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A note on characterization of symmetry about a point

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Abstract. Behboodian investigated, *inter alia*, the question of symmetry of a linear form in three independent and identically distributed random variables implying their symmetry about some point. By providing a counter-example, we point out a flaw in Behboodian's paper.

1 Introduction

Let $X_1,\ldots,X_n$ be a random sample from an absolutely continuous distribution with cumulative distribution function F and density function f. X is said to be symmetric about θ if and only if the random variables $X-\theta$ and $\theta-X$ have the same distribution. If $\theta=0$, we simply call X a symmetric random variable.

Randles *et al.* (1980) proposed an asymptotically distribution-free test for testing the symmetry of X about an unknown point θ. It is easy to prove that the symmetry of X about θ implies the symmetry of the random variable $X_\lambda^*=X_1-\lambda X_2-\bar{\lambda}X_3$ (about 0) where $0<\lambda\leqslant 1, \bar{\lambda}=1-\lambda$; and X_1, X_2 and X_3 are three independent copies of X. Their test, which is commonly known as the 'triples test', is based on the U statistic estimator $\hat{\eta}$ of the parameter

$$\eta=\Pr\{X_1+X_2-2X_3>0\}-\Pr\{X_1+X_2-2X_3<0\} \tag{1}$$

Obviously, under H_0, $\eta=0$.

Behboodian (1989) studies some properties of η. Also, Theorem 1 there asserts that for independent, identically distributed random variables X_1, X_2 and X_3—assumed to have a non-vanishing characteristic function (CHF.)—the symmetry of $X_1+X_2-2X_3$ is equivalent to the symmetry of X_1 about some point. The proof depends on the (false) conclusion that exponential functions are the only (continuous) solutions of the functional equation $f(2t)=\{f(t)\}^2$. The example below provides a family of CHFs, non-vanishing on $\mathbb{R}$ and satisfying the functional equation

$$\phi(2t)=\{\phi(t)\}^2\ \forall t\in\mathbb{R} \tag{2}$$

Then, the CHF of $X_1+X_2-2X_3$ is real-valued, being equal to

$$\{\phi(t)\}^2\phi(-2t)=|\phi(2t)|^2=\{\phi(-t)\}^2\phi(2t) \qquad (>0\ \forall t\in\mathbb{R})$$

We shall see below that there exists no real c such that $\phi(-t)=\phi(t)e^{ct}\ \forall t\in\mathbb{R}$, i.e. X_1 cannot be symmetric about any point. Thus we have a counter-example to the quoted assertion.

2. Example

Consider an infinitely divisible CHF ϕ with the Lévy representation

$$L(0,0,M,N) \equiv \log\phi(t) = \int_{-\infty}^{0}\left[e^{itu}-1-\frac{itu}{1+u^2}\right]dM(u) + \int_{0}^{\infty}\left[e^{itu}-1-\frac{itu}{1+u^2}\right]dN(u)$$

(for an explanation of the notation and other details, see Ramachandran, 1967, pp. 27–29), where M and N are subject to the following additional conditions:

$$M(u)=2M(2u) \text{ for } u<0 \qquad N(u)=2N(2u) \text{ for } u>0$$

$$M(-u)\not\equiv -N(u-) \text{ or } N(-u)\not\equiv -M(u-) \tag{3}$$

and

$$\int_{-\infty}^{0} h(u)\,dM(u)+\int_{0}^{\infty} h(u)\,dN(u)=0$$

where

$$h(u)=u^3/\{(1+u^2)(4+u^2)\} \tag{4}$$

Let $\psi(t)=\log\phi(t)$. Then

$$\begin{aligned}\psi(2t) &= \int_{-\infty}^{0}\left\{e^{2itv}-1-\frac{2itv}{1+v^2}\right\}dM(v) + \int_{0}^{\infty}\left\{e^{2itv}-1-\frac{2itv}{1+v^2}\right\}dN(v)\\ &= \int_{-\infty}^{0}\left\{e^{itu}-1-\frac{itu}{1+(u^2/4)}\right\}dM\left(\frac{u}{2}\right) + \int_{0}^{\infty}\left\{e^{itu}-1-\frac{itu}{1+(u^2/4)}\right\}dN\left(\frac{u}{2}\right)\\ &= 2\left[\int_{-\infty}^{0}\left\{e^{itu}-1-\frac{itu}{1+u^2/4)}\right\}dM(u) + \int_{0}^{\infty}\left\{e^{itu}-1-\frac{itu}{1+(u^2/4)}\right\}dN(u)\right]\end{aligned}$$

(by equation (3)). Hence, by equation (4)

$$\begin{aligned}2\psi(t)-\psi(2t) &= 6it\left[\int_{-\infty}^{0} h(u)\,dM(u)+\int_{0}^{\infty} h(u)\,dN(u)\right]\\ &= 0\ \forall t\in\mathbb{R}\end{aligned}$$

so that equation (2) holds.

However, $\tilde{\psi}(\cdot)=\psi(-.)$ also has a Lévy representation, i.e. $L(0,\ 0,\ \tilde{M},\ \tilde{N})$, with $\tilde{M}(-u)=-N(u)$ for $u>0$ and $\tilde{N}(-u)=-M(u-)$ for $u<0$. The uniqueness of the Lévy representation implies that a relation of the form $\tilde{\psi}(t)=\psi(t)+ict\ \forall t\in\mathbb{R}$ will hold, for some real c, if and only if $c=0$, $\tilde{M}=M$ and $\tilde{N}=N$: but, by our assumptions on M and N, at least one of the last two equalities is ruled out. Thus ϕ does not correspond to a distribution symmetric about some point, whereas $\{\phi(t)\}^2\phi(-2t)$ is real-valued.

A specific choice of M and N subject to equation (3) is given by

$$M(u)=\sum_{n=-\infty}^{\infty} 2^{-n}\delta_{-2^n}(u) \qquad N(u)=c\sum_{n=-\infty}^{\infty} 2^{-n}\bar{\delta}_{2^{(n+1/2)}}(u)$$

where $c<0$ is chosen so as to satisfy the last condition in equation (3): M and N are easily seen to satisfy the conditions of the Lévy representation as well as the other conditions in equation (3). Here, as usual, $\delta_a(u)=1$ for $u \geqslant a$ and 0 for $u<a$ and $\bar{\delta}_a(u)=1-\delta_a(u)$.

3 Some remarks

Rao & Shanbhag (1992) have shown that if X is assumed to be integrable with a non-vanishing CHF, then for any $0<\lambda<1$, $X_\lambda^*=X_1-\lambda X_2-\bar{\lambda}X_3$ is symmetric (about 0) if and only if X is symmetric about some point.

A sufficient condition for $\eta<0$ is that X is convex ordered with respect to $-X$ (see van Zwet (1964) for the concept of convex ordering). Thus the one-sided test of Randles *et al.* (1980) is consistent for testing symmetry against the alternative: X is convex ordered with respect to $-X$.

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References

BEHBOODIAN, J. (1989) A note on skewness and symmetry, *The Statistician*, 39, pp. 21–23.

RAMACHANDRAN, B. (1967) *Advanced Theory of Characteristic Functions* (Calcutta, Statistical Publishing Society).

RANDLES, R. H., FLIGNER, M. A., POLICELLO, G. E. & WOLFE, D. A. (1980) An asymptotically distribution-free test for symmetry versus asymmetry. *Journal of the American Statistical Association*, 75, pp. 168–172.

RAO, C. R. & SHANBHAG, D. N. (1992) Further observations on the integrated Cauchy functional equation, *Mathematische Nachrichten*, in press.

VAN ZWET, W. R. (1964) Convex Transformations of Random Variables (Amsterdam, Mathematische Centrum).