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Stochastic Comparisons and Dependence among Concomitants of Order Statistics

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Let (X_i, Y_i) $i=1, 2, ..., n$ be n independent and identically distributed random variables from some continuous bivariate distribution. If $X_{(r)}$ denotes the rth ordered X-variate then the Y-variate, $Y_{[r]}$, paired with $X_{(r)}$ is called the concomitant of the rth order statistic. In this paper we obtain new general results on stochastic comparisons and dependence among concomitants of order statistics under different types of dependence between the parent random variables X and Y. The results obtained apply to any distribution with monotone dependence between X and Y. In particular, when X and Y are likelihood ratio dependent, it is shown that the successive concomitants of order statistics are increasing according to likelihood ratio ordering and they are TP_2 dependent in pairs. If we assume that the conditional hazard rate of Y given $X=x$ is decreasing in x, then the concomitants are increasing according to hazard rate ordering and are dependent according to the right corner set increasing property. Finally, it is proved that if Y is stochastically increasing in X, then the concomitants of order statistics are stochastically increasing and are associated. Analogous results are obtained when the variables X and Y are negatively dependent. We also prove that if the hazard rate of the conditional distribution of Y given $X=x$ is decreasing in x and y, then the concomitants have DFR (decreasing failure rate) distributions and are ordered according to dispersive ordering. © 2000 Academic Press

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1. INTRODUCTION

Let $(X_1, Y_1), ..., (X_n, Y_n)$ be a random sample of size n from a continuous bivariate distribution. If we arrange the X's in ascending order as $X_{(1)} \leqslant X_{(2)} \leqslant \cdots \leqslant X_{(n)}$ then the Y's associated with these order statistics

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are denoted by $Y_{[1]}, Y_{[2]}, ..., Y_{[n]}$ and are called concomitants of order statistics. They are also known as induced order statistics in the literature. The concomitants are of interest in selection and prediction problems based on the ranks of the X's. For example, when k $(<n)$ individuals having the highest X-scores are selected, we may wish to know the behavior of the corresponding Y-scores. They are also of interest in a variety of estimation problems (see Bhattacharya, 1984, for details). Throughout this paper *increasing* means *nondecreasing*, and *decreasing* means *nonincreasing*. We assume that expectations are well defined and multiple integrals can be evaluated irrespective of order.

Let $f(y \mid x)$ denote the conditional pdf of Y given $X=x$ and let $f_{r_1, ..., r_k}(x_1, ..., x_k)$ denote the joint pdf of $X_{(r_1)}, ..., X_{(r_k)}$ with $1 \leqslant r_1 \leqslant \cdots \leqslant r_k \leqslant n$. Then, as discussed in Yang (1977), the joint pdf of the k-concomitants $(Y_{[r_1]}, ..., Y_{[r_k]})$ $(1 \leqslant k \leqslant n)$ is

$$f_{Y_{[r_1]}, ..., Y_{[r_k]}}(y_1, ..., y_k)$$
$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{x_k} \cdots \int_{-\infty}^{x_2} \prod_{i=1}^{k} f(y_i \mid x_i)\, f_{r_1, ..., r_k}(x_1, ..., x_k) \prod_{i=1}^{k} dx_i. \tag{1.1}$$

From this we obtain the marginal pdf of the rth concomitant $Y_{[r]}$ as

$$f_{Y_{[r]}}(y) = \int_{-\infty}^{+\infty} f(y \mid x)\, f_r(x)\, dx, \tag{1.2}$$

where f_r is the density of $X_{(r)}$.

Under the assumption that X and Y are linearly related, apart from an independent error term, the small sample theory of the concomitants of order statistics has been discussed in David (1973), O'Connell (1974), and Kim and David (1990). The asymptotic distribution theory for the bivariate normal distribution has been investigated by David and Galambos (1974). Bhattacharya (1974), Sen (1976), and Yang (1977) obtained results for the general asymptotic distribution theory of concomitants of order statistics. For a comprehensive review of this topic see Bhattacharya (1984) and David and Nagaraja (1998).

In this paper we consider the problems of stochastic comparisons and dependence among concomitants of order statistics. Intuitively, it is clear that when X and Y are positively (negatively) dependent, the $Y_{[i]}$'s should be increasing (decreasing) in some stochastic sense. There are several notions of stochastic ordering and dependence among random variables with varying degree of strength. By assuming different kinds of dependence between X and Y, we obtain various types of stochastic ordering and dependence results among the $Y_{[i]}$'s. The results obtained are general in the sense that they apply to any bivariate distribution with monotone dependence between the variables X and Y.

In the next section, we briefly review the various notions of stochastic ordering and dependence that will be used later on in this paper. In Section 3, we consider the stochastic orderings among concomitants of order statistics and in Section 4 we study their dependence properties under various types of dependence between X and Y. Concluding remarks are included in Section 5.

2. PRELIMINARIES

Let X and Y be random variables with distribution functions F and G, survival functions $\bar{F}$ and $\bar{G}$ density functions f and g, and hazard rates r_F $(=f/\bar{F})$ and r_G $(=g/\bar{G})$, respectively. X is said to be *stochastically* smaller than Y (denoted by $X \leqslant_{st} Y$) if $\bar{F}(x) \leqslant \bar{G}(x)$ for all x. A stronger notion of stochastic dominance is that of *hazard rate* ordering. X is said to be smaller than Y in hazard rate ordering (denoted by $X \leqslant_{hr} Y$) if $\bar{G}(x)/\bar{F}(x)$ is increasing in x. In case the hazard rates exist, $X \leqslant_{hr} Y$ if and only if $r_G(x) \leqslant r_F(x)$ for every x. X is said to be smaller than Y in *likelihood ratio* ordering (denoted by $X \leqslant_{lr} Y$) if $g(x)/f(x)$ is increasing in x. Finally, X is said to be smaller than Y in *mean residual life* (*MRL*) ordering (denoted by $X \leqslant_{mrl} Y$) if $\int_t^{+\infty} \bar{G}(x)\,dx / \int_t^{+\infty} \bar{F}(x)\,dx$ is increasing in t. In this case $\mu_F(x) \leqslant \mu_G(x)$ for every x, where $\mu_F(x) = E[X - x \mid X > x]$ denotes the mean residual life function of X. Similarly we define $\mu_G(x)$. When the supports of X and Y have a common left end-point, we have the following chain of implications among the above stochastic orders: $X \leqslant_{lr} Y \Rightarrow X \leqslant_{hr} Y \Rightarrow X \leqslant_{st} Y$. Also $X \leqslant_{hr} Y \Rightarrow X \leqslant_{mrl} Y$. For more details on stochastic orderings, see Chapter 1 of Shaked and Shanthikumar (1994).

When confronted with the problem of comparing dependent variables X and Y, Shanthikumar and Yao (1991) introduced the following criteria: Let

$$G_{st} = \{g\colon R^2 \to R : g(x, y) - g(y, x) \text{ increasing in } x\ \forall y\},$$

$$G_{hr} = \{g\colon R^2 \to R : g(x, y) - g(y, x) \text{ increasing in } x\ \forall y \leqslant x\},$$

$$G_{lr} = \{g\colon R^2 \to R : g(x, y) \geqslant g(y, x)\ \forall y \leqslant x\}.$$

Definition 2.1. X is said to be smaller than Y according to

(a) joint stochastic ordering (denoted by $X \overset{st:j}{\leqslant} Y$) if

$$E[g(X, Y)] \leqslant E[g(Y, X)], \tag{2.1}$$

for all $g \in G_{st}$;

(b) joint hazard rate ordering (denoted by $X \overset{hr:j}{\leqslant} Y$) if (2.1) holds for all $g \in G_{hr}$;

(c) joint likelihood ratio ordering (denoted by $X \overset{lr:j}{\leqslant} Y$) if (2.1) holds for all $g \in G_{lr}$.

We have the following chain of implications: $X \overset{lr:j}{\leqslant} Y \Rightarrow X \overset{hr:j}{\leqslant} Y \Rightarrow X \overset{st:j}{\leqslant} Y$.

As pointed out by Shanthikumar and Yao (1991), unless the random variables are independent, neither joint likelihood ratio ordering nor joint hazard rate ordering imply the corresponding usual ordering between their marginal distributions. However, all of these joint orderings imply $X \leqslant_{st} Y$. They have also extended these concepts to the multivariate case. Below we give the extension of the joint likelihood ratio ordering.

Let $\mathbf{x} = (x_1, ..., x_n)$ and $\mathbf{y} = (y_1, ..., y_n)$ be two vectors. We say that $\mathbf{x}$ *is better arranged than* $\mathbf{y}$ ($\mathbf{x} \overset{a}{\geqslant} \mathbf{y}$) if $\mathbf{x}$ can be obtained from $\mathbf{y}$ through successive pairwise interchanges of its components, with each interchange resulting in an increasing order of the two interchanged components. A function $g: \mathscr{R}^n \to \mathscr{R}$ that preserves the ordering $\overset{a}{\geqslant}$ is called an *arrangement increasing* function denoted by $g \in \mathscr{A}\mathscr{I}$ if $\mathbf{x} \overset{a}{\geqslant} \mathbf{y} \Rightarrow g(x) \geqslant g(y)$. See Marshall and Olkin (1979, p. 169) for more discussion of such functions.

Definition 2.2. Let f denote the joint density of $\mathbf{X}$. Then

$$X_1 \overset{lr:j}{\leqslant} X_2 \overset{lr:j}{\leqslant} \cdots \overset{lr:j}{\leqslant} X_n \Leftrightarrow f \in \mathscr{A}\mathscr{I}.$$

Notions of Dependence

There are several notions of positive and negative dependence between random variables and these have been discussed in detail in Barlow and Proschan (1981), Shaked (1977) and Lee (1985a, b). For a brief introduction, see Boland *et al.* (1996). The following concepts will be used later in this paper.

Definition 2.3. We say that a function $h(x, y)$ is *sign regular of order* 2 (SR_2) if $\varepsilon_1 h(x, y) \geqslant 0$ and

$$\varepsilon_2 \begin{vmatrix} h(x_1, y_1) & h(x_1, y_2) \\ h(x_2, y_1) & h(x_2, y_2) \end{vmatrix} \geqslant 0, \tag{2.2}$$

whenever $x_1 < x_2$, $y_1 < y_2$, and $\varepsilon_i \in \{-1, 1\}$ for $i = 1, 2$.

If the above relations hold with $\varepsilon_1 = +1$ and $\varepsilon_2 = +1$ then h is said to be *totally positive of order* 2 (TP_2); and if they hold with $\varepsilon_1 = +1$ and $\varepsilon_2 = -1$ then h is said to be *reverse regular of order* 2 (RR_2).

Let $X_1, ..., X_n$ be random variables with joint distribution function F and density f. For $s>0$, let $\gamma^{(s)}(t)$ be defined as follows:

$$\gamma^{(s)}(t)=\begin{cases}(-t)^{s-1}/\Gamma(s) & \text{if} \quad t\leqslant 0\\ 0 & \text{if} \quad t>0.\end{cases}$$

Define the n-fold integral $\psi_{k_1, ..., k_n}$ by

$$\psi_{k_1, ..., k_n}(x_1, ..., x_n)=\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{i=1}^{n} \gamma^{(k_i)}(x_i-t_i)\, dF(t_1, ..., t_n)$$

and define $\psi_{0, ..., 0}=f$. Also define $\psi_{0, ..., 0, k_{i+1}, ..., k_n}$ to be the $(n-i)$-fold integral

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{j=i+1}^{n} \gamma^{(k_j)}(x_j-t_j)\, g_i(x_1, ..., x_i)\, dF(t_{i+1}, ..., t_n \mid x_1, ..., x_i),$$

where g_i is joint density of $(X_1, ..., X_i)$ and $F(t_{i+1}, ..., t_n \mid x_1, ..., x_i)$ is the conditional distribution function of $(X_{i+1}, ..., X_n)$ given $X_1=x_1, ..., X_i=x_i$, for $k_{i+1}>0, ..., k_n>0$. Similarly we can define $\psi_{k_1, ..., k_n}$ with any subset of $\{k_1, ..., k_n\}$ consisting of zeros. Lee (1985a) introduced the following concept of positive dependence for the multivariate case which is an extension of the one studied by Shaked (1977) for the bivariate case:

Definition 2.4. The random vector $(X_1, ..., X_n)$ is said to be dependent by total positivity with degree $(k_1, ..., k_n)$, denoted by $DTP(k_1, ..., k_n)$, if $\psi_{k_1, ..., k_n}(x_1, ..., x_n)$ is TP_2 in pairs of $\{x_1, ..., x_n\}$.

The corresponding concept of negative dependence was introduced by Lee (1985b).

Definition 2.5. We say that (X, Y) is dependent by reverse regular of degree k_1 and k_2, denoted by $DRR(k_1, k_2)$, if $\psi_{k_1, k_2}(x, y)$ is RR_2.

As pointed out by Shaked (1977), two random variables X and Y are likelihood ratio (or TP_2) dependent if and only if X and Y are $DTP(0, 0)$ dependent. They are $DTP(0, 1)$ ($DRR(0, 1)$) dependent if the conditional hazard rate of Y given $X=x$, $r(y \mid x)$, is decreasing (increasing) in x. The random variables X and Y are $DTP(1, 1)$ dependent if the joint survival function $\bar{F}(x, y)=P[X>x, Y>y]$ of (X, Y) is TP_2. In this case the random variables X and Y are also said to be *right corner set increasing* ($RCSI$). The random variables X and Y are $DTP(0, 2)$ ($DRR(0, 2)$) dependent on whether the conditional mean residual life function of Y given $X=x$, $\mu(y \mid X=x)$, is increasing (decreasing) in x. We say that Y is stochastically increasing (decreasing) in X (denoted by $SI(Y \mid X)$)

$$
\begin{tikzcd}
 & & TP_2(X,Y) \arrow[d] & & \\
 & & r(y|x)\downarrow x \arrow[dll] \arrow[d] \arrow[drr] & & \\
\mu(y|X=x)\uparrow x \arrow[dr] & & SI(Y|X) \arrow[dl] \arrow[drr] & & RCSI(X,Y) \arrow[d] \\
 & E(Y|X=x)\uparrow x & & & A(X,Y)
\end{tikzcd}
$$

FIG. 1. Implications among notions of positive dependence.

($SD(Y \mid X)$) if $P[Y > y \mid X = x]$ is increasing (decreasing) in x for all y. Two random variables X and Y are said to be associated (denoted by $A(X, Y)$) if $\mathrm{Cov}(u(X, Y), v(X, Y)) \geqslant 0$ for all increasing functions u and v. Figure 1 shows the chain of implications that hold among the above notions of positive dependence. There are many other notions of positive and negative dependence, but we will not be discussing them here. See Karlin and Rinott (1980a, b) for many interesting examples of bivariate distributions which satisfy the above criteria of dependence.

Kim and David (1990) studied the dependence properties of concomitants of order statistics for the model

$$Y = g(X) + Z, \tag{2.3}$$

where X and Z are mutually independent and g is an increasing function. The next theorem establishes dependence of different types between X and Y under various conditions on the distribution of Z.

THEOREM 2.1. *Assume the model given by* (2.3) *with g increasing* (*decreasing*). *Then*

(a) *Y is stochastically increasing* (*decreasing*) *in X*,

(b) *if Z has a log-concave density, then X and Y are TP_2* (RR_2) *dependent*,

(c) *if Z is IFR, then X and Y are $DTP(0, 1)$* ($DRR(0, 1)$) *dependent*,

(d) *if Z is DMRL, then X and Y are $DTP(0, 2)$* ($DRR(0, 2)$) *dependent.*

(*A random variable is said to be DMRL if its mean residual life function is decreasing.*)

Proof. (a) Let f_Z denote the density of Z. Then

$$P[Y>y \mid X=x]=\int_y^{+\infty} f_Z(w-g(x))\,dw$$
$$=\bar{F}_Z(y-g(x)),$$

is increasing (decreasing) in x since g is an increasing (decreasing) function.

(b) As the conditional density of Y given X is $f(y \mid x)=f_Z(y-g(x))$, it follows that the joint density of X and Y is

$$f_{X,Y}(x,y)=f_Z(y-g(x))\,.\,f_X(x).$$

Since f_Z being PF_2 is equivalent to $h(y,x)=f_Z(y-x)$ being TP_2, it follows from Theorems A.3 and A.2 of Marshall and Olkin (1979, p. 488) that $f_{X,Y}$ is TP_2 when g is increasing and RR_2 when g is decreasing.

(c) As seen in the proof of part (a), $\bar{F}(y \mid x)=\bar{F}_Z(y-g(x))$. This is clearly TP_2 (RR_2) in (x, y) as $\bar{F}_Z(y-x)$ is TP_2 if Z is IFR and g is an increasing (decreasing) function.

(d) The proof is similar to that of part (c) and is omitted. ∎

We shall be repeatedly using the following lemma of Karlin (1968, p. 99) in the next sections.

LEMMA 2.1. *Let A, B, and C be subsets of the real line and let $L(x, z)$ be SR_2 for $x \in A$, $z \in B$ and $M(z, y)$ be SR_2 for $z \in B$, $y \in C$. Then $K(x, y)=\int L(x, z)\, M(z, y)\, d\mu(z)$ is SR_2 for $x \in A$, $y \in C$ and $\varepsilon_i(K)=\varepsilon_i(L)\times\varepsilon_i(M)$ $\forall i=1, 2$. Here μ is a sigma-finite measure.*

Thus according to Lemma 2.1 the composition of two TP_2 functions or two RR_2 functions is TP_2 and the composition of a RR_2 function and a TP_2 function is RR_2.

3. STOCHASTIC ORDERINGS AMONG CONCOMITANTS OF ORDER STATISTICS

In this section we consider the problem of stochastically comparing the concomitant $Y_{[i]}$'s under different kinds of dependence between X and Y. It is proved in the next theorem that if Y is stochastically increasing (decreasing) in X, then the concomitant variables $Y_{[i]}$'s are stochastically increasing (decreasing).

THEOREM 3.1.

(a)
$$SI(Y \mid X) \Rightarrow Y_{[i]} \overset{st:j}{\leqslant} Y_{[j]} \Rightarrow Y_{[i]} \leqslant_{st} Y_{[j]} \qquad \textit{for} \quad 1 \leqslant i < j \leqslant n, \tag{3.1}$$

(b)
$$SD(Y \mid X) \Rightarrow Y_{[i]} \overset{st:j}{\geqslant} Y_{[j]} \Rightarrow Y_{[i]} \geqslant_{st} Y_{[j]} \qquad \textit{for} \quad 1 \leqslant i < j \leqslant n. \tag{3.2}$$

Proof. (a) Let g be any element of G_{st}. That is, g is such that

$$g(y_2, y_1) - g(y_1, y_2) \text{ is increasing in } y_2 \ \forall y_1. \tag{3.3}$$

It is enough to show that for every such function g,

$$E[g(Y_{[j]}, Y_{[i]})] - E[g(Y_{[i]}, Y_{[j]})] \geqslant 0. \tag{3.4}$$

The L.H.S. of (3.4) after changing the order of integration is

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{x_2} \left[\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \{g(y_2, y_1) - g(y_1, y_2)\} f(y_1 \mid x_1) f(y_2 \mid x_2) \, dy_1 \, dy_2 \right]$$
$$\times f_{i,j}(x_1, x_2) \, dx_1 \, dx_2$$
$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{x_2} [E(g(Y_2 \mid X_2 = x_2, Y_1 \mid X_1 = x_1))$$
$$- E(g(Y_1 \mid X_1 = x_1, Y_2 \mid X_2 = x_2))]$$
$$\times f_{i,j}(x_1, x_2) \, dx_1 \, dx_2 \tag{3.5}$$

where (X_1, Y_1) and (X_2, Y_2) are two independent copies of (X, Y). Now $SI(Y \mid X)$ implies that for any $x_1 < x_2$, the expression inside the square brackets in (3.5) is non-negative for all $x_1 < x_2$. The required result now follows from this.

The result $Y_{[i]} \leqslant^{st} Y_{[j]}$ for $i < j$ follows from Theorem 4.9 of Shanthikumar and Yao (1991) as joint stochastic ordering between two random variables implies usual stochastic ordering between their marginal distributions.

(b) By the definition of $SD(Y \mid X)$, in this case the expression inside the square brackets in (3.5) is non-positive and hence the inequality in (3.4) will be reversed. ∎

In the next theorem, we make a stronger assumption on the dependence between X and Y and establish hazard rate ordering among the concomitants of order statistics.

THEOREM 3.2. *Let $r(y \mid x)$, the hazard rate of the conditional distribution of Y given $X=x$, be decreasing in x. Then for $1 \leqslant i < j \leqslant n$,*

$$\text{(a)} \quad Y_{[i]} \overset{hr:j}{\preccurlyeq} Y_{[j]},$$
$$\text{(b)} \quad Y_{[i]} \leqslant_{hr} Y_{[j]}.$$

The inequalities in (a) *and* (b) *are reversed in case $r(y \mid x)$ is increasing in x.*

(Note that (b) does not follow from (a) since, as shown in Shanthikumar and Yao, 1991, joint hazard rate ordering may not imply usual hazard rate ordering.)

Proof. (a) We prove the result for $r(y \mid x)$ decreasing in x. The proof is similar when it is increasing in x. We have to prove that under the given condition

$$E[g(Y_{[j]}, Y_{[i]})] - E[g(Y_{[i]}, Y_{[j]})] \geqslant 0 \tag{3.6}$$

for any bivariate function $g \in G_{hr}$, that is, for a function g satisfying $g(y_2, y_1) - g(y_1, y_2)$ increasing in y_2 for $y_2 \geqslant y_1$ and for which the expectations exist. As seen in Theorem 3.1, the L.H.S. of (3.6) is

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{x_2} [E(g(Y_2 \mid X_2=x_2, Y_1 \mid X_1=x_1)) - E(g(Y_1 \mid X_1=x_1, Y_2 \mid X_2=x_2))] \times f_{i,j}(x_1, x_2)\, dx_1\, dx_2, \tag{3.7}$$

where (X_1, Y_1) and (X_2, Y_2) are two independent copies of (X, Y). By the assumption that $r(y \mid x)$ is decreasing in x, $\{Y \mid X=x_2\} \geqslant_{hr} \{Y \mid X=x_1\}$ for $x_1 < x_2$. Hence the expression inside the square brackets in (3.7) is non-negative for $x_1 \leqslant x_2$. The required result follows from this.

(b) The survival function of $Y_{[i]}$ is

$$\bar{F}_{Y_{[i]}}(y) = \int_y^{+\infty} \left[\int_{-\infty}^{+\infty} f(y \mid x) f_i(x)\, dx \right] dy$$
$$= \int_{-\infty}^{+\infty} \bar{F}(y \mid x) f_i(x)\, dx,$$

where f_i is the pdf of $X_{(i)}$. Since the successive order statistics are increasing according to likelihood ratio ordering (cf. Shaked and Shanthikumar, 1994, p. 22), the function $f_i(x)$ is TP_2 in (x, i). Also the survival function $\bar{F}(y \mid x)$ is TP_2 (RR_2) in (x, y) if $r(y \mid x)$ is decreasing (increasing) in x. It follows from Lemma 2.1, that $\bar{F}_{Y_{[i]}}(y)$ is TP_2 or RR_2 in (i, y) depending upon whether $r(y \mid x)$ is decreasing or increasing in x. That is, $Y_{[i]} \leqslant_{hr} (\geqslant_{hr})\ Y_{[j]}$ for $1 \leqslant i < j \leqslant n$ if $r(y \mid x)$ is decreasing (increasing) in x. ▮

In case X and Y are TP_2 (likelihood ratio dependent) or RR_2 dependent, we get the following stronger result on the stochastic monotonicity of the $Y_{[i]}$'s.

THEOREM 3.3. *Suppose that X and Y are TP_2 dependent. Then*

$$\text{(a)} \quad Y_{[i]} \leqslant_{lr} Y_{[j]} \qquad \textit{for} \quad i < j,$$

$$\text{(b)} \quad Y_{[1]} \overset{lr:j}{\leqslant} \cdots \overset{lr:j}{\leqslant} Y_{[n]}.$$

The inequalities in (a) *and* (b) *are reversed in case X and Y are RR_2 dependent.*

Proof. (a) The density function of $Y_{[i]}$ is

$$f_{Y_{[i]}}(y) = \int_{-\infty}^{+\infty} f(y \mid x)\, f_i(x)\, dx.$$

As noted in the previous theorem, the function $f_i(x)$ is TP_2 in (x, i). The TP_2 (RR_2) condition on f is equivalent to $f(y \mid x)$ being TP_2 (RR_2) in (y, x). The required result immediately follows from Lemma 2.1.

(b) First we consider the case when the joint density of (X, Y) is TP_2. We have to prove that under this condition the joint density of $(Y_{[1]}, ..., Y_{[n]})$ is *arrangement increasing*. That is, if $\mathbf{u}$ and $\mathbf{y}$ are vectors of order n such that $\mathbf{u} >_a \mathbf{y}$, then

$$f_{Y_{[1]}, ..., Y_{[n]}}(\mathbf{u}) - f_{Y_{[1]}, ..., Y_{[n]}}(\mathbf{y}) \geqslant 0. \tag{3.8}$$

Clearly the L.H.S. of (3.8) is

$$n! \int_{-\infty}^{+\infty} \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_2} \left[\prod_{i=1}^{n} f(x_i, u_i) - \prod_{i=1}^{n} f(x_i, y_i) \right] \prod_{i=1}^{n} dx_i.$$

Now the function $\prod_{i=1}^{n} f(x_i, y_i)$ is arrangement increasing if $f_{X, Y}(x, y)$ is TP_2 in (x, y) (cf. Marshall and Olkin, 1979, F. 9. (a), p. 163). Therefore,

$$\prod_{i=1}^{n} f(x_i, u_i) \geqslant \prod_{i=1}^{n} f(x_i, y_i) \tag{3.9}$$

for all $\mathbf{x}$ such that $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$. This implies that $f_{Y_{[1]}, ..., Y_{[n]}}(\mathbf{u}) \geqslant f_{Y_{[1]}, ..., Y_{[n]}}(\mathbf{y})$ for all $\mathbf{u} >_a \mathbf{y}$. That is, $f_{Y_{[1]}, ..., Y_{[n]}} \in \mathcal{AI}$. This proves that $Y_{[1]} \overset{lr:j}{\preccurlyeq} \cdots \overset{lr:j}{\preccurlyeq} Y_{[n]}$.

Now consider the case when (X, Y) are RR_2 dependent. To establish the required result we have to prove that (3.9) holds whenever $\mathbf{y} >_a \mathbf{u}$.

Suppose $x_1 \leqslant \cdots \leqslant x_n$, $y_1 \leqslant \cdots \leqslant y_n$. Without loss of generality assume that $\mathbf{u} = (y_2, y_1, ..., y_n)$. Then $\mathbf{y} >_a \mathbf{u}$ and

$$\prod_{i=1}^{n} f(x_i, y_i) - \prod_{i=1}^{n} f(x_i, u_i)$$

$$= \prod_{i=3}^{n} f(x_i, u_i)[f(x_1, y_1) f(x_2, y_2) - f(x_1, y_2) f(x_2, y_1)]. \tag{3.10}$$

By the RR_2 property of (X, Y), the quantity inside the square brackets in (3.10) is non-positive. The rest of the proof follows as in part (a) proving thereby that $Y_{[1]} \overset{lr:j}{\succcurlyeq} \cdots \overset{lr:j}{\succcurlyeq} Y_{[n]}$. ∎

Remark. In the above theorem, (a) does not follows from (b) and vice versa since, as discussed in Shanthikumar and Yao (1991), joint likelihood ratio ordering may not imply the usual likelihood ratio ordering among the marginal distributions of the components of the random vector.

THEOREM 3.4. *Suppose that the conditional mean residual life of Y given $X = x$, $\mu(y \mid x)$, is increasing in x. Then for any $1 \leqslant i < j \leqslant n$, $Y_{[i]} \leqslant_{mrl} Y_{[j]}$. The inequality is reversed in case $\mu(y \mid x)$ is decreasing in x.*

Proof. We give the proof for the case when $\mu(y \mid x)$, is increasing in x. The proof is similar when it is decreasing. Since $Y_{[i]} \leqslant_{mrl} Y_{[j]}$ iff

$$\frac{\int_t^{+\infty} \bar{F}_{[j]}(y)\, dy}{\int_t^{+\infty} \bar{F}_{[i]}(y)\, dy} \quad \text{is increasing in } t,$$

it is enough to prove that the function

$$h(i, t) = \int_t^{+\infty} \bar{F}_{[i]}(y)\, dy = \int_{-\infty}^{+\infty} \left[\int_t^{+\infty} \bar{F}(y \mid x)\ dy \right] f_i(x)\, dx$$

is TP_2 in (i, t) if (X, Y) is $DTP(0, 2)$.

The required result follows from Lemma 2.1 since the function $\int_t^{+\infty} \bar{F}(y \mid x)\, dy$ is TP_2 in (t, x) if $\mu(y \mid x)$ is increasing in x and $f_i(x)$ is TP_2 in (x, i). ∎

Obviously $Y_{[i]} \leqslant_{mrl} Y_{[j]}$ implies that $E[Y_{[i]}] \leqslant E[Y_{[j]}]$ for $i < j$. However, as proved in the next theorem, this inequality holds under the weaker condition that $E[Y \mid X = x]$ is increasing in x.

THEOREM 3.5. *Suppose $E[Y \mid X = x]$ is increasing in x. Then*

$$E[Y_{[i]}] \leqslant E[Y_{[j]}] \qquad \textit{for} \quad i < j. \tag{3.11}$$

The inequality in (3.11) is reversed in case $E[Y \mid X = x]$ is decreasing in x.

Proof. $E[Y_{[i]}] = \int_{-\infty}^{+\infty} E[Y \mid X = x] \, f_{(i)}(x) \, dx = E[\psi(X_{(i)})]$, where $\psi(x) = E[Y \mid X = x]$. The required result now follows from this since $X_{(i)} \leqslant_{st} X_{(j)}$ for $i < j$ and $\psi(x)$ is assumed to be increasing in x. The inequality in (3.11) is reversed in case $\psi(x)$ is decreasing in x. ∎

Now we show that if the conditional distribution of Y given $X = x$ is *DFR* for each fixed x, then the concomitant $Y_{[i]}$'s are also *DFR* for $1 \leqslant i \leqslant n$.

THEOREM 3.6. *If $r(y \mid x)$, the conditional hazard rate of Y given $X = x$, is decreasing in y for each fixed x, then $Y_{[i]}$ has DFR distribution for $1 \leqslant i \leqslant n$.*

Proof. Since a mixture of *DFR* distributions is *DFR* (cf. Barlow and Proschan, 1981, p. 103), it follows from (1.2) and the assumption that $r(y \mid x)$ is decreasing in y for each fixed x that $Y_{[i]}$ has *DFR* distribution for $1 \leqslant i \leqslant n$. ∎

A random variable X is said to be less dispersed than another random variable Y (denoted by $X \leqslant_{disp} Y$) if $F^{-1}(\beta) - F^{-1}(\alpha) \leqslant G^{-1}(\beta) - G^{-1}(\alpha)$ whenever $0 < \alpha \leqslant \beta < 1$ where F^{-1} and G^{-1} are the right continuous inverses of the distribution functions F and G of X and Y, respectively. Bagai and Kochar (1986) proved that in case either X or Y is *DFR* and they have a common left end-point of their supports, then $X \leqslant_{hr} Y$ implies $X \leqslant_{disp} Y$. Using this result and the above theorem, we get,

THEOREM 3.7. *Suppose $r(y \mid x)$ is decreasing in x and y and the left end-point of the support of the conditional distribution of Y given $X = x$ does not depend on x. Then*

$$Y_{[i]} \leqslant_{disp} Y_{[j]} \qquad \textit{for} \quad i < j. \tag{3.12}$$

The inequality in (3.12) is reversed in case $r(y \mid x)$ is increasing in x for each fixed y.

Here is an example of a bivariate distribution which satisfies the conditions of this theorem.

EXAMPLE 3.1. Let (X_i, Y_i), $i=1, ..., n$ be a random sample from bivariate Pareto distribution (see Johnson and Kotz, 1972, p. 285), with density

$$f(x, y)=a(a+1)(\theta_1\theta_2)^{a+1}\,(\theta_2 x+\theta_1 y-\theta_1\theta_2)^{-(a+2)},$$
$$\text{for}\quad a>0,\ x>\theta_1>0\ ,\ y>\theta_2>0.$$

The conditional hazard rate of Y given X is

$$r(y \mid x)=\frac{\theta_1(a+1)}{\theta_1 y+\theta_2 x-\theta_1\theta_2},$$

which is decreasing in x as well as in y. It follows from Theorems 3.6 and 3.7 that each $Y_{[i]}$ has *DFR* distribution and that $Y_{[i]} \leqslant^{disp} Y_{[j]}$ for $i<j$.

4. DEPENDENCE AMONG CONCOMITANTS OF ORDER STATISTICS

In this section we discuss the dependence properties of concomitants of order statistics. By assuming different kinds of dependence between X and Y, we obtain successively stronger dependence results among the concomitant variables $Y_{[1]}, ..., Y_{[n]}$. We shall see that monotone (positive or negative) dependence between X and Y implies positive dependence among $Y_{[i]}$'s.

We need the following lemma proved in the Appendix.

LEMMA 4.1. *Let*

$$z(y_1, ..., y_n)=\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\prod_{i=1}^{n} K(x_i, x_{i+1})\prod_{i=1}^{n} h(x_i, y_i)\prod_{i=1}^{n} dx_i, \tag{4.1}$$

where

$$K(x, y)=\begin{cases}1 & \text{if}\quad x<y\\ 0 & \text{if}\quad x\geqslant y,\end{cases} \tag{4.2}$$

and where $x_{n+1}\equiv\infty$. Then $h(x, y)$ RR_2 or TP_2 in (x, y) implies that the function z is TP_2 in pairs.

Using this lemma we prove a general result on positive dependence among the concomitants of order statistics.

THEOREM 4.1. *If* (X, Y) *is* $DRR(0, m)$ *or* $DTP(0, m)$ *then* $(Y_{[1]}, ..., Y_{[n]})$ *is* $DTP(m, ..., m)$ *for all non-negative integers* m.

Proof. Suppose that (X, Y) is $DRR(0, m)$. Then by Definition 2.4, for $m > 0$ we have

$$\begin{aligned}
&\psi_{m, ..., m}(y_1, ..., y_n)\\
&\quad = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{i=1}^{n} \gamma^{(m)}(y_i - t_i)\, f_{Y_{[1]}, ..., Y_{[n]}}(t_1, ..., t_n) \prod_{i=1}^{n} dt_i\\
&\quad = n! \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{i=1}^{n} K(x_i, x_{i+1}) \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{i=1}^{n} \gamma^{(m)}(y_i - t_i)\\
&\qquad \times \prod_{i=1}^{n} \{f(t_i \mid x_i)\, f(x_i)\}\, dt_i \prod_{i=1}^{n} dx_i\\
&\quad = n! \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{i=1}^{n} K(x_i, x_{i+1}) \prod_{i=1}^{n} \psi_{0, m}(x_i, y_i) \prod_{i=1}^{n} dx_i,
\end{aligned}$$

where

$$\psi_{0, m}(x_i, y_i) = \int_{-\infty}^{+\infty} \gamma^{(m)}(y_i - t_i)\, f(t_i \mid x_i)\, f(x_i)\, dt_i,$$

and the function K is given by (4.2).

Since (X, Y) being $DRR(0, m)$ is equivalent to $\psi_{0, m}(x, y)$ being RR_2 in (x, y), it follows from Lemma 4.1 with $h(x_i, y_i)$ replaced by $\psi_{0, m}(x_i, y_i)$, that $\psi_{m, ..., m}(y_1, ..., y_n)$ is TP_2 in pairs.

Now let us consider the case when $m = 0$. In this case the function

$$\begin{aligned}
\psi_{0, ..., 0}(y_1, ..., y_n) &= f_{Y_{[1]}, ..., Y_{[n]}}(t_1, ..., t_n)\\
&= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{i=1}^{n} K(x_i, x_{i+1}) \prod_{i=1}^{n} f(x_i, y_i) \prod_{i=1}^{n} dx_i
\end{aligned}$$

is clearly seen to be $DTP(0, ..., 0)$. The proof follows from Lemma 4.1 with $h(x, y) = f(x, y)$.

When (X, Y) is $DTP(0, m)$, then the function $\psi_{0, m}(x, y)$ is TP_2. The required result follows similarly using the TP_2 part of Lemma 4.1. ∎

The following results are immediate consequences of the above theorem.

COROLLARY 4.1. (i) *If* X *and* Y *are* TP_2 *or* RR_2 *dependent, then the joint density of* $(Y_{[1]}, ..., Y_{[n]})$ *is* TP_2 *in pairs.*

(ii) *If the conditional hazard rate of* Y *given* $X=x$ *is monotone in* x, *then the concomitants* $(Y_{[1]}, ..., Y_{[n]})$ *are* $DTP(1, ..., 1)$. *In particular* $Y_{[i]}$ *and* $Y_{[j]}$ *are RCSI for* $i \neq j \in \{1, ..., n\}$.

We show in the next theorem that if Y is either stochastically increasing or stochastically decreasing in X, then the $Y_{[i]}$'s are associated.

THEOREM 4.2. *If* Y *is stochastically monotone in* X, *then* $Y_{[1]}, ..., Y_{[n]}$ *are associated.*

Proof. We give the proof for the case when Y is stochastically decreasing in X. The proof for the other case follows on the same lines.

Consider arbitrary increasing real-valued functions M and N defined on R^n. Then by the definition of associated variables it is enough to show that

$$\mathrm{Cov}(M(\mathbf{Y}_{[\ ]}), N(\mathbf{Y}_{[\ ]})) \geqslant 0,$$

whenever it exists. Now

$$\begin{aligned}\mathrm{Cov}(M(\mathbf{Y}_{[\ ]}), N(\mathbf{Y}_{[\ ]})) = {} & \mathrm{Cov}(E[M(\mathbf{Y}_{[\ ]}) \mid \mathbf{X}_{(\ )}], E[N(\mathbf{Y}_{[\ ]}) \mid \mathbf{X}_{(\ )}]) \\ & + E(\mathrm{Cov}[M(\mathbf{Y}_{[\ ]}) \mid \mathbf{X}_{(\ )}, N(\mathbf{Y}_{[\ ]}) \mid \mathbf{X}_{(\ )}]), \quad (4.3)\end{aligned}$$

where $\mathbf{Y}_{[\ ]} = (Y_{[1]}, ..., Y_{[n]})$ and $\mathbf{X}_{(\ )} = (X_{(1)}, ..., X_{(n)})$.

Note that the concomitants $(Y_{[1]}, ..., Y_{[n]})$ given the order statistics are conditionally independent and for $1 \leqslant i \leqslant n$ the conditional distribution of $Y_{[i]}$ given $X_{(i)} = x$ is the same as that of Y given $X = x$ (Bhattacharya, 1984). Therefore,

$$\{\mathbf{Y}_{[\ ]} \mid \mathbf{X}_{(\ )} = \mathbf{x}_{(\ )}\} \overset{st}{\geqslant} \{\mathbf{Y}_{[\ ]} \mid \mathbf{X}_{(\ )} = \mathbf{x}'_{(\ )}\} \quad (4.4)$$

for $\mathbf{x}_{(\ )} \leqslant \mathbf{x}'_{(\ )}$ if Y is stochastically decreasing in X. Here $\overset{st}{\geqslant}$ denotes *multivariate stochastic ordering* and we have used the fact for independent random variables, component-wise stochastic ordering implies multivariate stochastic ordering (cf. Shaked and Shanthikumar, 1994, Chapter 4).

It follows from (4.4) that $E[M(\mathbf{Y}_{[\ ]}) \mid \mathbf{X}_{(\ )} = \mathbf{x}]$ is decreasing in $\mathbf{x}$ since M is increasing. The first term in the R.H.S. of (4.3) is the covariance between two decreasing functions of order statistics (which are associated) and hence is nonnegative.

The second term in the R.H.S. of (4.3) is also nonnegative since covariance between two increasing functions of independent random variables is nonnegative. ∎

We consider again the model $Y = g(X) + Z$, where g is monotone and Z is independent of X. Kim and David (1990) proved that if g is increasing, then

$(Y_{[1]}, ..., Y_{[n]})$ are associated. Under the additional condition that Z has log-concave density, they proved that the joint density of $(Y_{[1]}, ..., Y_{[n]})$ is MTP_2 (cf. Karlin and Rinott, 1980).

THEOREM 4.3. *Let $(X_1, Y_1), ..., (X_n, Y_n)$ be a random sample from a bivariate distribution satisfying the model $Y = g(X) + Z$, where g is monotone and Z is independent of X. Then*

(a) *if the density of Z is log-concave, then the joint density of $(Y_{[1]}, ..., Y_{[n]})$ is TP_2 in pairs,*

(b) *if Z is IFR, then $(Y_{[1]}, ..., Y_{[n]})$ are $DTP(1, ..., 1)$,*

(c) *if Z is DMRL, then $(Y_{[1]}, ..., Y_{[n]})$ are $DTP(2, ..., 2)$.*

Proof. The proofs follow immediately from Theorem 2.1, Theorem 4.1, and Corollary 4.1. ∎

Note that the concomitant $Y_{[j]}$'s are associated under the conditions of Corollary 4.1, and Theorems 4.2 and 4.3, and as a consequence $\mathrm{Cov}(Y_{[i]}, Y_{[j]}) \geqslant 0$ for $i, j \in \{1, ..., n\}$. However, as shown below this result holds under a rather weaker condition that $E[Y \mid X = x]$ is monotone in x.

THEOREM 4.4. *Let $(X_1, Y_1), ..., (X_n, Y_n)$ be a random sample from a distribution for which $E[Y \mid X = x]$ is monotone in x. Then $\mathrm{Cov}(Y_{[i]}, Y_{[j]}) \geqslant 0$ for all $i, j \in \{1, ..., n\}$.*

Proof. From Yang (1977),

$$\begin{aligned}\mathrm{Cov}(Y_{[i]}, Y_{[j]}) &= \mathrm{Cov}[E(Y_1 \mid X_1 = X_{(i)}), E(Y_1 \mid X_1 = X_{(j)})] \\ &= \mathrm{Cov}[\psi(X_{(i)}), \psi(X_{(j)})] \geqslant 0\end{aligned}$$

since $\psi(x) = E(Y \mid X = x)$ is monotone and $X_{(i)}$ and $X_{(j)}$ are associated. ∎

5. CONCLUDING REMARKS

In this paper we have obtained new results on stochastic orderings and dependence among concomitants of order statistics from bivariate distributions which have various types of monotone dependence structures. While we are not aware of any previous results on stochastic monotonocity among comcomitants, the results on dependence among concomitants of order statistics were known only for certain special types of models. The results obtained in this paper are general in the sense that they apply to any particular distribution with any monotone dependence between the variables X and Y. It has been proved that if Y is stochastically increasing in X, then $Y'_{[i]}s$ are stochastically increasing and are associated. However, under a stronger condition that the conditional hazard rate of Y given $X=x$ is decreasing in x, it is proved that the $Y_{[i]}$'s are increasing according to hazard rate ordering and they are dependent according to $DTP(1, ..., 1)$ criteria. In particular, in this case, $Y_{[i]}$ and $Y_{[j]}$ are $RCSI$ for $i \neq j \in \{1, ..., n\}$. In case X and Y are TP_2 dependent, the successive $Y'_{[i]}s$ are increasing according to likelihood ratio ordering and their joint density is TP_2 in pairs. Analogous results on stochastic orderings among the concomitants of order statistics are obtained when the variables X and Y have monotone negative dependence. Surprisingly, in this case also the $Y_{[i]}$'s are positively dependent. We also prove that when the conditional hazard rate of Y given $X=x$ is decreasing in y for each fixed x, then $Y_{[i]}$'s have DFR distributions. If in addition, the above conditional hazard rate is monotone in x as well for each y, then the concomitants are ordered according to dispersive ordering. These results may have potential applications in the study of small sample properties of various estimates and tests for independence based on concomitants of order statistics.

6. APPENDIX

To prove Lemma 4.1 we first prove the following result which may also be of independent interest. It is a modified version of Theorem 5.1 of Karlin (1968, p. 123).

LEMMA A.1. Suppose λ, x, and ζ traverse the ordered sets Λ, X, and Z, respectively, and consider the function $f(\lambda, x, \zeta)$ satisfying the following conditions: (a) $f(\lambda, x, \zeta) > 0$ and f is TP_2 in (λ, x); (b) $f(\lambda, x, \zeta)$ is RR_2 in (λ, ζ) as well as in (x, ζ) for all λ, x and ζ. Then the function $h(\lambda, x) = \int_Z f(\lambda, x, \zeta)\, d\mu(\zeta)$, defined on $\Lambda \times X$ is TP_2 in (λ, x). Here μ represents a σ-finite measure.

Proof. We have to prove that for $\lambda_1 < \lambda_2$ and $x_1 < x_2$, $h(\lambda_2, x_2)\, h(\lambda_1, x_1) - h(\lambda_2, x_1)\, h(\lambda_1, x_2) \geqslant 0$. That is,

$$\int_Z f(\lambda_2, x_2, \zeta)\, d\mu(\zeta) \int_Z f(\lambda_1, x_1, \zeta)\, d\mu(\zeta) - \int_Z f(\lambda_2, x_1, \zeta)\, d\mu(\zeta) \int_Z f(\lambda_1, x_2, \zeta)\, d\mu(\zeta) \geqslant 0. \quad \text{(A.1)}$$

After some simplifications the L.H.S. of (A.1) can be written as

$$\int_{-\infty}^{+\infty}\int_{u}^{+\infty} [f(\lambda_2, x_2, \zeta)\, f(\lambda_1, x_1, u) - f(\lambda_2, x_1, \zeta)\, f(\lambda_1, x_2, u) + f(\lambda_2, x_2, u)\, f(\lambda_1, x_1, \zeta) - f(\lambda_2, x_1, u)\, f(\lambda_1, x_2, \zeta)]\, d\mu(\zeta)\, d\mu(u). \tag{A.2}$$

We shall show that the expression inside the square bracket in (A.2) is nonnegative. By assumption (a),

$$\frac{f(\lambda_2, x_2, \zeta)}{f(\lambda_2, x_1, \zeta)} - \frac{f(\lambda_1, x_2, \zeta)}{f(\lambda_1, x_1, \zeta)} \geqslant 0,$$

It follows after some algebraic manipulations that

$$\begin{aligned}&[f(\lambda_2, x_2, \zeta)\, f(\lambda_1, x_1, u) - f(\lambda_2, x_1, \zeta)\, f(\lambda_1, x_2, u)]\\ &\quad + \frac{f(\lambda_2, x_1, \zeta)\, f(\lambda_1, x_1, u)}{f(\lambda_2, x_1, u)\, f(\lambda_1, x_1, \zeta)}\\ &\quad \times [f(\lambda_2, x_2, u)\, f(\lambda_1, x_1, \zeta) - f(\lambda_2, x_1, u)\, f(\lambda_1, x_2, \zeta)]\\ &\geqslant 0.\end{aligned} \tag{A.3}$$

Note that for $\zeta > u$ the ratio in the L.H.S. of (A.3) is at most one since f is RR_2 in λ and ζ. Now since f is TP_2 in (λ, x) and RR_2 in (x, ζ), it can be shown that the quantity inside the square bracket in the second term of (A.3) is nonnegative, from which the result follows. ▌

Proof of Lemma 4.1. Suppose that $h(x, y)$ is RR_2. By Lemma 2.1 the innermost integral in (4.1),

$$g_1(x_2, y_1) = \int_{-\infty}^{+\infty} K(x_1, x_2)\, h(x_1, y_1)\, dx_1,$$

is RR_2 in (x_2, y_1) since K is TP_2 and h is RR_2. The next integral in (4.1) is

$$g_2(x_3, y_1, y_2) = \int_{-\infty}^{+\infty} K(x_2, x_3)\, g_1(x_2, y_1)\, h(x_2, y_2)\, dx_2. \tag{A.4}$$

Again by Lemma 2.1, the function g_2 in (A.4) is TP_2 in (y_1, y_2), RR_2 in (x_3, y_1) and in (x_3, y_2). We prove the desired result by induction. Define for $i=1, ..., n$,

$$g_i(x_{i+1}, y_1, ..., y_i) = \int_{-\infty}^{+\infty} g_{i-1}(x_i, y_1, ..., y_{i-1})\, h(x_i, y_i)\, K(x_i, x_{i+1})\, dx_i \tag{A.5}$$

Assume that g_{i-1} is TP_2 in (y_j, y_k), RR_2 in (x_i, y_j) for $j, k \in \{1, ..., i-1\}$. Using Lemma 2.1, the function g_i is RR_2 in (y_i, x_{i+1}), TP_2 in (y_i, y_j) and RR_2 in (x_{i+1}, y_j) for $j \in \{1, ..., i-1\}$. It remains to show that g_i is TP_2 in (y_j, y_k) for $j, k \in \{1, 2, ..., i-1\}$. For fixed $(y_1, ..., y_{j-1}, y_{j+1}, ..., y_{k-1}, y_{k+1}, ..., y_{i-1}, y_i)$ and x_{i+1} the function

$$m(x_i, y_j, y_k) = h(x_i, y_i) \times K(x_i, x_{i+1}) \times g_{i-1}(x_i, y_1, ..., y_{i-1})$$

is TP_2 in (y_j, y_k), RR_2 in (x_i, y_j) and (x_i, y_k). Now from Lemma A.1 it follows that g_i is TP_2 in (y_j, y_k) for $j, k \in \{1, ..., i-1\}$. That is, $g_n(x_{n+1}, y_1, ..., y_n) = z(y_1, ..., y_n)$ is TP_2 in (y_i, y_j) for $i, j \in \{1, ..., n\}$. This proves the required result.

The proof when h is TP_2 follows on the same lines using Lemma 2.1 and Theorem 5.1 of Karlin (1968, p. 123). ∎

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