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Title: Communications in statistics: theory and methods.
Article: Bagai and Kochar: A distribution-free test for the equality of failure rates due to two competing risks.
Volume: 18 (1989), no. 1, 107--120
Number: 1
Date: 1989
Pages: 107-120
Imprint: [New York, Marcel Dekker] 1976-
ISSN: 0361-0926

Verified: <TN:109874><ODYSSEY:131.252.130.129/ILL> OCLC
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COMMUN. STATIST.-THEORY METH. 18(1), 107-120 (1989)

A DISTRIBUTION–FREE TEST FOR THE EQUALITY OF FAILURE RATES DUE TO TWO COMPETING RISKS

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Key Words and Phrases: *U–statistics; sign statistic; Wilcoxon signed rank statistic; asymptotic relative efficiency.*

ABSTRACT

A distribution–free test has been proposed for testing the equality of two failure rates in the competing risks set up when the information available is only the causes of failure and the observed times of failure. The proposed test is consistent and unbiased. The test performs well as compared to sign test for a wide spectrum of alternatives, whereas for the proportional hazards model the sign test is seen to do better.

1. INTRODUCTION

Consider the competing risks set up wherein a unit is subject to failure due to any of several risks, although the actual failure is attributed to only one of these and is called the cause of failure. Suppose that there are two risks operating and the notional lifetimes of a unit under these two risks are denoted by X and Y, respectively. The actual information available is $T = \min(X,Y)$, the time at which the unit fails and the cause of failure indicated by $\delta = I\{X > Y\}$, $I\{A\}$ being the indicator function of the set A. Such data also arise when two components with lifetimes X and Y are arranged in series and T denoting the lifetime of this series system. Cox (1959) has considered some more examples where the above type of data arise.

We assume in this paper that X and Y are independent and absolutely continuous random variables with distribution functions F and G such that $F(0) = G(0) = 0$. It is to be kept in mind that the assumption of independence cannot be tested on the bases of the competing risks set up due to nonidentifiability problems. It must be verified externally to the data. This assumption of independence of the risks may not always be appropriate. However, Keyfitz et al. (1972) have argued that risks to human life can be grouped into four nonoverlapping sets, which are independent of each other. In such situations it is appropriate to use methods based on independence assumption while dealing with these pooled risks.

Let $\bar{F}$ and $\bar{G}$ be the survival functions, f and g the probability density functions; and $r_F = f/\bar{F}$ and $r_G = g/\bar{G}$, the failure rates of the two risks X and Y, respectively. Let $X_1,\ldots,X_n$ and $Y_1,\ldots,Y_n$ be two independent random samples from F and G, respectively denoting the hypothetical times to failure of the n individuals in the sample under the two risks. However, due to the competing risks set up, we observe only $(T_1,\delta_1),\ldots,(T_n,\delta_n)$, where $T_i = \min(X_i,Y_i)$ denotes the time to failure and $\delta_i = I\{X_i > Y_i\}$ indicates the cause of failure of the ith unit. On the basis of this observed information, we wish to test the null hypothesis

$$H_0 : r_F(x) = r_G(x), \text{ for every } x, \text{ that is,}$$
$$F(x) = G(x), \text{ for every } x$$

against the alternative

$$H_A : r_F(x) \le r_G(x), \text{ for every } x, \text{ and with a strict inequality over a set of nonzero probability.}$$

The problem of testing for the equality of the failure rates of two competing risks in industrial reliability, arises when we wish to determine if one of the two components put in operation in series is more reliable (in the sense of a smaller failure rate throughout) than the other. If we were interested in the stochastic dominance of one survival function by the other only, one could use graphical methods based on the sample survival functions obtained form the competing risks data. These would be the product limit estimators obtained from the two samples censoring each other, (see, Kalbfleisch and Prentice (1980) p. 162). However, the graphical methods would not enable us to associate any error rates with our decision. As shown in Kochar (1979) ordering in terms of failure rates is finer than the ordering of survival functions in the sense that the former implies the latter. Graphical methods for comparison in terms of failure rates would be based on estimates of the failure rates which are typically a lot more complicated to handle. It will be difficult to gain any meaningful insight through graphical comparisons alone.

It is felt that if we can realistically assume the exponential distribution for the two risks, then we should use appropriate optimal test. It would be based on the estimates of the scale parameters of the underlying distributions and can be equivalently carried out in terms of the sign statistic discussed below. However, the parametric theory becomes very complicated if the distribution is thought to be some other life distribution. See, for example, David and Moeschberger (1978). Also, in many situations the experimenter may not have knowledge of the distributions involved precise enough to follow the parametric theory. So we feel that there is a strong case for the development of appropriate nonparametric tests for this problem.

We present below a distribution –free test for the equality of failure rates in the competing risks set up, which is simple to carry out and has good efficiency.

When complete independent random samples from F and G are available, tests for this testing problem have been given by Chikkagoudar and

Shuster (1974), Kochar (1979 and 1981), Joe and Proschan (1984), Cheng (1985) and Bagai and Kochar (1986). But these tests are not applicable in above setup. The classical sign test may be applied in this case as the competing risks data is sufficient for its use. The use of the sign test and some other distribution–free tests when the alternative hypothesis specifies stochastic dominance have been considered in Bagai (1986) and Bagai, Deshpandé and Kochar (1987).

The new test is introduced in the second section. In Section 3, the exact as well as the asymptotic distribution of the proposed test statistic has been considered. Section 4 is devoted to asymptotic relative efficiency comparisons. Finally, in the last section, we consider an example to illustrate the use of the proposed test.

2. THE PROPOSED TEST

Under the alternative H_A the failure rate due to the first risk is required to be uniformly smaller than the failure rate due to the second risk. Thus, under the alternative H_A, we would expect the failures due to risk II to occur quicker or at an earlier stage, as compared to the failures due to risk I. Thus, if we have (X_1,Y_1) and (X_2,Y_2) as the hypothetical failure times on two individuals due to risk I and risk II, respectively, then arrangements of the type Y's preceding X's [YYXX] and Y's being sandwiched between X's [XYYX] tend to favor the alternative H_A, whereas arrangements of the type [XXYY] and [YXXY] would tend to favor a smaller failure rate due to the second risk. See, Kochar (1979) for more details. He considered the functional

$$\Delta(F,G) = P[YYXX] + P[XYYX] - P[XXYY] - P[YXXY] \qquad (2.1)$$

for testing H_0 against H_A. Here P[YYXX] means

$$P[\{Y_1 < Y_2 < X_1 < X_2\} \cup \{Y_2 < Y_1 < X_1 < X_2\} \cup \{Y_1 < Y_2 < X_2 < X_1\}$$

$\cup \{Y_2 < Y_1 < X_2 < X_1\}]$, where X_1, X_2, Y_1 and Y_2 are independent observations, the first two from F and the latter two from G. Under H_0, $\Delta(F,G) = 0$ but under the alternative, $\Delta(F,G) > 0$. He proposed a test based on the U–statistic estimator of $\Delta(F,G)$ based on the complete samples. Below we propose a U–statistic estimator of $\Delta(F,G)$ which can be evaluated in the competing risks setup. Let (X_i,Y_i) and (X_j,Y_j) be two independent pairs from which the available information is (T_i,δ_i) and (T_j,δ_j). The following arrangements of X's and Y's are possible in terms of T's and δ's.

	$\delta_i=1, \delta_j=1$	$\delta_i=1, \delta_j=0$	$\delta_i=0, \delta_j=1$	$\delta_i=0, \delta_j=0$
$T_i>T_j$	$Y_j<Y_i<X_i$	$X_j<Y_i<X_i$	$Y_j<X_i<Y_i$	$X_j<X_i<Y_i$
	$Y_j<X_j$	$X_j<Y_j$	$Y_j<X_j$	$X_j<Y_j$
$T_i<T_j$	$Y_i<Y_j<X_j$	$Y_i<X_j<Y_j$	$X_i<Y_j<X_j$	$X_i<X_j<Y_j$
	$Y_i<X_i$	$Y_i<X_i$	$X_i<Y_i$	$X_i<Y_i$

These arrangements lead us to consider the following kernel

$$\varphi^*\{(T_i,\delta_i),(T_j,\delta_j)\} = \begin{cases} 1 \text{ if } \delta_i = 1 \text{ and } T_i > T_j \\ \quad \text{or } \delta_j = 1 \text{ and } T_j > T_i \\ -1, \text{ otherwise} \end{cases} \qquad (2.2)$$

Let U^* be the corresponding U–statistic defined by

$$U^* = [\tbinom{n}{2}]^{-1} \sum_{1 \le i < j \le n}\sum \varphi^*\{(T_i,\delta_i),(T_j,\delta_j)\} \qquad (2.3)$$

It is easy to see that $E(U^*) = E[\varphi^*\{(T_i,\delta_i),(T_j,\delta_j)\}] = \Delta(F,G)$. One can base a test on U^* for testing H_0 against H_A. We, however, consider an equivalent statistic

$$U = [\binom{n}{2}]^{-1} \sum\sum_{1\le i<j\le n} \varphi\{(T_i,\delta_i),(T_j,\delta_j)\} \tag{2.4}$$

where

$$\varphi\{(T_i,\delta_i),(T_j,\delta_j)\} = \begin{cases} 1 \text{ if } \delta_i = 1 \text{ and } T_i > T_j \\ \quad \text{or } \delta_j = 1 \text{ and } T_j > T_i \\ 0, \text{otherwise} \end{cases} \tag{2.5}$$

Large values of U are significant for testing H_0 against the one sided alternative H_A. Let $R_1,\dots,R_n$ be the ranks of $T_1,\dots,T_n$. Then U can also be expressed as

$$\binom{n}{2}U = \sum_{i=1}^{n} (R_i-1)\delta_i \tag{2.6}$$

Clearly, $\sum_{i=1}^{n} \delta_i$ is the sign statistic and $\sum_{i=1}^{n} R_i\delta_i$ is the Wilcoxon signed rank type statistic in the present context.

3. DISTRIBUTION OF U

Let $T_{(1)} < T_{(2)} < \dots < T_{(n)}$ denote the ordered T_i's. Let

$$W_j = \begin{cases} 1 \text{ if } T_{(j)} \text{ corresponds to a Y–observation} \\ 0 \text{ otherwise} \end{cases}$$

Then from (2.6), we find that U can be expressed as

$$S = \binom{n}{2}U = \sum_{j=1}^{n} (j-1)W_j \tag{3.1}$$

We shall find the moment generating function of S under H_0 to discuss the exact null distribution of the test statistic. To that end, we shall need the following results.

Result 1: (Armitage (1959) and Allen (1963)). the pair (X,Y) follows the proportional hazards model $\bar{G}(X) = \{\bar{F}(x)\}^{\beta}$ for a positive β if and only if $T = \min(X.Y)$ and $\delta = I\{X>Y\}$ are independent.

Result 2: Under the proportional hazards model $W_1,W_2,\dots,W_n$ are independent and identically distributed with $P[W_i=1] = p = P[X > Y]$, $i = 1,2,\dots,n$, $0 < p < 1$.

With the help of these results, we can prove the following.

Theorem 3.1. Under the proportional hazards model the moment generating function of $S = \binom{n}{2}U$ is

$$M_S(t) = \prod_{j=1}^{n-1} [(1-p) + pe^{jt}] \tag{3.2}$$

Proof: The moment generating function of S under proportional hazards model is

$$\begin{aligned} M_S(t) &= E[e^{tS}] \\ &= \prod_{j=1}^{n} E[\exp\{t(j-1)W_j\}]. \\ &= \prod_{j=1}^{n} [(1-p) + pe^{t(j-1)}] \\ &= \prod_{j=1}^{n-1} [(1-p) + pe^{tj}]. \end{aligned}$$

Corollary 3.1. Under H_0

$$M_S(t) = \frac{1}{2^{n-1}} \prod_{j=1}^{n-1} (1+e^{tj}). \qquad (3.3)$$

From the moment generating function, we can easily obtain the mean and the variance of U under proportional hazards model as

$$E[U] = p \text{ and } Var[U] = \frac{2}{3} pq \frac{(2n-1)}{n(n-1)}$$

Under H_0, i.e., when $p = \frac{1}{2}$, we have

$$E[U] = \frac{1}{2} \text{ and } Var[U] = \frac{2n-1}{6n(n-1)}$$

It can be shown that under H_0, the distribution of U is symmetric about its mean 1/2.

It is interesting to note that the moment generating function of the Wilcoxon Signed Rank statistic under H_0 is of the form (3.3) with $n-1$ replaced by n (see, Hettmansperger (1984), p. 35). Hence the tables for the critical points of the Wilcoxon Signed Rank statistic which are extensively available in the literature (for example, Hollander and Wolfe (1973)) can be used for the statistic S by making appropriate changes in the value of n.

The proof of the following theorem follows easily from the well–known properties of U–statistics (see, Puri and Sen 1971)).

Theorem 3.2. The asymptotic distribution of $n^{1/2}[U-E(U)]$ as $n \to \infty$ is normal with mean zero and variance $4\xi_1$, where

$$\xi_1 = E[\psi^2(X_1,Y_1)] - E^2(U)$$
$$\psi(x_1,y_1) = E(\Phi(x_1,y_1,X_2,Y_2)].$$

Under H_0, $E(U) = 1/2$ and $\lim_{n\to\infty} n\sigma_U^2 = 1/3$.

It is easily seen that the test based on rejecting H_0 for large values of U is consistent and unbiased for testing again H_A (Bagai 1986).

4. ASYMPTOTIC RELATIVE EFFICIENCIES

In order to compare the Pitman asymptotic relative efficiencies, we parametrize the problem in the following way. Let $G(x) = F_\theta(x)$, $F(x) = F_0(x)$, where F is absolutely continuous and θ is a positive real number such that $r_{F_0}(x) \le r_{F_\theta}(x)$ for all x and strict inequality over a set of nonzero probability for every $\theta > 0$.

It can be shown that the sign test $nU_1 = \sum_{i=1}^{n} \delta_i$ is the locally most powerful rank test for the proportional hazards model. The asymptotic relative efficiencies of the U test relative to the sign test U_1 are compared for the following alternatives

$H_1 : r_{F_\theta}(x) = (\theta+1)r_F(x)$, (the proportional hazards model)

$H_2 : \bar{F}_\theta(x) = (1-\theta)\bar{F}(x) + \theta\bar{F}(x)[1-F^k(x)]$, $k > 1/2$

$H_3 : \bar{F}_\theta(x) = \exp[-\{x+\theta(x+e^{-x}-1)\}]$, (Makeham distribution)

that is, $r_{F_\theta}(x) = 1+\theta(1-e^{-x})$

$H_4 : \bar{F}_\theta(x) = \bar{F}(x) \exp[-\theta/2\{\log \bar{F}(x)\}^2]$,

that is, $r_{F_\theta}(x) = r_F(x)[1-\theta \log \bar{F}(x)]$

(linearly increasing failure rate when F is exponential).

All these alternatives belong to H_A since H_A holds if and only if $\bar{F}(x)/\bar{F}_\theta(x)$ is nondecreasing in x, (see, for example, Kochar (1981). H_1 includes, in particular, the alternative of two exponential distributions with ordered scale parameters. The other alternatives are standard in the literature for this type of problem, (see, Joe and Proschan (1984)).

The following table gives the asymptotic relative efficiencies $e(U,U_1)$ of the U–test with respect to the sign test U_1 for the above alternatives.

TABLE I

Asymptotic Relative Efficiencies of U relative to the sign statistic U_1

Alternative	$e(U,U_1)$
H_1	.75
H_2, k = 1	.48
k = 2	1.08
k = 3	1.53
H_3	1.47
H_4	1.69

It is seen from the above table that the newly proposed test U performs very well as compared to its competitor sign test for Makeham and linear failure rate distribution. For the proportional hazards model the sign test is seen to be better.

5. AN EXAMPLE

In most reliability situations questions of interest are, if a certain component is improved what will be its effect on the performance of the system? How can we estimate the true failure rate due to a particular cause of interest under the complicating presence of other causes of failure? Is the failure rate due to one cause smaller than the failure rate due to the other cause? Here we illustrate the use of the sign statistic U_1 and the statistic U in answering the last question.

We consider a set of mortality data available in Hoel (1972). The data was obtained from a laboratory experiment on RFM strain male mice which had received a radiation dose of 300r at any age of 5–6 weeks and were kept in a conventional environment. We considered only two major risks of death:

TABLE II

Ages at death for 99 RFM conventional male mice which received a radiation dose of 300r at the age 5–6 weeks due to cancer and due to all other causes

Other causes

40	42	51	62	163	179	206	222	228	249
252	282	324	333	341	366	385	407	420	431
441	461	462	482	517	517	524	564	567	586
619	620	621	622	647	651	686	761	763	

Cancer

159	189	191	198	200	207	220	235	245
250	256	261	265	266	280	317	318	343
356	383	399	403	414	428	432	495	525
536	549	552	554	557	558	571	586	594
596	605	612	621	628	631	636	643	647
648	649	661	663	666	670	695	697	700
705	712	713	738	748	753			

first is cancer and the second risk is the combination of all other risks into a single group. Biologists believe that the disease considered is lethal and is independent of other causes of failure. Thus, the basic assumptions of the model are reasonably well satisfied.

Let $X_1,\ldots,X_{99}$ and $Y_1,\ldots,Y_{99}$ denote the latent times to death of the mice due to other causes and cancer, respectively. In our terminology $\delta_i = 1$ when the death of the ith mice is due to cancer and $\delta_i = 0$, otherwise. We are interested in a comparison between the failure rates due to the two risks.

For exploratory data analysis, one can plot the estimators of the failure rates under the random censoring model as X's randomly censor Y's and vice–versa under the competing risks model. However, these estimators are difficult to implement in practice. As noted earlier the alternative H_A implies that X's are stochastically larger than Y's. So, we plot the Kaplan–Meier estimators (Kalbfleisch and Prentice (1980) pp. 12–13) of the survival functions $\bar{F}$ and $\bar{G}$, respectively, in Figure I below. From this, we get an indication that probably the failure rate due to cancer is higher than due to other causes.

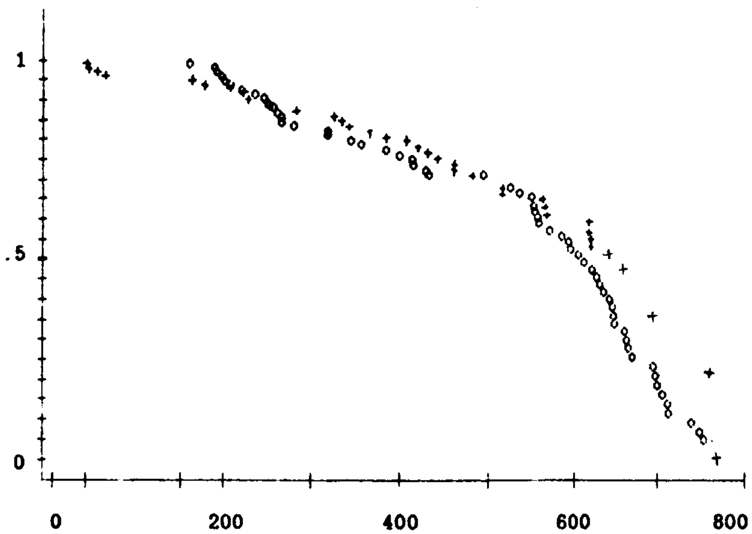

Figure I. Kaplan–Meier estimators of the survival functions
+ for other causes
o for cancer

We find that the standardized values of U_1 and U are 2.1 and 2.76, respectively. We compare these values with the asymptotic critical values and conclude that the null hypothesis H_0 is rejected by both the tests at 5% level. There is evidence to conclude that the failure rate due to other causes is smaller than the failure rate due to cancer. We note that, in this example, the new test leads to a smaller asymptotic p–value than the sign test.

ACKNOWLEDGEMENTS

The authors are grateful to the referees for their suggestions which led to an improved version of the paper. The paper was finalized while Subhash C. Kochar was visiting the University of Northern Illinois and the University of Iowa and Jayant V. Deshpandé was visiting the University of California, Santa Barbara.

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Received by Editorial Board member January 1988; Revised August 1988.

Recommended by S. Kocherlakata, University of Manitoba, CANADA.

Refereed by M. S. Chikkagoudar, Karnatak University, Karnatak, INDIA.

COMMUN. STATIST.-THEORY METH. 18(1), 121-146 (1989)

NONPARAMETRIC EMPIRICAL BAYES ESTIMATION OF THE DISTRIBUTION FUNCTION AND THE MEAN

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Key Words and Phrases: estimation, distribution function, mean, Dirichlet Process Prior, Bayes, Empirical Bayes, Bayes Risk, relative savings loss, asymptotic optimality, Dirichlet Invariant Process.

ABSTRACT

The paper considers nonparametric empirical Bayes estimation of the distribution function and the mean both for finite and infinite populations. The proposed estimators are optimal within the class of estimators which are weighted averages of empirical distribution functions when the underlying prior is a Dirichlet process prior. Comparison with certain rival estimators is made. The results are also obtained for estimable parameters of degree one with a Dirichlet invariant prior.

1. INTRODUCTION

Korwar and Hollander (1976) considered nonparametric empirical Bayes estimation of the distribution function based on Dirichlet process priors. In a Dirichlet process analysis, a n-dimensional random vector $\underset{\sim}{X} = (X_1, \ldots, X_n)$ is viewed as a random sample of size n from some random probability measure P on some measurable space $(\mathfrak{X}, \mathfrak{B})$. Let α be a non-null σ-additive finite measure on $(\mathfrak{X}, \mathfrak{B})$. We say P has a Dirichlet process prior with parameter α, if for every measurable