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PARTIAL ORDERINGS OF DISTRIBUTIONS BASED ON RIGHT-SPREAD FUNCTIONS

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Abstract

In this paper we introduce a quantile dispersion measure. We use it to characterize different classes of ageing distributions. Based on the quantile dispersion measure, we propose a new partial ordering for comparing the spread or dispersion in two probability distributions. This new partial ordering is weaker than the well known dispersive ordering and it retains most of its interesting properties.

Keywords: Partial orderings; spread; mean residual life function; dispersive ordering

AMS 1991 Subject Classification: Primary 62N05
Secondary 62E10

1. Introduction

The stochastic comparison of distributions has been an important area of research in many diverse areas of statistics and probability. Many different types of stochastic orders have been studied in the literature; a comprehensive discussion of them is available in Shaked *et al.* (1994). It is often easy to make value judgements when such orderings exist. For example, if X and Y are two random variables with their distribution functions F_X and F_Y satisfying $F_X(x) \leq F_Y(x)$ for every x, then we say that Y is stochastically smaller than X ($Y \overset{\text{st}}{\preceq} X$). Stochastic ordering between two probability distributions, if it holds, is more informative than simply comparing their means or medians only. If $Y \overset{\text{st}}{\preceq} X$, then every quantile of the distribution of X is smaller than the corresponding quantile of the distribution of Y, and any reasonable measure of location will be smaller for Y than for X. It is well known that Y is stochastically smaller than X if and only if $E[h(Y)] \leq E[h(X)]$ for every non-decreasing function h, provided the expectations exist.

Similarly, if one wishes to compare the dispersion or spread between two distributions, the simplest way would be to compare their standard deviations or some such other measures of dispersion. However, such a comparison is based only on two single numbers, and therefore it is often not very informative. In addition to this, the standard deviations of the distributions may not exist or they may not be the appropriate quantities to compare in some situations. A

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more informative way will be to compare their interquantile differences of all orders at the same time.

Let Q_X (Q_Y) be the left continuous inverse (quantile function) of F_X (F_Y) defined by $Q_X(p) = \inf\{x : F_X(x) \geq p\}, 0 < p < 1$. If

$$Q_Y(q) - Q_Y(p) \leq Q_X(q) - Q_X(p) \tag{1.1}$$

holds for every $0 < p < q < 1$, then we say that Y is less dispersed than X and denote this partial ordering by $Y \overset{\text{disp}}{\preceq} X$. It requires the difference of any two quantiles of Y to be smaller than the difference of the corresponding quantiles of X. An equivalent definition of $Y \overset{\text{disp}}{\preceq} X$ is to require $(Q_Y \circ F_X)(x) - x$ to be non-decreasing in x. Dispersive ordering can also be expressed in terms of failure (or hazard) rates (if they exist). Let $r_X(\cdot)$ and $r_Y(\cdot)$ denote the failure rates of X and Y, respectively. Then $Y \overset{\text{disp}}{\preceq} X$ if and only if

$$r_X(Q_X(p)) \leq r_Y(Q_Y(p)) \quad \forall p \in (0, 1). \tag{1.2}$$

A consequence of $X \overset{\text{disp}}{\preceq} Y$ is that $\text{var}(X) \leq \text{var}(Y)$. Also $X \overset{\text{disp}}{\preceq} Y$ implies $E[|X_1 - X_2|] \leq E[|Y_1 - Y_2|]$, where $X_1, X_2 (Y_1, Y_2)$ are two independent copies of $X(Y)$. For more details on dispersive ordering, see Chapter 2.B of Shaked *et al.* (1994).

In this paper, we introduce a new partial ordering for spread which is weaker than dispersive ordering. For this we introduce a quantile dispersion function: the right spread function. We study its properties in Section 2. In Section 3, we use RS functions to characterize various ageing classes of life distributions. In the last section we introduce a new functional partial ordering to compare two probability distributions in terms of their dispersions and study its properties. It is seen that the new ordering is weaker than dispersive ordering.

2. The right spread function and its properties

Let X be a random variable with distribution function (d.f.) F and with finite mean μ_F. The right spread function (RS function) of X is defined by

$$\begin{aligned} S_X^+(p) &= E[(X - Q_X(p))^+] \\ &= E[\max\{X - Q_X(p), 0\}] \\ &= \int_{Q_X(p)}^{\infty} \overline{F}_X(t)\, dt, \end{aligned} \tag{2.1}$$

where $\overline{F}_X = 1 - F_X$ denotes the survival function of X and $Q_X(p)$ is the quantile function of X.

We shall call S_X^+ the *right spread* (RS) function of the random variable X to distinguish it from the spread function $S_X(p) = E[|X - Q_X(p)|]$ as studied in Muńoz-Pérez (1990).

The RS function is non-decreasing and, if $\mu_F < \infty$ and X is a non-negative random variable, then its limit is μ_F when $p \downarrow 0$. The RS function of a random variable X is closely related to its mean residual life function, which is defined by

$$\mu_F(x) = E[X - x | X > x] = \int_x^{\infty} \bar{F}(t)\, dt / \bar{F}(x).$$

In this next theorem we give some important properties of the RS function.

Theorem 2.1. *Let X be a continuous random variable with d.f. F which is strictly increasing on its support. Then:*

(b)

$$\mu_F(Q_X(u)) = S_X^+(u)/(1-u); \tag{2.2}$$

(c)

$$\int_0^1 [S_X^+(u)/(1-u)]^2\,du = \mathrm{var}(X), \tag{2.3}$$

provided $Q_X(0) = 0$;

(d) *if X_1 and X_2 are two independent copies of X, then*

$$\int_0^1 S_X^+(u)\,du = \tfrac{1}{2}E[|X_1 - X_2|] \tag{2.4}$$

(e)

$$r_F(Q_X(u)) = -\left[\frac{d}{du}S_X^+(u)\right]^{-1} \quad \forall u \in (0,1). \tag{2.5}$$

Proof. (a), (b) These are easy to show.
(c)

$$\begin{aligned}\int_0^1 [S_X^+(u)/(1-u)^2]\,du &= \int_0^1 \mu_F^2(Q_X(u))\,du \\ &= \int_0^\infty \mu_F^2(t)\,dF(t) \\ &= \mathrm{var}(X).\end{aligned}$$

The last equality follows from Lemma 7.4 of Pyke (1965).
(d)

$$\begin{aligned}E[|X_1 - X_2|] &= 2\iint_{x_1 \le x_2} (x_2 - x_1) f(x_1) f(x_2)\,dx_1\,dx_2 \\ &= 2\int_{-\infty}^{\infty}\int_{x_1}^{\infty} (x_2 - x_1) f(x_1) f(x_2)\,dx_2\,dx_1 \\ &= 2\int_{-\infty}^{\infty} \bar{F}(x_1)\left[\int_{x_1}^{\infty}(x_2 - x_1)\frac{f(x_2)}{\bar{F}(x_1)}\,dx_2\right] f(x_1)\,dx_1 \\ &= 2\int_{-\infty}^{\infty} \bar{F}(x_1)\mu_F(x_1) f(x_1)\,dx_1 \\ &= 2\int_0^1 (1-v)\mu_F(Q_X(v))\,dv \\ &= 2\int_0^1 S_X^+(v)\,dv.\end{aligned}$$

(e) This is easy to show.

3. Characterizing partial orderings of life distributions in terms of right spread functions

We shall assume in this section that all distributions under consideration are life distributions with the property that $F(0) = 0$. In reliability and survival analysis, several non-parametric classes of ageing distributions like increasing failure rate (IFR), increasing failure rate average (IFRA), new better than used (NBU), decreasing mean residual life (DMRL), new better than used in expectation (NBUE) and so on, have been studied. In all these criteria, we compare the relative performance of a unit under consideration with that of an exponential distribution with the same mean. Remember that a unit with exponential life distribution does not age with time. See Hollander and Proschan (1984) for an excellent review of different notions of ageing. Kochar and Wiens (1987) and Kochar (1989) generalized these notions of ageing to compare the relative ageing of two arbitrary life distributions. We review some of them here. We assume throughout that all distributions being considered have finite means, and are strictly increasing on their supports.

Definition 3.1.

(i) *X is said to be more IFR than Y* ($X \overset{\text{IFR}}{\preceq} Y$) if

$$\frac{r_X(Q_X(p))}{r_Y(Q_Y(p))} \quad \text{is non-decreasing in } p \text{ for } p \in (0, 1). \tag{3.1}$$

Or equivalently, if the function $(Q_Y \circ F_X)(x)$ is convex.

(ii) *X is said to be more DMRL than Y* ($X \overset{\text{DMRL}}{\preceq} Y$) if

$$\frac{\mu_X(Q_X(p))}{\mu_Y(Q_Y(p))} \quad \text{is non-decreasing in } p \in (0, 1). \tag{3.2}$$

(iii) *X is said to be more NBUE than Y* ($X \overset{\text{NBUE}}{\preceq} Y$) if

$$\frac{\mu_X(Q_X(p))}{\mu_Y(Q_Y(p))} \leq \frac{E[X]}{E[Y]} \quad \forall p \in (0, 1). \tag{3.3}$$

In the above definitions, if Y has negative exponential distribution, then

$$X \overset{\mathcal{P}}{\preceq} Y \text{ if and only if } X \text{ has ageing property } \mathcal{P}$$

for $\mathcal{P} \in$ {IFR, DMRL, NBUE}.

Now we show that these partial orderings of distributions can be conveniently expressed in terms of their RS functions. The IFR ordering as given in the Definition 3.1 is also called *convex ordering*, a concept initially introduced by van Zwet (1964). If (3.1) holds, we say that X is convex ordered with respect to Y.

Theorem 3.1. *Let X and Y be two non-negative random variables with RS functions S_X^+ and S_Y^+, respectively. Then*

(a)

$$X \overset{\text{IFR}}{\preceq} Y \iff \frac{\frac{d}{dp}S_X^+(p)}{\frac{d}{dp}S_Y^+(p)} \text{ is non-increasing in } p \text{ for } p \in (0,1) \tag{3.4}$$

$$\iff (S_X^+ \circ S_Y^{+^{-1}})(t) \text{ is convex in } t \text{ for } t \in (0,\infty). \tag{3.5}$$

(b)

$$X \overset{\text{DMRL}}{\preceq} Y \iff \frac{S_X^+(p)}{S_Y^+(p)} \text{ is non-increasing in } p \text{ for } p \in (0,1)$$

$$\iff (S_X^+ \circ S_Y^{+^{-1}})(t) \text{ is star-shaped in } t.$$

(c)

$$X \overset{\text{NBUE}}{\preceq} Y \iff \frac{S_X^+(p)}{S_Y^+(p)} \le \frac{E[X]}{E[Y]}. \tag{3.6}$$

Proof. (a) From (2.5),

$$r_F(Q_X(u)) = -\left[\frac{d}{du}S_X^+(u)\right]^{-1} \quad \forall u \in (0,1).$$

The required result follows from this and (3.1).

(b) From (2.2),

$$\mu_F(Q_X(u)) = S_X^+(u)/(1-u).$$

The required result follows from this and (3.2).

(c) From Kochar and Wiens (1987), we have that

$$X \overset{\text{NBUE}}{\preceq} Y \iff \frac{\mu_F(Q_X(u))}{\mu_G(Q_Y(u))} \le \frac{\mu_F}{\mu_G} \tag{3.7}$$

The result follows from this and (2.2).

Note that the RS function of the negative exponential distribution with mean μ is $\mu(1-u)$. By taking Y as an exponential random variable in the above theorem, we get the following corollary.

A real-valued function ϕ defined on [0, 1] is said to be convex at 1 if and only if

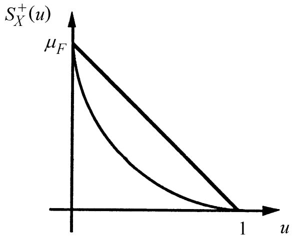
$$\phi(\lambda u + (1-\lambda)1) \le \lambda\phi(u) + (1-\lambda)\phi(1) \quad \forall u \in [0,1] \text{ and } \forall \lambda \in [0,1].$$

This means that any chord joining $(1, \phi(1))$ to any other point $(u, \phi(u))$ in the curve ϕ lies above it (see Figure 2).

Corollary 3.1.

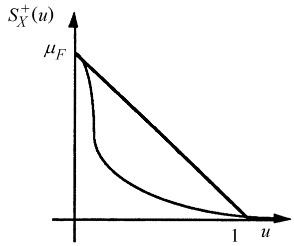
(a) *F is IFR if and only if $S_X^+(p)$ is a convex function of p.*

(b) *F is DMRL if and only if $S_X^+(p)$ is a convex function at* 1.



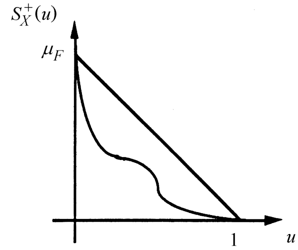
IFR

FIGURE 1.



DMRL

FIGURE 2.



NBUE

FIGURE 3.

(c) *F is NBUE if and only if $S_X^+(p) \leq (1-p)E[X]$.*

Proof. (a) We have that

$$r_F(Q_X(u)) = -\left\{\frac{d}{du} S_X^+(u)\right\}^{-1}, \quad \forall u \in (0,1).$$

Now F is IFR if, and only if, $r_F(Q_X(\cdot))$ is a non-decreasing function, so $\frac{d}{du} S_X^+(u)$ is non-decreasing in u. Therefore, $S_X^+(u)$ is a convex function. Any chord joining two points on the curve S_X^+ lies above it (see Figure 1).

(b) F is DMRL if, and only if $S_X^+(u)/(1-u)$, is decreasing in u. If $S_X^+(u)$ is a convex function at 1, then

$$\lambda S_X^+(u) \geq S_X^+(\lambda u + (1-\lambda)) \qquad \forall \lambda \in [0,1].$$

If we take $u = u_1$ and $\lambda = [(1-u_2)/(1-u_1)]$ with $u_1 \leq u_2$ we obtain

$$S_X^+(u_2) \leq \frac{1-u_2}{1-u_1} S_X^+(u_1),$$

so that $S_X^+(u)/(1-u)$ is decreasing in u.

If $S_X^+(u)/(1-u)$ is decreasing in u, then

$$S_X^+(u_2) \leq [(1-u_2)/(1-u_1)] S_X^+(u_1) \quad \text{for } u_1 \leq u_2.$$

Note that $[(1-u_2)/(1-u_1)]$ varies from 0 to 1 as u_2 varies from 1 to u_1. For fixed λ and u, taking $u_1 = u$ and $u_2 = \lambda u + (1-\lambda)$, we have that

$$S_X^+(\lambda u + (1-\lambda)) \leq \lambda S_X^+(u).$$

Any chord joining $(1,0)$ to any point on the curve S_X^+ lies above it (see Figure 2).

(c) Since for the exponential distribution with mean μ_G, $S_Y^+(u) = \mu_G(1-u)$, the required result follows from (3.6).

4. Right spread out ordering

Muñoz-Pérez (1990) has shown that $Y \overset{\text{disp}}{\leq} X$ if and only if the random variable $(X - Q_X(p))^+$ is stochastically greater than the random variable $(Y - Q_Y(p))^+$ for every $0 < p < 1$, where $(Z)^+ = \max\{Z, 0\}$. If, instead of stochastic ordering, we compare the means of $(X - Q_X(p))^+$ and $(Y - Q_Y(p))^+$, we have the following new notion of dispersive ordering.

Definition 4.1. Y is less right spread out than X ($Y \overset{\text{RS}}{\preceq} X$) if

$$S_Y^+(p) \le S_X^+(p), \quad \forall p \in (0,1). \tag{4.1}$$

It follows immediately that $Y \overset{\text{disp}}{\preceq} X$ implies $Y \overset{\text{RS}}{\preceq} X$. Thus the RS ordering is weaker than the dispersive ordering. In the light of (2.2) the RS ordering can be equivalently expressed in terms of the mean residual life functions as

$$Y \overset{\text{RS}}{\preceq} X \Leftrightarrow \mu_Y(Q_Y(p)) \le \mu_X(Q_X(p)), \quad \forall p \in (0,1). \tag{4.2}$$

Compare this with (1.2), which expresses dispersive ordering in terms of failure rates at quantiles of the same orders.

It is easy to see that $X \overset{\text{RS}}{\preceq} aX$ for $a \ge 1$. The RS ordering is location-free in the sense that $Y \overset{\text{RS}}{\preceq} X \Leftrightarrow Y + c \overset{\text{RS}}{\preceq} X$ for any real c. It follows from Theorem 2.1 that if X and Y are non-negative random variables with finite second moments, then $Y \overset{\text{RS}}{\preceq} X$ implies that $\text{var}(X) \le \text{var}(Y)$ as well as $E[|Y_1 - Y_2|] \le E[|X_1 - X_2|]$, where $X_1, X_2 (Y_1, Y_2)$ are two independent copies of $X(Y)$. The next theorem states that even dispersive ordering can be conveniently expressed in terms of RS functions.

Theorem 4.1. $Y \overset{\text{disp}}{\preceq} X \Leftrightarrow S_X^+(p) - S_Y^+(p)$ *is non-increasing in* $p \in (0,1)$.

Proof. Since

$$\frac{d}{dp} S_X^+(p) = -[r_X(Q_X(p))]^{-1}, \quad \text{for } p \in (0,1),$$

the required result follows from (1.2).

We have seen earlier that there is a close relationship between the RS function and the mean residual life function of a random variable. We show below that, under some conditions, RS ordering implies *mean residual life (mrl) ordering* and *vice versa*. First we recall the definition of mrl ordering.

Definition 4.2. A random variable Y is said to be smaller than another random variable X in the mean residual life ordering sense ($Y \overset{\text{mrl}}{\preceq} X$) if

$$\mu_Y(t) \le \mu_X(t), \quad \forall t.$$

See Joag-dev *et al.* (1995) for properties of mrl ordering.

Theorem 4.2. *Let X and Y be two random variables such that $Y \overset{\text{st}}{\preceq} X$. Then*

(a) *if $Y \overset{\text{mrl}}{\preceq} X$ and either X or Y has increasing mean residual life (IMRL) distribution, then $Y \overset{\text{RS}}{\preceq} X$;*

(b) *if $Y \overset{\text{RS}}{\preceq} X$ and either X or Y has decreasing mean residual life (DMRL) distribution, then $Y \overset{\text{mrl}}{\preceq} X$.*

Proof. (a) Assume that X is IMRL.
Since $Y \overset{\text{st}}{\preceq} X$, $Q_Y(p) \leq Q_X(p)\ \forall p \in (0, 1)$.
Now

$$\begin{aligned} Y \overset{\text{mrl}}{\preceq} X &\Leftrightarrow \mu_Y(Q_Y(p)) \leq \mu_X(Q_Y(p)), \quad \forall p \in (0, 1) \\ &\Longrightarrow \mu_Y(Q_Y(p)) \leq \mu_X(Q_X(p)), \quad \forall p \in (0, 1) \\ &\Leftrightarrow Y \overset{\text{RS}}{\preceq} X. \end{aligned}$$

(b) The proof of this part is on the same lines.

Bagai and Kochar (1986) proved similar results for relations between dispersive ordering and hazard rate ordering. The next theorem gives sufficient conditions under which NBUE ordering implies RS ordering.

Theorem 4.3. *Let* $X \overset{\text{NBUE}}{\preceq} Y$ *and* $E[X] \leq E[Y]$, *then* $X \overset{\text{RS}}{\preceq} Y$.

Proof. It follows from (2.3) that $X \overset{\text{NBUE}}{\preceq} Y$ if and only if, for $p \in (0, 1)$, we have

$$\frac{S_X(p)}{S_Y(p)} \leq \frac{E[X]}{E[Y]}.$$

Since $E[X] \leq E[Y]$, it follows that $S_X^+(p) \leq S_Y^+(p)$, for all $p \in (0, 1)$. Hence the result.

This theorem extends a similar result of Ahmed *et al.* (1986) from super-additive ordering to NBUE ordering.

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