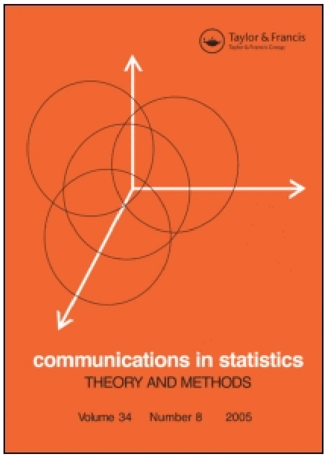


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# Stochastic Orderings of Order Statistics of Independent Random Variables with Different Scale Parameters

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*This is an article on recent results on stochastic comparisons of order statistics of n independent random variables differing in their scale parameters. Most of the results obtained so far are for the Weibull and the Gamma distributions.*

**Keywords** Hazard rate ordering; Majorization; $p$-larger ordering; Proportional hazards family; Schur functions.

**AMS Subject Classification** Primary 60K10; Secondary 60E15.

## 1. Introduction

Let $X_1, \ldots, X_n$ be $n$ random variables and let $X_{(i)}$ denote their $i$th order statistic, $i = 1, \ldots, n$. Order statistics arise naturally at number of places in applications. A $k$-out-of-$n$ system of $n$ components functions if at least $k$ of the $n$ components function. The time of a $k$-out-of-$n$ system of $n$ components with lifetimes $X_1, \ldots, X_n$ corresponds to the $(n-k+1)$th order statistic. Thus, the study of lifetimes of $k$-out-of-$n$ systems is equivalent to studying the stochastic properties of order statistics. In particular, a 1-out-of-$n$ system corresponds to a parallel system and an $n$-out-of-$n$ system corresponds to a series system. A lot of work has been done in the literature on different aspects of order statistics when the observations are independently and identically distributed (i.i.d.). In many practical situations, like in reliability theory, however, the observations are not necessarily i.i.d. Because of the complicated nature of the problem, not much work has been done for the non i.i.d. case. Some interesting partial ordering results on order statistics when the parent observations are independent with proportional hazard rates have been obtained by Pledger and

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Proschan (1971), Proschan and Sethuraman (1976), Boland et al. (1994), Dykstra et al. (1997), and Khaledi and Kochar (2000, 2005), among others. Also see Belzunce et al. (2001) and Boland et al. (2002) for general results on order statistics. In this article we review some recently obtained results on stochastic comparisons of order statistics when the parent observations are independent but with different scale parameters.

Now we introduce notations and recall some definitions. Throughout this article, *increasing* means *non decreasing* and *decreasing* means *non increasing*; and we shall be assuming that all distributions under study are absolutely continuous. Let $X$ and $Y$ be univariate random variables with distribution functions $F$ and $G$, survival functions $\overline{F}$ and $\overline{G}$, density functions $f$ and $g$, and hazard rates $r_F$ $(=f/\overline{F})$ and $r_G$ $(=g/\overline{G})$, respectively. Let $l_X$ $(l_Y)$ and $u_X$ $(u_Y)$ be the left and the right endpoints of the support of $X(Y)$. The random variable $X$ is said to be *stochastically* smaller than $Y$ (denoted by $X \leq_{st} Y$) if $\overline{F}(x) \leq \overline{G}(x)$ for all $x$. This is equivalent to saying that $Eg(X) \leq Eg(Y)$ for any increasing function $g$ for which expectations exist. $X$ is said to be smaller than $Y$ in *hazard rate* ordering (denoted by $X \leq_{hr} Y$) if $\overline{G}(x)/\overline{F}(x)$ is increasing in $x \in (-\infty, max(u_X, u_Y))$. In case the hazard rates exist, it is easy to see that $X \leq_{hr} Y$, if and only if, $r_G(x) \leq r_F(x)$ for every $x$. Note that hazard rate ordering implies stochastic ordering.

A random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is said to be smaller than another random vector $\mathbf{Y} = (Y_1, \ldots, Y_n)$ according to multivariate stochastic ordering (denoted by $\mathbf{X} \overset{st}{\preceq} \mathbf{Y}$) if $h(\mathbf{X}) \leq_{st} h(\mathbf{Y})$ for all increasing functions $h$. It is easy to see that multivariate stochastic ordering implies component-wise stochastic ordering. For more details on stochastic orderings, see Chs. 1 and 4 of Shaked and Shanthikumar (1994) and Müller and Stoyan (2002).

One of the basic tools in establishing various inequalities in statistics and probability is the notion of majorization. Let $\{x_{(1)} \leq \cdots \leq x_{(n)}\}$ denote the increasing arrangement of the components of a vector $\mathbf{x} = (x_1, \ldots, x_n)$. A vector $\mathbf{x}$ is said to majorize another vector $\mathbf{y}$ (written $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$) if $\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}$ for $j = 1, \ldots, n-1$ and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$. Functions that preserve the majorization ordering are called Schur-convex functions. The vector $\mathbf{x}$ is said to majorize the vector $\mathbf{y}$ weakly (written $\mathbf{x} \overset{w}{\succeq} \mathbf{y}$) if $\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}$ for $j = 1, \ldots, n$.

Bon and Paltanea (1999) have considered a pre-order on $\mathbb{R}^{+^n}$, which they call as a *p-larger order*.

**Definition 1.1.** A vector $\mathbf{x}$ in $\mathbb{R}^{+^n}$ is said to be $p$-larger than another vector $\mathbf{y}$ also in $\mathbb{R}^{+^n}$ (written $\mathbf{x} \overset{p}{\succeq} \mathbf{y}$) if $\prod_{i=1}^{j} x_{(i)} \leq \prod_{i=1}^{j} y_{(i)}$, $j = 1, \ldots, n$.

Let $\log(\mathbf{x})$ denote the vector of logarithms of the coordinates of $\mathbf{x}$. It is easy to verify that

$$\mathbf{x} \overset{p}{\succeq} \mathbf{y} \Leftrightarrow \log(\mathbf{x}) \overset{w}{\succeq} \log(\mathbf{y}). \tag{1.1}$$

It is known that $\mathbf{x} \overset{m}{\succeq} \mathbf{y} \Longrightarrow (g(x_1), \ldots, g(x_n)) \overset{w}{\succeq} (g(y_1), \ldots, g(y_n))$ for all concave functions $g$ (cf. Marshall and Olkin, 1979, p. 115). From this and (1.1), it follows that when $\mathbf{x}, \mathbf{y} \in \mathbb{R}^+$

$$\mathbf{x} \overset{m}{\succeq} \mathbf{y} \Longrightarrow \mathbf{x} \overset{p}{\succeq} \mathbf{y}.$$

The converse is, however, not true. For example, the vectors $(0.2, 1, 5) \overset{p}{\succeq} (1, 2, 3)$ but majorization does not hold between these two vectors. Marshall and Olkin (1979) provide extensive and comprehensive details on the theory of majorization and its applications in statistics.

The proof of the main result in this article uses the following result.

**Theorem 1.1** (Marshall and Olkin, 1979, p. 57)**.** *Let $I \subset \mathbb{R}$ be an open interval and let $\phi : I^n \to \mathbb{R}$ be continuously differentiable. Necessary and sufficient conditions for $\phi$ to be Schur-convex on $I^n$ are: $\phi$ is symmetric on $I^n$ and for all $i \neq j$,*

$$(z_i - z_j)[\phi_{(i)}(z) - \phi_{(j)}(z)] \geq 0 \quad \text{for all } z \in I^n,$$

*where $\phi_{(i)}(z)$ denotes the partial derivative of $\phi$ with respect to its $i$th argument.*

In Sec. 2, we stochastically compare order statistics corresponding to two sets of independent Weibull as well as gamma random variables with a common shape parameter but when their scale parameters majorize each other. In Sec. 3, we obtain bounds on the hazard rate and the variance of the lifetime of a parallel system with different independent Weibull distributions.

## 2. Stochastic Comparisons of Order Statistics from Weibull and Gamma Distributions

Weibull and gamma distributions are perhaps the most commonly used distributions in reliability theory and life testing. The probability density function of a Weibull random variable with shape parameter $\alpha(>0)$ and scale parameter $\lambda(>0)$ is

$$f(x, \alpha, \lambda) = \alpha x^{\alpha-1} \lambda^{\alpha} e^{-(\lambda x)^{\alpha}}, \quad x > 0$$

and the probability density function of a gamma random variable $\alpha(>0)$ and scale parameter $\lambda(>0)$ is

$$g(x, \alpha, \lambda) = \lambda^{\alpha} x^{\alpha-1} e^{-\lambda x} / \Gamma(\alpha), \quad x > 0.$$

We shall use the notations $W(\alpha, \lambda)$ to denote a Weibull random variable and $G(\alpha, \lambda)$ to denote a gamma random variable with shape parameter $\alpha$ and scale parameter $\lambda$. These are very flexible families of distributions, having decreasing, constant and increasing failure rates when $0 < \alpha < 1$, $\alpha = 1$, and $\alpha > 1$, respectively.

In this section we study the stochastic properties of order statistics associated with independent random variables $X_1, \ldots, X_n$ when

(a) $X_i \sim W(\alpha, \lambda_i)$ for $i = 1, \ldots, n$,
(b) $X_i \sim G(\alpha, \lambda_i)$ for $i = 1, \ldots, n$.

It is of interest to investigate the effect on the survival function, the hazard rate function, and other characteristics of the time to failure of a system consisting of such components when we switch the vector $(\lambda_1, \ldots, \lambda_n)$ to another vector say, $(\lambda_1^*, \ldots, \lambda_n^*)$. An assumption often made in reliability models is that the components have lifetimes with proportional hazards. Let $X_i$ denote the lifetime of the $i$th component of a reliability system with survival function $\overline{F}_i(t)$, $i = 1, \ldots, n$.

Then they have proportional hazard rates (PHR) if there exist constants $\lambda_1, \ldots, \lambda_n$ and a (cumulative hazard) function $R(t) \geq 0$ such that $\overline{F}_i(t) = e^{-\lambda_i R(t)}$ for $i = 1, \ldots, n$. If $X_1, \ldots, X_n$ are independent random variables such that $X_i \sim W(\alpha, \lambda_i)$ for $i = 1, \ldots, n$, then they belong to the PHR family with $R(t) = t^\alpha$ and a *new* parameter vector $(\mu_1, \ldots, \mu_n)$, where $\mu_i = \lambda_i^\alpha$, $i = 1, \ldots, n$, but not with the original parameters.

Pledger and Proschan (1971) proved the following result for the PHR model which contains exponential distributions as a special case.

**Theorem 2.1.** *Let* $(X_1, \ldots, X_n)$ *and* $(X_1^*, \ldots, X_n^*)$ *be two random vectors of independent lifetimes with proportional hazards and with* $(\lambda_1, \ldots, \lambda_n)$ *and* $(\lambda_1^*, \ldots \lambda_n^*)$ *as the constants of proportionality, respectively. Then*

$$\boldsymbol{\lambda} \overset{m}{\succeq} \boldsymbol{\lambda}^* \Longrightarrow X_{(i)} \geq_{st} X_{(i)}^*, \quad i = 1, \ldots, n. \tag{2.1}$$

Proschan and Sethuraman (1976) extended this result from componentwise stochastic ordering to multivariate stochastic ordering. That is, under the assumptions of Theorem 2.1, they proved that

$$(X_{(1)}, \ldots, X_{(n)}) \overset{st}{\succeq} (X_{(1)}^*, \ldots, X_{(n)}^*). \tag{2.2}$$

It follows from Theorem 2.1 that in the case of Weibull distributions with a common shape parameter $\alpha$ and with scale parameters as $(\lambda_1, \ldots, \lambda_n)$ and $(\lambda_1^*, \ldots \lambda_n^*)$, (2.1) and (2.2) hold if $(\lambda_1^\alpha, \ldots, \lambda_n^\alpha) \overset{m}{\succeq} (\lambda_1^{*\alpha}, \ldots, \lambda_n^{*\alpha})$. Khaledi and Kochar (2005) and Sun and Zhang (2005), respectively, proved that a similar result also holds in Weibull and gamma cases when the two original vectors of scale parameters majorize each other and $0 < \alpha \leq 1$.

**Theorem 2.2.** *Let* $(X_1, \ldots, X_n)$ *and* $(X_1^*, \ldots, X_n^*)$ *be two independent random vectors with*

(A) $X_i \sim W(\alpha, \lambda_i)$ *and* $X_i^* \sim W(\alpha, \lambda_i^*)$, $i = 1, \ldots, n$ *or*
(B) $X_i \sim G(\alpha, \lambda_i)$ *and* $X_i^* \sim G(\alpha, \lambda_i^*)$, $i = 1, \ldots, n$.

*Then for* $0 < \alpha \leq 1$,

$$\boldsymbol{\lambda} \overset{m}{\succeq} \boldsymbol{\lambda}^* \Rightarrow (X_{(1)}, \ldots, X_{(n)}) \overset{st}{\succeq} (X_{(1)}^*, \ldots, X_{(n)}^*).$$

*Proof.* (A) First we prove the result for $n = 2$. According to Theorem 5.4.13 of Barlow and Proschan (1975), in order to prove the required result, it is sufficient to prove that for $0 < \alpha \leq 1$,

(a) $X_{(1)} \geq_{st} X_{(1)}^*$
(b) for $x \leq x'$, $\{X_{(2)} \mid X_{(1)} = x\} \leq_{st} \{X_{(2)} \mid X_{(1)} = x'\}$ and
(c) $\{X_{(2)} \mid X_{(1)} = x\} \geq_{st} \{X_{(2)}^* \mid X_{(1)}^* = x\}$.

Proving (a) is equivalent to proving that $\overline{F}_{X_{(1)}}(x)$, the survival function of $X_{(1)}$ is Schur-convex in $(\lambda_1, \lambda_2)$. To prove it, we use Theorem 1.1. The partial derivative of $\overline{F}_{X_{(1)}}(x)$ with respect to $\lambda_i$ is

$$\frac{\partial \overline{F}_{X_{(1)}}(x)}{\partial \lambda_i} = -\alpha \lambda_i^{\alpha-1} x^\alpha e^{-x^\alpha(\lambda_1^\alpha + \lambda_2^\alpha)}, \quad i = 1, 2.$$

This leads to

$$(\lambda_1 - \lambda_2)\left(\frac{\partial \overline{F}_{X_{(1)}}}{\partial \lambda_1} - \frac{\partial \overline{F}_{X_{(2)}}}{\partial \lambda_2}\right) \geq 0,$$

thus proving (a).

The conditional survival function of $X_{(2)} \mid X_{(1)} = x$,

$$\begin{aligned}\overline{F}_{X_{(2)} \mid X_{(1)}=x}(z) &= \frac{\lambda_1^\alpha e^{-(x\lambda_1)^\alpha-(z\lambda_2)^\alpha} + \lambda_2^\alpha e^{-(x\lambda_2)^\alpha-(z\lambda_1)^\alpha}}{(\lambda_1^\alpha + \lambda_2^\alpha)(e^{-(x\lambda_1)^\alpha-(x\lambda_2)^\alpha})} \\ &= \frac{\lambda_1^\alpha}{\lambda_1^\alpha + \lambda_2^\alpha} e^{-(z\lambda_2)^\alpha+(x\lambda_2)^\alpha} + \frac{\lambda_2^\alpha}{\lambda_1^\alpha + \lambda_2^\alpha} e^{-(z\lambda_1)^\alpha+(x\lambda_1)^\alpha}\end{aligned}$$

is increasing in $x$, thus proving (b). Proving (c) is equivalent to proving that $\overline{F}_{X_{(2)}|X_{(1)}=x}(z)$ is Schur convex in $(\lambda_1, \lambda_2)$. Its partial derivatives with respect to $\lambda_1$ and $\lambda_2$, respectively, are

$$\begin{aligned}\frac{\partial}{\partial \lambda_1}\overline{F}_{X_{(2)}|X_{(1)}=x}(z) &= \frac{\alpha\lambda_1^{\alpha-1}(\lambda_1^\alpha + \lambda_2^\alpha) - \alpha\lambda_1^{2\alpha-1}}{(\lambda_1^\alpha + \lambda_2^\alpha)^2} e^{-\lambda_2^\alpha(z^\alpha - x^\alpha)} \\ &\quad - \frac{\alpha\lambda_1^{\alpha-1}\lambda_2^\alpha}{(\lambda_1^\alpha + \lambda_2^\alpha)^2} e^{-\lambda_1^\alpha(z^\alpha - x^\alpha)} - (z^\alpha - x^\alpha)\alpha\lambda_1^{\alpha-1} e^{-\lambda_1^\alpha(z^\alpha - x^\alpha)} \frac{\lambda_2^\alpha}{\lambda_1^\alpha + \lambda_2^\alpha},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial \lambda_2}\overline{F}_{X_{(2)}|X_{(1)}=x}(z) &= \frac{\alpha\lambda_2^{\alpha-1}(\lambda_1^\alpha + \lambda_2^\alpha) - \alpha\lambda_2^{2\alpha-1}}{(\lambda_1^\alpha + \lambda_2^\alpha)^2} e^{-\lambda_1^\alpha(z^\alpha - x^\alpha)} \\ &\quad - \frac{\alpha\lambda_2^{\alpha-1}\lambda_1^\alpha}{(\lambda_1^\alpha + \lambda_2^\alpha)^2} e^{-\lambda_2^\alpha(z^\alpha - x^\alpha)} - (z^\alpha - x^\alpha)\alpha\lambda_2^{\alpha-1} e^{-\lambda_2^\alpha(z^\alpha - x^\alpha)} \frac{\lambda_1^\alpha}{\lambda_1^\alpha + \lambda_2^\alpha}.\end{aligned}$$

Now the difference between these two derivatives is

$$\begin{aligned}&\frac{\partial}{\partial \lambda_1}\overline{F}_{X_{(2)}|X_{(1)}=x}(z) - \frac{\partial}{\partial \lambda_2}\overline{F}_{X_{(2)}|X_{(1)}=x}(z) \\ &\quad = \frac{\alpha\lambda_1^{\alpha-1}\lambda_2^{\alpha-1}}{(\lambda_1^\alpha + \lambda_2^\alpha)^2}\{e^{-\lambda_2^\alpha(z^\alpha - x^\alpha)}(\lambda_1 + \lambda_2 + \lambda_1(z^\alpha - x^\alpha)(\lambda_1^\alpha + \lambda_2^\alpha)) \\ &\quad\quad - e^{\lambda_1^\alpha(z^\alpha - x^\alpha)}(\lambda_1 + \lambda_2 + \lambda_2(z^\alpha - x^\alpha)(\lambda_1^\alpha + \lambda_2^\alpha))\}.\end{aligned}$$

If $\lambda_1 > \lambda_2$, then $e^{-\lambda_2^\alpha(z^\alpha - x^\alpha)} \geq e^{-\lambda_1^\alpha(z^\alpha - x^\alpha)}$, since $z > x$. If $\lambda_1 < \lambda_2$, then the above inequality is reversed. That is,

$$(\lambda_1 - \lambda_2)\left(\frac{\partial}{\partial \lambda_1}\overline{F}_{X_{(2)}|X_{(1)}=x}(z) - \frac{\partial}{\partial \lambda_2}\overline{F}_{X_{(2)}|X_{(1)}=x}(z)\right) \geq 0.$$

The proof of part (c) again follows from Theorem 1.1. This completes the proof in the case of $n = 2$. The proof for $n > 2$ follows from this and using similar kind of arguments as used in Theorem 3.4 of Proschan and Sethuraman (1976).

(B) Using similar kind of arguments as used to prove (A), Sun and Zhang (2005) proved this result.

For comparing two series systems with independent Weibull components, Khaledi and Kochar (2005) proved the following stronger result.

**Theorem 2.3.** *Let $X_1, \ldots, X_n$ be independent random variables with $X_i \sim W(\alpha, \lambda_i)$, $i = 1, \ldots, n$. Let $X_1^*, \ldots, X_n^*$ be another set of independent random variables with $X_i^* \sim W(\alpha, \lambda_i^*)$, $i = 1, \ldots, n$. Then $\boldsymbol{\lambda} \overset{m}{\succeq} \boldsymbol{\lambda}^*$ implies that $X_{(1)} \geq_{hr} X_{(1)}^*$ for $0 < \alpha \leq 1$ and $X_{(1)} \leq_{hr} X_{(1)}^*$ for $\alpha > 1$.*

*Proof.* The hazard rate of $X_{(1)}$ is

$$r_{X_{(1)}}(x; \lambda_1, \ldots \lambda_n) = \sum_{i=1}^{n} \alpha x^{\alpha-1} \lambda_i{}^{\alpha}.$$

The function $g(\lambda) = \alpha x^{\alpha-1}\lambda_i^{\alpha}$ is concave (convex) in $\lambda$ for $0 < \alpha \leq 1$ $(\alpha \geq 1)$. It follows from Proposition C.1. of Marshall and Olkin (1979, p. 64), that $\sum_{i=1}^{n} g(\lambda_i)$ is Schur concave (convex). This completes the proof. □

Khaledi and Kochar (2005) proved that for the largest order statistic, the conclusion of Theorem 2.1 holds under the weaker $p$-larger ordering. This result is formally stated below.

**Theorem 2.4.** *Let $X_1, \ldots, X_n$ be independent random variables with $X_i$ having survival function $\overline{F}^{\lambda_i}(x)$, $i = 1, \ldots, n$. Let $X_1^*, \ldots, X_n^*$ be another set of independent random variables with $X_i^*$ having survival function $\overline{F}^{\lambda_i^*}(x)$, $i = 1, \ldots, n$. Then*

$$\boldsymbol{\lambda} \overset{p}{\succeq} \boldsymbol{\lambda}^* \Longrightarrow X_{(n)} \geq_{st} X_{(n)}^*.$$

Since for any $\alpha > 0$,

$$(\lambda_1, \ldots, \lambda_n) \overset{p}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*) \Leftrightarrow (\lambda_1^{\alpha}, \ldots, \lambda_n^{\alpha}) \overset{p}{\succeq} (\lambda_1^{*\alpha}, \ldots, \lambda_n^{*\alpha}),$$

the above theorem leads to the following corollary.

**Corollary 2.1.** *Let $X_1, \ldots, X_n$ be independent random variables with $X_i \sim W(\alpha, \lambda_i)$, $i = 1, \ldots, n$. Let $X_1^*, \ldots, X_n^*$ be another set of independent random variables with $X_i^* \sim W(\alpha, \lambda_i^*)$, $i = 1, \ldots, n$. Then for any $\alpha > 0$,*

$$\boldsymbol{\lambda} \overset{p}{\succeq} \boldsymbol{\lambda}^* \Longrightarrow X_{(n)} \geq_{st} X_{(n)}^*.$$

Khaledi and Kochar (2000) had earlier proved a special case of the above corollary when $\alpha = 1$. It will be interesting to see whether the above result can be extended to other order statistics. Boland et al. (1994) have proved that in case $\alpha = 1$, such a result does not hold for parallel systems with more than two components.

Sun and Zhang (2005) also proved the following result for gamma random variables.

**Theorem 2.5.** *Let $(X_1, \ldots, X_n)$ and $(X_1^*, \ldots, X_n^*)$ be two independent random vectors with $X_i \sim G(\alpha, \lambda_i)$ and $X_i^* \sim G(\alpha, \lambda_i^*)$, $i = 1, \ldots, n$. Then for $\alpha \geq 1$, $\boldsymbol{\lambda} \overset{m}{\succeq} \boldsymbol{\lambda}^*$ implies $X_{(1)} \leq_{st} X_{(1)}^*$ and $X_{(n)} \geq_{st} X_{(n)}^*$.*

## 3. Comparisons with the i.i.d. Case

Dykstra et al. (1997) and Khaledi and Kochar (2000) studied the problem of stochastically comparing the largest order statistic of a set of $n$ independent and non-identically distributed exponential random variables with that corresponding to a set of $n$ independent and identically distributed exponential random variables. In particular, Khaledi and Kochar (2000) proved the following result.

**Theorem 3.1.** *Let $X_1, \ldots, X_n$ be independent exponential random variables with $X_i$ having hazard rate $\lambda_i$, $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be a random sample of size n from an exponential distribution with common hazard rate $\tilde{\lambda} = (\prod_{i=1}^n \lambda_i)^{1/n}$. Then*

(a) $X_{(n)} \geq_{hr} Y_{(n)}$;
(b) $X_{(n)} \geq_{disp} Y_{(n)}$.

In Theorem 3.3, we extend this result from exponential to the PHR model. To prove this, we need the following result due to Rojo and He (1991).

**Theorem 3.2.** *Let X and Y be two random variables such that $X \leq_{st} Y$. Then $X \leq_{disp} Y$ implies that $\gamma(X) \leq_{disp} \gamma(Y)$ where $\gamma$ is a non decreasing convex function.*

**Theorem 3.3.** *Let $X_1, \ldots, X_n$ be independent random variables with $X_i$ having survival function $\overline{F}^{\lambda_i}(x), i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be a random sample of size n from a distribution with survival function $\overline{F}^{\tilde{\lambda}}(x)$, where $\tilde{\lambda} = (\prod_{i=1}^n \lambda_i)^{1/n}$. Then*

(a) $X_{(n)} \geq_{hr} Y_{(n)}$ *; and*
(b) *if F is DFR, then* $X_{(n)} \geq_{disp} Y_{(n)}$.

*Proof.* (a) Let $H(x) = -\log \overline{F}(x)$ denote the cumulative hazard of $F$. Let $Z_i = H(X_i)$, $i = 1, \ldots, n$ and $W_i = H(Y_i)$, $i = 1, \ldots, n$. Since the $X_i$'s follow the PHR model, it is easy to show that $Z_i$ is exponential with hazard rate $\lambda_i, i = 1, \ldots, n$. Similarly, $W_i$ is exponential with hazard rate $\tilde{\lambda}, i = 1, \ldots, n$. It follows from Theorem 3.1(a) that $Z_{(n)} \geq_{hr} W_{(n)}$. Using this fact (since $H^{-1}$, the right inverse of $H$, is non decreasing), it is easy to show that $H^{-1}(Z_{(n)}) \geq_{hr} H^{-1}(W_{(n)})$ from which part (a) follows.

(b) Theorem 3.1(a) and (b), respectively, imply that $Z_{(n)} \geq_{st} W_{(n)}$ and $Z_{(n)} \geq_{disp} W_{(n)}$. The function $H^{-1}(x)$ is convex, since $F$ is *DFR*, and is non decreasing. Using these observations, it follows from Theorem 3.2 that $H^{-1}(Z_{(n)}) \geq_{disp} H^{-1}(W_{(n)})$ which is equivalent to $X_{(n)} \geq_{disp} Y_{(n)}$. □

We show with the help of the next example that the DFR condition in the above theorem cannot be dispensed with.

**Example 3.1.** Let $X_1$ and $X_2$ be independent random variable with $X_i$ having survival function $\overline{F_i}(x) = (1 - x)^{\lambda_i}$, $0 \leq x \leq 1$, $i = 1, 2$. Let $Y_1$ and $Y_2$ be independent random variables with common survival function $\overline{G}(x) = (1-x)^{(\lambda_1\lambda_2)^{1/2}}$, $0 \leq x \leq 1$. Let $\lambda_1 = 1$ and $\lambda_2 = 4$. Under this setting, it is easy to find that $var(X_{(2)}) = \frac{43}{720} < \frac{11}{225} = var(Y_{(2)})$, from which it follows that part (b) of Theorem 3.3 may not hold for the case when $F$, the baseline distribution, is not *DFR*. Note that in this example $F$ being uniform distribution on (0, 1) is IFR.

**Corollary 3.1.** *Let $X_1, \ldots, X_n$ be independent random variables with $X_i \sim W(\alpha, \lambda_i)$, $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be a random sample of size n from a $W(\alpha, \tilde{\lambda})$ distribution, where $\tilde{\lambda} = (\prod_{i=1}^n \lambda_i)^{1/n}$. Then*

(a) *for any $\alpha > 0$, $X_{(n)} \geq_{hr} Y_{(n)}$*
(b) *for $0 < \alpha \leq 1$, $X_{(n)} \geq_{disp} Y_{(n)}$.*

*Proof.* The proof follows from Theorem 3.3 since for any $\alpha > 0$, the geometric mean of $\lambda_1^\alpha, \ldots, \lambda_n^\alpha$ is $\tilde{\lambda}^\alpha$ and the fact that the Weibull distribution is DFR when $0 < \alpha \leq 1$. □

It will be interesting to investigate whether results parallel to Corollaries 2.1 and 3.1 hold in the case of gamma and other distributions.

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