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# SOME NEW RESULTS ON STOCHASTIC COMPARISONS OF PARALLEL SYSTEMS

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**Abstract**

Let $X_1, \dots, X_n$ be independent exponential random variables with $X_i$ having hazard rate $\lambda_i$, $i = 1, \dots, n$. Let $Y_1, \dots, Y_n$ be a random sample of size $n$ from an exponential distribution with common hazard rate $\tilde{\lambda} = (\prod_{i=1}^n \lambda_i)^{1/n}$, the geometric mean of the $\lambda_i$s. Let $X_{n:n} = \max\{X_1, \dots, X_n\}$. It is shown that $X_{n:n}$ is greater than $Y_{n:n}$ according to dispersive as well as hazard rate orderings. These results lead to a lower bound for the variance of $X_{n:n}$ and an upper bound on the hazard rate function of $X_{n:n}$ in terms of $\tilde{\lambda}$. These bounds are sharper than those obtained by Dykstra *et al.* ((1997), *J. Statist. Plann. Inference* **65,** 203–211), which are in terms of the arithmetic mean of the $\lambda_i$s. Furthermore, let $X_1^*, \dots, X_n^*$ be another set of independent exponential random variables with $X_i^*$ having hazard rate $\lambda_i^*$, $i = 1 \dots, n$. It is proved that if $(\log \lambda_1, \cdots, \log \lambda_n)$ weakly majorizes $(\log \lambda_1^*, \cdots, \log \lambda_n^*)$, then $X_{n:n}$ is stochastically greater than $X_{n:n}^*$.

*Keywords:* Exponential distribution; hazard rate ordering; dispersive ordering; Schur functions; majorization; $p$-larger ordering

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## 1. Introduction

Order statistics play an important role in reliability theory. The time to failure of a $k$-out-of-$n$ system of $n$ components corresponds to the $(n - k + 1)$th order statistic. In particular, the lifetime of a parallel system is the same as the largest order statistic. Series and parallel systems are the simplest examples of coherent systems and they have been studied extensively in the literature in the case where the components are independent and identically distributed. But in real life, systems are usually made up of components with non-identically distributed lifetimes. Since the distribution theory becomes quite complicated then, fewer results are available in the general case.

The exponential distribution plays a very important role in statistics. Because of its non-aging property, it has many nice properties and it often gives very convenient bounds on survival probabilities and other characteristics of interest for systems with non-exponential components. Pledger and Proschan (1971) studied the problem of stochastically comparing the order statistics of non-identically distributed independent exponential random variables with those corresponding to independent and identically distributed exponential random variables. This topic has been followed up by many researchers including Proschan and Sethuraman (1976), Boland *et al.* (1994), Dykstra *et al.* (1997), Boland *et al.* (1998), Bon and Paltanea

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(1999), and Kochar and Ma (1999), among others. In this note we obtain some new results on stochastic comparisons of parallel systems.

Let us denote the density function, the distribution function, the survival function, and the hazard rate of a random variable $X$ by $f_X$, $F_X$, $\overline{F}_X$ and $r_X$, respectively. A random variable $X$ is said to be *stochastically* larger than another random variable $Y$ (denoted by $X \geq_{\text{st}} Y$) if $\overline{F}_X(x) \geq \overline{F}_Y(x)$ for all $x$. A stronger notion of stochastic dominance is that of *hazard rate* ordering. We say that $X$ is larger than $Y$ in *hazard rate* ordering (denoted by $X \geq_{\text{hr}} Y$) if $\overline{F}_X(x)/\overline{F}_Y(x)$ is non-decreasing in $x$. A random variable $X$ is said to be more dispersed than another random variable $Y$ (denoted by $X \geq_{\text{disp}} Y$) if $F_X^{-1}(\beta) - F_X^{-1}(\alpha) \geq F_Y^{-1}(\beta) - F_Y^{-1}(\alpha)$ whenever $0 < \alpha \leq \beta < 1$, where $F_X^{-1}$ and $F_Y^{-1}$ are the right continuous inverses of the distribution functions $F_X$ and $F_Y$, respectively. For more details regarding these stochastic orders, see Chapter 1 and Section 2.B of Shaked and Shanthikumar (1994).

Let $\{x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}\}$ denote the increasing arrangement of the components of the vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$. The vector $\boldsymbol{x}$ is said to majorize the vector $\boldsymbol{y}$ (written $\boldsymbol{x} \succeq^{\text{m}} \boldsymbol{y}$) if $\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}$ for $j = 1, \ldots, n-1$ and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$. Functions that preserve the majorization ordering are called Schur convex functions. See Marshall and Olkin (1979, Chapter 3) for properties and more details. The vector $\boldsymbol{x}$ is said to majorize the vector $\boldsymbol{y}$ weakly (written $\boldsymbol{x} \succeq^{\text{w}} \boldsymbol{y}$) if $\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}$ for $j = 1, \ldots, n$.

Recently Bon and Paltanea (1999) have considered a new pre-order on $\mathbb{R}$, which they call a *p-larger order*. A vector $\boldsymbol{x}$ in $\mathbb{R}$ is said to be $p$-larger than another vector $\boldsymbol{y}$ also in $\mathbb{R}$ (written $\boldsymbol{x} \succeq^{\text{p}} \boldsymbol{y}$) if $\log(\boldsymbol{x}) \succeq^{\text{w}} \log(\boldsymbol{y})$, where $\log(\boldsymbol{x})$ denotes the vector of the logarithms of the coordinates of $\boldsymbol{x}$. It is known that $\boldsymbol{x} \succeq^{\text{m}} \boldsymbol{y} \Longrightarrow (g(x_1), \ldots, g(x_n)) \succeq^{\text{w}} (g(y_1), \ldots, g(y_n))$ for all concave functions $g$ (cf. Marshal and Olkin (1979), p. 115). Since log is a concave function, it follows that for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}$, if $\boldsymbol{x} \succeq^{\text{m}} \boldsymbol{y}$ then $\boldsymbol{x} \succeq^{\text{p}} \boldsymbol{y}$. The converse is, however, not true. For example, $(0.2, 1, 5) \succeq^{\text{p}} (1, 2, 3)$ but majorization does not hold between these two vectors.

We shall denote by $Z_{i:n}$ the $i$th order statistic of a set of $n$ random variables $Z_1, \ldots, Z_n$. Let $X_1, \ldots, X_n$ be independent exponential random variables with $X_i$ having hazard rate $\lambda_i$, $i = 1, \ldots, n$ and let $Y_1 \ldots, Y_n$ be a random sample of size $n$ from an exponential distribution with hazard rate $\overline{\lambda} = \sum_{i=1}^{n} \lambda_i / n$. Dykstra *et al.* (1997) proved that $X_{n:n} \geq_{\text{disp}} Y_{n:n}$ and $X_{n:n} \geq_{\text{hr}} Y_{n:n}$. These results give a lower bound for the variance of $X_{n:n}$ and an upper bound on the hazard rate of $X_{n:n}$ in terms of those of $Y_{n:n}$. In the next section we improve upon these bounds by replacing $\overline{\lambda}$ with $\tilde{\lambda} = (\prod_{i=1}^{n} \lambda_i)^{1/n}$, the geometric mean of the λs.

Let $X_1^*, \ldots, X_n^*$ be another set of independent exponential random variables with $Z_i$ having hazard rate $\lambda_i^*$. Pledger and Proschan (1971) showed that if $\boldsymbol{\lambda} \succeq^{\text{m}} \boldsymbol{\lambda}^*$ then $X_{i:n} \geq_{\text{st}} X_{i:n}^*$, $i = 1, \ldots, n$. We prove in Section 2 that for $i = n$ the same result continues to hold under the $p$-larger ordering.

## 2. Main results

To prove the main theorem in this section we shall need the following lemma whose proof can be easily verified.

**Lemma 2.1.** *For $z > 0$, the functions $g(z) = (1 - \mathrm{e}^{-z})/z$ and $\psi(z) = z^2 \mathrm{e}^{-z}/(1 - \mathrm{e}^{-z})^2$ are both decreasing.*

**Theorem 2.1.** *Let $X_1, \ldots, X_n$ be independent exponential random variables with $X_i$ having hazard rate $\lambda_i$, $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be a random sample of size n from an exponential distribution with common hazard rate $\tilde{\lambda} = (\prod_{i=1}^{n} \lambda_i)^{1/n}$. Then $X_{n:n} \geq_{\text{disp}} Y_{n:n}$ and $X_{n:n} \geq_{\text{hr}} Y_{n:n}$.*

*Proof.* Note that

$$X_{n:n} \geq_{\text{disp}} Y_{n:n} \quad \Leftrightarrow \quad f_{X_{n:n}}(x) \leq f_{Y_{n:n}}(F^{-1}_{Y_{n:n}} F_{X_{n:n}}(x)) \qquad \forall x \geq 0. \tag{2.1}$$

One can see after some simplifications that (2.1) is equivalent to

$$\sum_{i=1}^{n} \frac{\lambda_i}{1-\mathrm{e}^{-\lambda_i x}} - n\prod_{i=1}^{n}\biggl(\frac{\lambda_i}{1-\mathrm{e}^{-\lambda_i x}}\biggr)^{1/n} \leq \sum_{i=1}^{n}\lambda_i - n\prod_{i=1}^{n}(\lambda_i)^{1/n} \qquad \forall x \geq 0. \tag{2.2}$$

To prove that (2.2) holds for all $\lambda_i > 0,\ i = 1, \dots, n$, it is sufficient to show that the left-hand side of (2.2), denoted by $h(x)$, is non-decreasing in $x$ since then, for $x \geq 0$,

$$h(x) \leq \lim_{t\to+\infty} h(t) = \sum_{i=1}^{n}\lambda_i - n\prod_{i=1}^{n}(\lambda_i)^{1/n},$$

the right-hand side of (2.2).

The derivative of $h(x)$ is

$$\begin{aligned} h'(x) &= \biggl(\sum_{i=1}^{n}\frac{\lambda_i \mathrm{e}^{-\lambda_i x}}{1-\mathrm{e}^{-\lambda_i x}}\biggr)\biggl(\prod_{i=1}^{n}\frac{\lambda_i}{1-\mathrm{e}^{-\lambda_i x}}\biggr)^{1/n} - \sum_{i=1}^{n}\frac{\lambda_i^2\mathrm{e}^{-\lambda_i x}}{(1-\mathrm{e}^{-\lambda_i x})^2} \\ &\geq \biggl(\sum_{i=1}^{n}\frac{\lambda_i \mathrm{e}^{-\lambda_i x}}{1-\mathrm{e}^{-\lambda_i x}}\biggr)\biggl(\frac{n}{\sum_{i=1}^{n}(1-\mathrm{e}^{-\lambda_i x})/\lambda_i}\biggr) - \sum_{i=1}^{n}\frac{\lambda_i^2\mathrm{e}^{-\lambda_i x}}{(1-\mathrm{e}^{-\lambda_i x})^2}, \end{aligned}$$

since the geometric mean of a set of numbers is always greater than or equal to its harmonic mean. Hence $h'(x)$ will be non-negative if we can prove that for $z_i > 0,\ i = 1, \dots, n$,

$$n\sum_{i=1}^{n}\frac{z_i\mathrm{e}^{-z_i}}{1-\mathrm{e}^{-z_i}} \geq \biggl(\sum_{i=1}^{n}\frac{z_i^2\mathrm{e}^{-z_i}}{(1-\mathrm{e}^{-z_i})^2}\biggr)\biggl(\sum_{i=1}^{n}\frac{1-\mathrm{e}^{-z_i}}{z_i}\biggr). \tag{2.3}$$

The inequality in (2.3) follows immediately from Čebyšev's inequality (Mitrinović (1970, Theorem 1, p. 36)), Lemma 2.1, and by writing

$$\frac{z_i\mathrm{e}^{-z_i}}{1-\mathrm{e}^{-z_i}} = \biggl(\frac{z_i^2\mathrm{e}^{-z_i}}{(1-\mathrm{e}^{-z_i})^2}\biggr)\biggl(\frac{1-\mathrm{e}^{-z_i}}{z_i}\biggr).$$

This proves that $h(x)$ is non-decreasing in $x$ and hence the result.

The proof of $X_{n:n} \geq_{\text{hr}} Y_{n:n}$ is identical to that of Theorem 2.1 (b) of Dykstra *et al.* (1997) and is omitted.

**Corollary 2.1.** *Under the conditions of Theorem 2.1:* (a) *the hazard rate $r_{X_{n:n}}$ of $X_{n:n}$ satisfies*

$$r_{X_{n:n}}(x; \boldsymbol{\lambda}) \leq \frac{n\tilde{\lambda}(1-\exp(-\tilde{\lambda}x))^{n-1}\exp(-\tilde{\lambda}x)}{1-(1-\exp(-\tilde{\lambda}x))^n}\ ;$$

(b)

$$\operatorname{var}(X_{n:n}; \boldsymbol{\lambda}) \geq \frac{1}{\tilde{\lambda}^2}\sum_{i=1}^{n}\frac{1}{(n-i+1)^2}.$$

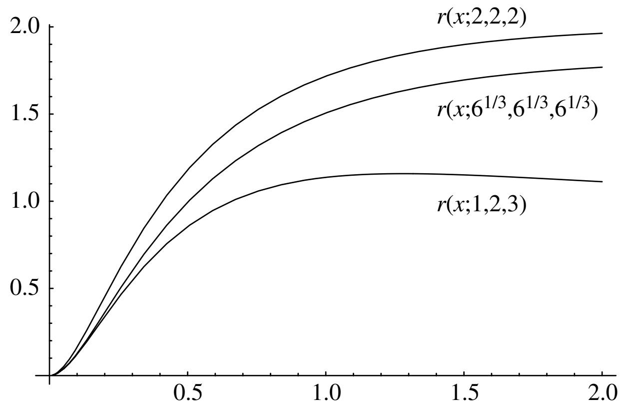


FIGURE 1: Graphs of hazard rates of $X_{3:3}$ ($\boldsymbol{\lambda}_1 = (1, 2, 3)$).

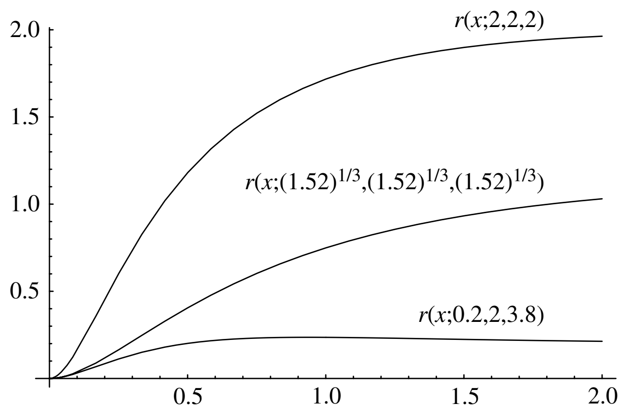


FIGURE 2: Graphs of hazard rates of $X_{3:3}$ ($\boldsymbol{\lambda}_2 = (0.2, 2, 3.8)$).

The new bounds given by Corollary 2.1 are better than those obtained by Dykstra *et al.* (1997), since $r_{Y_{n:n}}$ is a non-decreasing function of $\tilde{\lambda}$ and the fact that the geometric mean of the $\lambda_i$s, is smaller than their arithmetic mean.

In Figures 1 and 2, we plot the hazard rates of parallel systems of three exponential components along with the upper bounds as given by Dykstra *et al.* (1997) and the ones given by Corollary 2.1(a). The vector of parameters in Figure 1 is $\boldsymbol{\lambda}_1 = (1, 2, 3)$ and that in Figure 2 is $\boldsymbol{\lambda}_2 = (0.2, 2, 3.8)$. Note that $\boldsymbol{\lambda}_2 \succeq^{\mathrm{m}} \boldsymbol{\lambda}_1$. It appears from these figures that the improvements on the bounds are relatively more if the $\lambda_i$s are more dispersed in the sense of majorization. This fact follows because the geometric mean is Schur concave whereas the arithmetic mean is Schur constant and the hazard rate of a parallel system of i.i.d. exponential components with common parameter $\tilde{\lambda}$ is increasing in $\tilde{\lambda}$.

To prove our next theorem, we shall need the following result of Marshall and Olkin (1997, p. 59).

**Lemma 2.2.** *A real valued function $\phi$ on the set $A \subset \mathbb{R}^n$ satisfies*

$$\boldsymbol{x} \succeq^{\mathrm{w}} \boldsymbol{y} \text{ on } A \Longrightarrow \phi(\boldsymbol{x}) \geq \phi(\boldsymbol{y})$$

*if and only if $\phi$ is decreasing and Schur convex on $A$.*

**Theorem 2.2.** *Let $X_1, \ldots, X_n$ be independent exponential random variables with $X_i$ having hazard rate $\lambda_i$, $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be another set of independent exponential random variables with $Y_i$ having hazard rate $\lambda_i^*$, $i = 1, \ldots, n$. Then*

$$\boldsymbol{\lambda} \succeq^{\mathrm{p}} \boldsymbol{\lambda}^* \Longrightarrow X_{n:n} \geq_{\mathrm{st}} Y_{n:n}.$$

*Proof.* The survival function of $X_{n:n}$ can be written as

$$\overline{F}_{X_{n:n}}(x) = 1 - \prod_{i=1}^{n}(1 - \mathrm{e}^{-\exp(a_i)x}), \tag{2.4}$$

where $a_i = \log \lambda_i$, $i = 1, \ldots, n$.

Using Lemma 2.2, we find that it is enough to show that the function given by (2.4) is Schur convex and decreasing in the $a_i$s. Now

$$\frac{\partial \overline{F}_{X_{n:n}}}{\partial a_i} = -\prod_{i=1}^{n}(1 - \mathrm{e}^{-\exp(a_i)x})\left(\frac{x\mathrm{e}^{a_i}\mathrm{e}^{-\exp(a_i)x}}{1 - \mathrm{e}^{-\exp(a_i)x}}\right).$$

To prove Schur convexity we have to show that, for $i \neq j$,

$$(a_i - a_j)\left(\frac{\partial \overline{F}_{X_{n:n}}}{\partial a_i} - \frac{\partial \overline{F}_{X_{n:n}}}{\partial a_j}\right) \geq 0,$$

(cf. Marshall and Olkin (1979, p. 57)). That is,

$$x(a_i - a_j)\left(\prod_{i=1}^{n}(1 - \mathrm{e}^{-\exp(a_i)x})\right)\left(\frac{\mathrm{e}^{a_j}\mathrm{e}^{-\exp(a_j)x}}{1 - \mathrm{e}^{-\exp(a_j)x}} - \frac{\mathrm{e}^{a_i}\mathrm{e}^{-\exp(a_i)x}}{1 - \mathrm{e}^{-\exp(a_i)x}}\right) \geq 0, \quad \text{for } i \neq j. \tag{2.5}$$

It is easy to see that the function $b\mathrm{e}^{-bx}/(1 - \mathrm{e}^{-bx})$ is decreasing in $b$. Replacing $b$ with $\mathrm{e}^{a_i}$, it follows that the function $\mathrm{e}^{a_i}\mathrm{e}^{-\exp(a_i)x}/(1 - \mathrm{e}^{-\exp(a_i)x})$ is also decreasing in $a_i$ for $i = 1, \ldots, n$. This proves (2.5). The partial derivative of $\overline{F}_{X_{n:n}}$ with respect to $a_i$ is negative, which in turn implies that the survival function of $X_{n:n}$ is decreasing in $a_i$ for $i = 1, \ldots, n$. This completes the proof.

**Remark 2.1.** Theorem 2.2 strengthens a similar result of Pledger and Proschan (1971) who proved that in the context of Theorem 2.2,

$$\boldsymbol{\lambda} \succeq^{\mathrm{m}} \boldsymbol{\lambda}^* \Longrightarrow X_{n:n} \geq_{\mathrm{st}} Y_{n:n}. \tag{2.6}$$

**Remark 2.2.** Boland *et al.* (1994) showed with the help of a counterexample that, for $n > 2$, (2.6) cannot be strengthened from stochastic ordering to hazard rate ordering. Since majorization implies $p$-larger ordering, it follows that, in general, Theorem 2.2 can not be extended to hazard rate ordering.

**Remark 2.3.** Bon and Paltanea (1999) have recently obtained similar results for convolutions of independent exponential random variables.

As shown in the next example, the results of Theorem 2.2 cannot be extended to other order statistics.

**Example 2.1.** Let $X_1$, $X_2$, $X_3$ be independent exponential random variables with $\boldsymbol{\lambda} = (0.1, 1, 7.9)$ and let $Y_1, Y_2, Y_3$ be independent exponential random with $\boldsymbol{\lambda}^* = (1, 2, 5)$. It is easy to see that $\boldsymbol{\lambda} \succeq^{\mathrm{p}} \boldsymbol{\lambda}^*$. Then $X_{1:3}$ and $Y_{1:3}$ have exponential distributions with respective hazard rates 9 and 8, which implies that $Y_{1:3} \geq_{\mathrm{st}} X_{1:3}$.

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