vegetable, and the type 9 entree.

A financial application of multiple-choice models is presented in the next example.

EXAMPLE 7.3

Multinational Investments The investment opportunities of a multinational corporation are described in Table 7.4. The board of directors has specified that: (1) at least two of the European investment must be accepted, (2) one of the two South American investments must be accepted, and (3) no more than one investment in Africa can be accepted. Furthermore, the firm has only $40 million available to finance the investments.

REQUIRED

1. Verbally state a multiple-choice model that can be used to determine which of the investment opportunities to accept.
2. Provide an algebraic statement of the model.

SOLUTION

1. Select investments to be accepted to

Maximize total expected profit

subject to:

(a) The number of European investments that are accepted ≥ 2
(b) The number of South American investments that are accepted = 1
(c) The number of African investments that are accepted ≤ 1
(d) The total cost of the investments that are accepted ≤ $40 (million)

2. Let x_i be a zero-one integer-valued variable, where $x_i = 1$ if investment i is accepted and $x_i = 0$ if investment i is not accepted.

TABLE 7.4

Investment	Location	Cost (in millions)	Expected profit (in millions)
1	Europe	$10	$1.0
2	Europe	8	0.9
3	Europe	8	0.9
4	South America	16	2.0
5	South America	12	1.4
6	Africa	4	0.2
7	Africa	6	0.5
8	Africa	16	2.1

Maximize $z = 1.0x_1 + 0.9x_2 + 0.9x_3 + 2.0x_4 + 1.4x_5 + 0.2x_6 + 0.5x_7 + 2.1x_8$

subject to:
(a) $x_1 + x_2 + x_3 \geq 2$
(b) $x_4 + x_5 = 1$
(c) $x_6 + x_7 + x_8 \leq 1$
(d) $10x_1 + 8x_2 + 8x_3 + 16x_4 + 12x_5 + 4x_6 + 6x_7 + 16x_8 \leq 40$
(all variables to be zero-one integer-valued variables) ■

An important feature of many multiple-choice problems in business is that in many applications the choice that is made in one category must be coordinated with a choice that is made in another category. *If*, for example, an appliance retailer selects a specific manufacturer's brand of washing machine for inclusion in the retailer's inventory, *then* the retailer might have to coordinate the choice of a brand of clothes dryer in order to have compatible styles. Note that the coordination of choices among categories in multiple-choice problems can take the form of *if-then* reasoning.

KEY CONCEPT

In many multiple-choice situations in business, it is necessary to coordinate a choice made in one category with a choice made in another category. In many instances, if-then constraints are used to assure the necessary coordination.

To illustrate this concept, imagine that an aircraft company is planning a new medium-range aircraft. The firm can use any one of the three types of airframes and any one of the two types of engines that are listed in Table 7.5. Because of technical considerations, the airframe choice must be coordinated with the engine choice.

One constraint that would appear in a multiple-choice model is "one airframe must be selected." Letting x_1, x_2, and x_3 be zero-one integer-valued variables that indicate whether or not a type 1, type 2, or type 3 airframe is selected, this constraint can be expressed as

$$x_1 + x_2 + x_3 = 1$$

Similarly, one engine must be selected. This requirement can be represented by the constraint

$$x_a + x_b = 1$$

where x_a and x_b are zero-one integer-valued variables that indicate whether or not a type A or type B engine is selected.

It is also necessary to coordinate the airframe and engine choices. A constraint is needed to guarantee that *if a type 3 airframe is selected, then a type B engine must also be selected*. This can be accomplished by the constraint:

$$x_b - x_3 \geq 0$$

TABLE 7.5

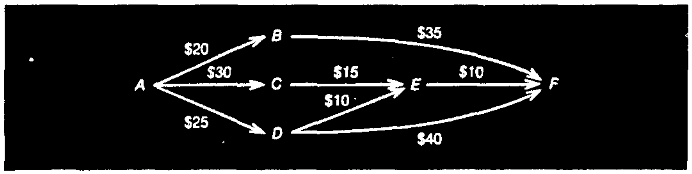
Component	Remarks
Airframes:	
Type 1	*If* a type 3 airframe is selected,
Type 2	*then* a type *B* engine must also be
Type 3	selected.
Engines:	
Type *A*	
Type *B*	

Note that if a type 3 airframe is selected, x_3 will have a value of 1. To satisfy the constraint in this case, x_b will also have to be assigned a value of 1, so that $x_b - x_3$ will equal zero. Note, also, that if a type 3 airframe is *not* selected ($x_3 = 0$), then the constraint will be satisfied if $x_b = 0$ or if $x_b = 1$; this means that when a type 3 airframe is not selected, a type *B* engine may or may not be selected (for use with either a type 1 or a type 2 airframe).

Example 7.4 introduces a multiple-choice problem that requires if-then constraints.

EXAMPLE 7.4

If-Then Constraints in a Logistics Problem The Pratt Corporation must ship a large gas turbine from City *A* to City *F* and can select any of the routes that are shown on the accompanying map. On the map, the numbers indicate transportation costs along the route segments. The firm wants to select a route that will minimize total costs.



REQUIRED

1. Why is this problem a multiple-choice problem? Give some examples of the if-then requirements that are implicit.
2. Verbally state an appropriate integer programming model.
3. Provide an algebraic formulation of the model.

SOLUTION

1. The problem is a multiple-choice problem because, for example, decision makers must select one of three available route segments leaving City *A*. *If*, for example, the route segment from *A* to *B* is selected, *then* the route segment from *B* to *F* must also be selected. This is an example of an if-then requirement.

2. Minimize the total cost of the route segments that are selected

subject to:

(a) One of the three route segments out of City *A* must be selected
(b) If the $A \rightarrow B$ route segment is selected, then the $B \rightarrow F$ segment must also be selected
(c) If the $A \rightarrow C$ route segment is selected, then the $C \rightarrow E$ segment must also be selected
(d) If the $A \rightarrow D$ segment is selected, then either the $D \rightarrow E$ or the $D \rightarrow F$ segment must be selected
(e) If either the $C \rightarrow E$ or the $D \rightarrow E$ segment is selected, then the $E \rightarrow F$ segment must be selected

3. Let zero-one integer-valued variables of the form x_{ij} indicate whether or not a route segment from City *i* to City *j* is selected.

Minimize $z = 20x_{ab} + 30x_{ac} + 25x_{ad} + 35x_{bf} + 15x_{ce} + 10x_{de} + 10x_{ef} + 40x_{df}$

subject to:
(a) $x_{ab} + x_{ac} + x_{ad} = 1$
(b) $x_{ab} = x_{bf}$ or $x_{ab} - x_{bf} = 0$
(c) $x_{ac} = x_{ce}$ or $x_{ac} - x_{ce} = 0$
(d) $x_{ad} = x_{de} + x_{df}$ or $x_{ad} - x_{de} - x_{df} = 0$
(e) $x_{ce} + x_{de} = x_{ef}$ or $x_{ce} + x_{de} - x_{ef} = 0$
(all variables are zero-one integer-valued variables) ■

Business Applications of Fixed-Charge Models

Suppose that the XYZ Corporation needs an additional \$200,000 to finance a new venture. The firm can obtain the required funds in any one of, or any combination of, the following ways:

1. The firm can borrow money from a bank at an interest rate of 15%; the loan will be paid off at the end of the year with interest.
2. The firm can sell additional shares of stock. If the firm sells stock, it will incur a fixed attorney's fee of \$5000 for registering the stock issue with the Securities and Exchange Commission, plus clerical expenses in the amount of \$0.02 per dollar of stock sold. The \$5000 attorney's fee is called a fixed charge.

A fixed charge is a cost that is incurred if more than zero units of something is produced or sold. The fixed charge does not increase proportionally with the units of something that is produced or sold.

A ***fixed charge*** is a cost that is incurred if more than zero units of something is produced or sold. The fixed charge does not increase proportionally with the units of something that is produced or sold. Figure 7.2 shows how the fixed charge influences the total cost of selling stock (the cost of a bank loan is *not* represented in the figure).

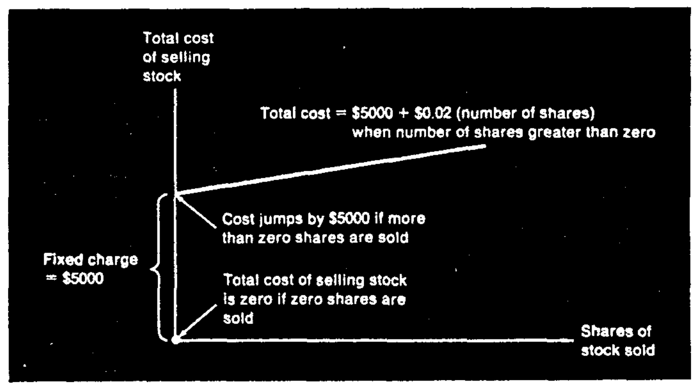


FIG. 7.2

We would like to develop an expression that represents the total cost of obtaining funds by borrowing from the bank and/or by selling stock. For this purpose, let

x_1 = dollars borrowed from the bank at 15% interest
x_2 = dollars obtained by selling stock

An *incorrect* expression is shown; why is the expression invalid?

$$\text{Total cost} = \underbrace{0.15x_1}_{\text{Cost of funds from bank}} + \underbrace{5000 + 0.02x_2}_{\text{Cost of funds from issue of stock}} \qquad \text{Incorrect expression}$$

The expression is incorrect because even if no stock is sold ($x_2 = 0$), the total cost will still include the \$5000 attorney's fee. If $x_2 = 0$, then the \$5000 fixed charge should not be included.

To model this situation correctly, it is necessary to introduce a new zero-one integer-valued variable, which will be represented by the symbol, Δ.* A correct expression for total cost can now be stated as

$$\text{Total cost} = 0.15x_1 + 5000\Delta + 0.02x_2 \qquad \text{Correct expression}$$

where $\Delta = 0$, if $x_2 = 0$
$\Delta = 1$, if $x_2 > 0$

The overall problem facing the XYZ Corporation is to determine how many dollars to borrow from the bank (x_1) and how many dollars to obtain by selling stock (x_2) in order to obtain the additional \$200,000 needed. The firm would like to minimize the total cost of the funds. The problem can be verbally stated as: find values for x_1, x_2, and Δ to

Minimize total cost

subject to:

(a) Funds borrowed from bank plus funds from stock sale = \$200,000
(b) Δ = 0, if funds from stock sale = 0
Δ = 1, if funds from stock sale > 0
(Δ is a zero-one integer-valued variable)

The objective function and constraint (a) are easily converted to algebraic expressions. Constraint (b) requires a clever formulation trick, that will guarantee that $\Delta = 1$ when $x_2 > 0$. The model is shown below.

Minimize $z = \$0.15x_1 + \$5000\Delta + \$0.02x_2$

subject to: (a) $x_1 + x_2 = 200{,}000$
(b) $200{,}000\Delta - x_2 \geq 0$
(Δ is a zero-one integer-valued variable)

Note that by constraint (a), the largest possible value for x_2 is 200,000. By constraint (b), if x_2 takes on any value greater than 0 (but not exceeding 200,000),

* In lieu of Δ, a symbol such as x_3 could also be used to represent the zero-one integer-valued variable. The symbol Δ was chosen so that the zero-one variable will be especially prominent in the model.

Δ will have to assume a value of 1 to satisfy the constraint. This means that if any stock is sold, Δ will be forced to have a value of 1, and the fixed charge of \$5000 will be incurred in the objective function.

Next, suppose that in an optimal solution, $x_2 = 0$. When $x_2 = 0$, constraint (b) can be satisfied if Δ has either a value of 0 or 1. Which of these two feasible values will be assigned to Δ in the optimal solution? Remember that the integer programming algorithm searches for values that *minimize z*. Given a choice of assigning either a value of 0 or a value of 1 to Δ, the algorithm will assign a value of 0 so that z will have the smallest possible value. Thus, if no stock is sold ($x_2 = 0$), Δ will be assigned a value of 0 and the fixed charge will not be incurred. This technique will work whenever the objective is to be *minimized*.

KEY CONCEPT

If an objective function includes a fixed-charge expression, such as

Minimize $z = k_1\Delta + k_2x + \cdots$

where k_1 = a fixed charge
Δ = a zero-one integer-valued variable that is to equal 1 when $x > 0$
k_2 = a variable cost

add a constraint of the form

$M\Delta - x \geq 0$

where M is a constant that has a value at least as large as the largest possible value of x. The constraint will force Δ to have a value of 1 whenever x has a value that is greater than 0.

EXAMPLE 7.5

Fixed Charges in Production Scheduling The Acme Press has contracted to print 400,000 pages for an advertising brochure. The pages can be printed on any or all of three presses. The job is a special rush order that must be completed before a noon deadline. Data are shown below.

Press	Cost per page	Number of pages that can be printed before noon deadline
1	\$0.05	200,000
2	0.04	250,000
3	0.02	300,000

Press number 3 is a high-speed press that is currently used for single-color printing. Because the advertising is to be printed in three colors, press 3 will have have to be cleaned and set up for a three-color run. The fixed cost of setting up press 3 for a three-color run is \$200.

REQUIRED

1. Provide a verbal statement of an integer programming model that can be used to determine the number of pages to print on each press.

2. Provide an algebraic formulation of the model. Let M represent a constant that will be used in a constraint to force a zero-one variable to have a value of 1 if (in the optimal solution) more than zero pages are printed on press 3.
3. What is the maximum number of pages that might be printed on press 3? What is the smallest suitable value for the constant M?

SOLUTION

1. Minimize the total cost of pages printed on each press, including the \$200 fixed setup charge for press 3 (if press 3 is used)

   subject to:

   (a) The total number of pages to be printed = 400,000
   (b) The number of pages printed on press 1 ≤ 200,000
   (c) The number of pages printed on press 2 ≤ 250,000
   (d) The number of pages printed on press 3 ≤ 300,000
   (e) The fixed setup cost is incurred if more than zero pages are printed on press 3

2. Let

x_1 = pages to be printed on press 1

x_2 = pages to be printed on press 2

x_3 = pages to be printed on press 3

Δ = a zero-one integer-valued variable

where $\Delta = 1$ if $x_3 > 0$ and $\Delta = 0$ if $x_3 = 0$

Minimize $z = 0.05x_1 + 0.04x_2 + 200\Delta + 0.02x_3$

subject to:
(a) $x_1 + x_2 + x_3 = 400{,}000$
(b) $x_1 \le 200{,}000$
(c) $x_2 \le 250{,}000$
(d) $x_3 \le 300{,}000$
(e) $M\Delta - x_3 \ge 0$
(x_1, x_2, x_3 are to be integer valued; Δ is to be a zero-one integer-valued variable)

3. By constraint (d), the maximum number of pages that can be printed on press 3 is 300,000. In constraint (e), M must have a value that is at least as large as the largest possible value for x_3 (300,000); therefore, the value of M must be no smaller than 300,000. ■

Models for Business Problems with Vanishing Restrictions

Consider a marketer who is contemplating the introduction of a product in Sweden, Australia, and Peru. Also, suppose that Peruvian authorities require foreign businesses to post a bond for \$200,000 if they want to conduct business in Peru.

In the initial planning stage, the marketer must pay attention to the requirement of posting a bond, but it must also be recognized that this restriction vanishes if it is subsequently decided not to introduce the product in Peru. This section presents a method for including constraints in a model with provisions for making the constraints nonrestrictive (in effect, the constraints "vanish") if, in the optimal solution, choices are made that do not require the constraint to hold.

To illustrate the modeling technique, which is similar to the technique that was introduced for the fixed-charge problem, suppose that a real estate executive is considering the development of a tract of land. The site will hold up to a total of 500 houses and/or luxury apartments. A local zoning law requires the number of apartments to be less than or equal to the number of houses, but this restriction is waived if the developer contributes \$100,000 for school expenses that are not equally shared by apartment dwellers and home owners. In other words, the constraint, "number of apartments ≤ number of houses," can "vanish," or become nonrestrictive if the developer contributes \$100,000 to the school board. The developer anticipates a profit of \$500 for each house and \$650 for each luxury apartment.

A verbal statement of the real estate executive's problem is: determine how many houses to build and how many apartments to build and whether or not a donation should be made to the school board to

Maximize total profits, less donation (if made)

subject to:
(a) Number of houses plus number of apartments ≤ 500
(b) Number of apartments ≤ number of houses
where this constraint is to vanish if the donation is made

KEY CONCEPT

Constraints can be made to vanish by augmenting the constraint with the term, $M\Delta$, where M is a number and Δ is a zero-one integer-valued variable. Depending on the sign (+ or −) of M and whether or not Δ has a value of 1, the augmented constraint will either impose a restriction on the solution or allow other variables to assume values without the restriction.

To provide an algebraic formulation of the model, define the following variables:

x_1 = number of houses to be built

x_2 = number of apartments to be built

Δ = a zero-one integer-valued variable, where $\Delta = 1$ means that a donation is made

The model is

Maximize $z = \$500x_1 + \$650x_2 - \$100{,}000\Delta$

subject to:
(a) $x_1 + x_2 \le 500$
(b) $x_2 \le x_1 + 500\Delta$ or $-x_1 + x_2 - 500\Delta \le 0$
(x_1 and x_2 to be integer valued; Δ is a zero-one integer-valued variable)

Constraint (b) includes a formulation trick. If the donation is not made, $\Delta = 0$, and the constraint $x_2 \le x_1 + 500\Delta$ reduces to $x_2 \le x_1 + 500(0)$, or $x_2 \le x_1$. This is precisely the condition that must be satisfied if the donation is not made. On the other hand, suppose that $\Delta = 1$; in this case, the constraint reduces to $x_2 \le x_1 + 500(1)$, or $x_2 \le x_1 + 500$. Even if no houses are built ($x_1 = 0$), the constraint will allow x_2 to take on a value up to 500; that is, the restriction that $x_2 \le x_1$ vanishes when $\Delta = 1$.

TABLE 7.6

Product	Pounds required per gallon of product		Profit per gallon of product
	Material *A*	Material *B*	
1	5	20	$30
2	15	4	25
Pounds of material available	100	300	

EXAMPLE 7.6

Insurance Tacoma Chemicals uses materials *A* and *B* to produce products 1 and 2. Details are shown in Table 7.6. Product 1 is highly explosive, and an existing insurance contract requires that no more than 5 gallons of product 1 be produced; however, this restriction will be waived if the firm purchases supplemental insurance at a cost of $400.

REQUIRED

1. Provide a verbal statement of an integer programming model that can be used to determine the amount of each type of product that should be produced and whether or not the supplemental insurance should be purchased.
2. Let x_1 = gallons of product 1 to be produced, x_2 = gallons of product 2 to be produced, Δ = a zero-one integer-valued variable, where $\Delta = 1$ means that the supplemental insurance is to be purchased. M will represent a positive-valued constant. Provide an algebraic formulation of the model.
3. What is the smallest appropriate value for M?

SOLUTION

1. Maximize total profits less the cost of insurance (if purchased)

   subject to:

   (a) The amount of material *A* used in products 1 and 2 ≤ 100
   (b) The amount of material *B* used in products 1 and 2 ≤ 300
   (c) $x_1 \leq 5$, *but this restriction is to vanish if the supplemental insurance is purchased*

2. Maximize $z = \$30x_1 + \$25x_2 - \$400\Delta$

   subject to: (a) $5x_1 + 15x_2 \leq 100$
   (b) $20x_1 + 4x_2 \leq 300$
   (c) $x_1 \leq 5 + M\Delta$ or $x_1 - M\Delta \leq 5$
   (Δ is a zero-one integer-valued variable)

3. Suppose that only product 1 is produced ($x_2 = 0$). In this case, constraint (a) reduces to $5x_1 + 0 \leq 100$, or $x_1 \leq 20$. Constraint (b) would become $20x_1 + 0 \leq 300$, or $x_1 \leq 15$. Accordingly, the largest possible value for x_1 that will not violate either constraint (a) or constraint (b) is $x_1 = 15$. The value for M in constraint (c) must be big enough to allow x_1 to have a value as large as 15, when $\Delta = 1$. The smallest value for M that will meet this requirement is $M = +10$. ■

Exercises

7.1 Types of Integer Programming Models

(a) In Example 7.2, a model is presented for a retailing application. Is this model a pure-integer programming model? Is it a zero-one integer programming model?
(b) In Example 7.6, a model is presented for an insurance application. Is this model a zero-one integer programming model? If not, what type of model (mixed or pure) is it?

7.2 Types of Integer Programming Models

(a) In Example 7.3, a model is presented for selecting multinational investments. What type (mixed, pure, and/or zero-one) of integer programming model is it?
(b) In Example 7.5, a model is presented for a production-scheduling application. What type (mixed, pure, and/or zero-one) of integer programming model is it?

7.3 Auditing Teams

Kronkhauser, Golden, and Smith, a public accounting firm, has received a request to audit the estate of a prominent individual who is being considered for appointment to a sensitive governmental position. The managing partner would like to assemble an auditing team that has maximum total auditing experience. The auditing team will travel by a private business jet that has a passenger-load capacity of 1000 pounds. Staff accountants who are available for assignment to the audit team are shown below.

Staff accountant	Approximate weight, pounds	Auditing experience, years
Linda Nelson	105	1.2
Susan Mayo	120	5.8
Karen Dubronski	130	6.5
George Oswald	150	3.2
William Masterson	170	10.0
Donald Crowder	185	0.5
John Zushinski	210	2.5
Roger Kramer	214	14.0

Formulate a knapsack model for determining who will become a member of the auditing team.

7.4 Logistics

National Transit operates buses between major cities and carries commercial packages on a space-available basis. A departing bus has room for up to 350 cubic feet of packages. In addition, the packages that are included cannot exceed a total weight of 700 pounds. Packages awaiting shipment are described in the table.

Package	Volume, cubic feet	Weight, pounds
1	40	60
2	25	100
3	130	250
4	180	120
5	200	340

Formulate a knapsack-type model for selecting packages to be included in the shipment. National Transit would like to maximize the total number of packages shipped.

7.5 New-Product Development

A marketer must select individuals from the following list for inclusion on a new-product development task force.

Person	Name	Experience
1	Sue Smith	1.5
2	Jane Jones	2.0
3	Carol Barns	5.0
4	Thomas Jones	2.0
5	William Dullis	2.5
6	Robert Mayo	3.6

Let x_i represent zero-one integer-valued variables, where $x_i = 1$ means that person i will be appointed to the task group.

(a) Provide an algebraic constraint for the condition: The task group must consist of at least five individuals.
(b) Provide an algebraic constraint for the condition: No more than two males are to be on the task group.
(c) Provide an algebraic constraint for the condition: The total experience of the individuals selected for the task group must be at least 12 years.
(d) Provide an algebraic constraint for the condition: The task group must not include two individuals having the same last name.
(e) Provide an algebraic constraint for the condition: If Sue Smith is appointed to the task group, then William Dullis and/or Robert Mayo must be appointed to the task group.

7.6 Investment Portfolios

A financial advisor has brought the following investments to your attention.

Investment	Type	Cost
1	stock	$100
2	stock	150
3	stock	200
4	bond	100
5	bond	100
6	annuity	300

Let x_i represent zero-one integer-valued variables, where $x_i = 1$ means that investment i will be accepted for your portfolio.

(a) Provide an algebraic constraint for the condition: At least three investments must be accepted.
(b) Provide an algebraic constraint for the condition: The cost of all accepted investments must not exceed $800.
(c) Provide an algebraic constraint for the condition: If investment 1 is accepted, investment 2 must *not* be accepted.

(d) Provide algebraic constraints for the condition: If investment 1 is accepted, then none of the bonds can be accepted. *Hint:* Use two constraints to model the condition.
(e) Provide an algebraic constraint for the condition: The total cost of all stocks that are accepted must not be less than the total cost of all bonds and annuities that are accepted.

7.7 Product Development

Deutsches Autoverken is planning to introduce a new low-fuel-consumption taxicab that is to be of minimal total cost. Some design alternatives are described in the table.

Component	Cost	Remarks
Engines:		
Diesel	$225	Not compatible with type *A* suspension
Electric	180	Not compatible with type 1 or 2 transmissions
Rotary	300	
Transmissions:		
Type 1	$150	
Type 2	200	
Type 3	200	Not compatible with type *B* suspension
Suspensions:		
Type *A*	$180	
Type *B*	230	

Formulate a multiple-choice integer programming model for coordinating the choice of an engine, a transmission, and a suspension.

7.8 Hotel Management

The banquet manager for the Aristocrat Hotel must select a soup, a salad, and an entree for a forthcoming banquet. The total cost of items selected must not exceed $3.25 per serving. In addition, the manager would like to maximize the total popularity rating of the items selected.

Item	Cost per serving	Popularity rating	Remarks
Soups:			
A	$0.25	15	Not compatible with salad *C*
B	0.40	40	
Salads:			
C	$0.50	35	
D	0.75	40	Not compatible with entree *F*
Entrees:			
E	$2.00	20	
F	2.50	70	
G	2.25	40	

Formulate a multiple-choice integer programming model for optimizing the banquet menu.

7.9 Fixed Charges in Production Scheduling

In Example 7.5, a model is presented for fixed charges in production scheduling. After examining the model, the plant manager (who is occasionally wrong) declared, "The optimal solution is to print 150,000 pages on press 2 and 250,000 pages on press 3."

(a) Given the plant manager's "optimal" solution, what, according to constraint (e), is a feasible value for Δ?
(b) Is the plant manager's solution a *feasible* solution?
(c) Given the plant manager's "optimal" solution, what will be the value of the objective, z?
(d) Has the manager really found the optimal solution?

7.10 Fixed Charges in Production Scheduling

In Example 7.5, a model is presented for fixed charges in production scheduling. After examining the model, the assistant plant manager (who is frequently wrong) declared, "The optimal solution is to print 250,000 pages on press 2, and the remaining pages should be printed on press 1."

(a) Given the assistant plant manager's "optimal" solution, what value(s) for Δ will *satisfy* constraint (e)?
(b) Given the assistant plant manager's solution, what value will be assigned to Δ? *Hint*: Recall that costs are to be minimized.
(c) Has the assistant plant manager really found the optimal solution?

7.11 Real Estate

Ourco Enterprises plans to obtain enough land, in one or more locations, to construct 1500 vacation homes. Data on available tracts of land are shown in the table.

Tract	Cost of tract (in thousands)	Number of homes that can be accommodated	Average cost/home for development* (in thousands)
1	$150	700	$0.8
2	200	500	1.0
3	300	1200	0.4

* Cost of installing roads and utilities.

Formulate a fixed-charge model for determining the number of vacation homes to be built on each tract. Ourco wants to minimize the total costs for tract acquisition and development.

7.12 Energy Management

Alpha Corporation is planning a new production facility that will produce 5000 small metal components each week for a recently won defense contract. The contract requires management to employ energy conservation methods. Three different heat-treating furnaces are available.

Furnace	BTUs needed to bring furnace up to temperature	BTUs required to heat-treat one component	Maximum capacity: components/week
1	50,000	60	3000
2	40,000	100	4000
3	20,000	200	6000

At the end of each week the furnaces are shut down for the weekend and brought back up to heat-treating temperature the following Monday morning.

Formulate a model for determining the number of components to heat-treat in each type of furnace each week to minimize total energy (British thermal units) consumption.

7.13 Stock Market Transactions

The following stocks are listed on the Pottsville Stock Exchange:

Stock	Price per share	Annual dividend
1	$100	$ 5
2	150	10
3	200	12

An investor has $100,000 that can be used to purchase stocks and pay commissions required by the Pottsville Stock Exchange. The Pottsville Stock Exchange charges a commission of 1% of the value of the shares purchased, plus a fixed fee of $100 if fewer than 800 shares are purchased. The $100 fixed fee is waived if 800 or more shares are purchased.

The investor must purchase enough shares of various stocks to provide an annual dividend income of at least $5000 and would like to satisfy this investment goal by purchasing quantities of each stock in a manner that minimizes the commissions and fees that are paid to the Pottsville Stock Exchange.

Develop a suitable model. *Hint*: Let x_1, x_2, and x_3 represent, respectively, the number of shares of stock 1, 2, and 3 that are to be purchased. Let Δ be a zero-one integer-valued variable that is to have a value of 1 when the total number of shares purchased is less than 800. Consider using a constraint of the form $x_1 + x_2 + x_3 \geq 800 - M\Delta$ in your model, where M represents a constant. You are to provide a suitable numerical value for M.

7.14 Insurance

Example 7.6 provides a model in which Tacoma Chemicals could produce no more than 5 gallons of product 1 unless the firm purchased supplemental insurance at a cost of $400.

Now suppose that the insurance company informs Tacoma that it must purchase one supplemental insurance policy for $400 if it produces more than 5 gallons of product 1 and another supplemental policy, costing $250, if it produces more than 2 gallons of product 2. Revise the model presented in Example 7.6.

Integer Programming Additional Problems

1. Solve the following integer programming problem:

$$\text{MIN } C = 3X + 4Y + 8Z$$
$$X + 2Y + 4Z \geq 5$$
$$2X + 4Y + 6Z \geq 6$$
$$7X + 9Y + 6Z \geq 14$$

2. The Esterly Corp. needs to build two Western region warehouses to better serve its customers in that area. The construction cost (in millions of dollars) and the average yearly delivery delay cost (in thousands of dollars per year) are shown in the following table:

Warehouse Location	Construction Cost	Delivery Delay Cost
Denver	4.1	150
Los Angeles	4.9	100
Phoenix	3.2	170
San Francisco	5.1	110
Seattle	3.6	180

Which two locations should be selected to minimize the total yearly delivery delay cost and not exceed a construction budget of $9 million?

3. The Eastern sales region must assign three sales managers to three sales offices. Its objective is to find the assignments that maximize the total yearly sales of all three offices. Naturally, only one person can be assigned to each sales office. The expected yearly sales (in millions of dollars) if each individual is assigned to each office are as follows:

	Albany	Sales Office Boston	Chicago
Smith	20	23	17
Turner	15	16	14
Unger	17	19	16

The relocation expense budget for all three moves is $200,000. The costs (in thousands of dollars) of relocating each individual to each location are as follows:

	Albany	Sales Office Boston	Chicago
Smith	65	50	40
Turner	80	65	70
Unger	90	70	80

which individual should you assign to each sales office?

Integer Programming
Additional Problems

4 A presidential candidate must decide which states to visit in the 10 days before the election. Naturally the goal is to increase the nı ber of votes by the largest possible amount. The relevant data ar follows:

State	Vote Increase by Visit	Days Required for Visit
A	100,000	4
B	20,000	3
C	40,000	3
D	90,000	4
E	30,000	3
F	10,000	1

a) Which states should be visited?
b) How many votes will be generated by these visits?

5 A presidential candidate is down to the last five days of the campa and the race is close. Only $300,000 remains in the campaign bud Three key states appear likely to swing the election one way or other. Each state can be visited, or a saturation TV ad series caı purchased. The candidate's staff has made up the following estima

State	Action	Vote Increase	Days Required	Cost
J	Visit	100,000	4	200,000
J	Ads	50,000	0	100,000
K	Visit	80,000	4	150,000
K	Ads	40,000	0	90,000
L	Visit	20,000	1	45,000
L	Ads	15,000	0	30,000

In no case will both the ad series and a visit be scheduled in the s. state.

a) Which states should be visited?
b) In which states should the ad series be purchased?
c) How many votes will be generated by these visits and ads?

Integer Programming
Additional Problems

6 Anderson Industries, a leading manufacturer of car stereo components, is considering locating several new assembly plants in the Southwest. These new plants must have a combined production capacity of at least 450,000 units per year and must have a combined operating cost that does not exceed $7.5 million per year. The selection team has narrowed the list to four sites in Phoenix, Tucson, Santa Fe, and El Paso. Anderson's management has established a construction cost limit of $45 million for the project. This money will be used to build the combination of plants that provides the maximum yearly profit.

Any one of three different-sized plants can be built on the Phoenix site, but only one size plant can be built on each of the other sites. All estimates of production capacity, yearly operating cost, yearly profit, and construction cost are as follows:

Plant Site Location	Production Capacity (units/yr)	Operating Cost ($/yr)	Yearly Profit ($/yr)	Construction Cost ($)
Phoenix				
Size 1	250,000	4,000,000	400,000	25,000,000
Size 2	225,000	3,500,000	350,000	22,000,000
Size 3	200,000	3,000,000	300,000	19,000,000
Tucson	200,000	3,000,000	325,000	22,000,000
Santa Fe	200,000	2,500,000	325,000	20,000,000
El Paso	225,000	3,000,000	275,000	23,000,000

a) Which sites should be selected?

b) If the Phoenix site is selected, which size plant should be constructed?

c) How much yearly profit results from this solution?