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SOME COMPETITORS OF TESTS BASED ON POWERS OF RANKS FOR THE TWO-SAMPLE PROBLEM

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SUMMARY. Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ be two independent random samples from two populations with absolutely continuous distribution functions $F(x)$ and $G(x)$, respectively. We wish to test the null hypothesis $H_0 : F(x) = G(x)$ against the alternative $H_A : F(x) \leqslant G(x)$ for all real x. We define a function which takes value 1 if max $(y_1, \ldots, y_d) \leqslant$ max $(x_1, \ldots, x_c)$ and zero otherwise for fixed c and d such that $1 \leqslant c \leqslant n$, $1 \leqslant d \leqslant m$. Let $U_{c,d}$ be the U-statistic based on this kernel. This statistic is proposed to be used as the test statistic. It is shown that for fixed k, $2 \leqslant k < \min(n, m)$, all the tests $U_{c, k-c}$, $1 \leqslant c \leqslant k-1$ are equally efficient in the Pitman ARE sense and all of them have Pitman ARE one w.r.t. the test based on powers of ranks statistic $A_k = n^{-1} \sum_{i=1}^{n} R_i^{k-1}$, $k > 1$. Here R_i is the rank of X_i in the combined increasing arrangement of X's and Y's.

1. Introduction

Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ be two independent random samples from two populations with absolutely continuous distribution functions $F(x)$ and $G(x)$, respectively. We wish to test the null hypothesis,

$$H_0 : F(x) = G(x) \text{ for every real } x, \quad \ldots \quad (1.1)$$

against the slippage or stochastic order alternative

$$H_A : F(x) \leqslant G(x) \text{ for every real } x, \quad \ldots \quad (1.2)$$

and strict inequality over a set of nonzero probability.

Let $N = n+m$ and let R_i be the rank of X_i $(i = 1, \ldots, n)$ in the combined increasing arrangement of X's and Y's. Tamura (1963) has proposed tests based on the sum of powers of rank statistics

$$A_k = n^{-1} \sum_{i=1}^{n} \{R_i/(N+1)\}^{k-1},$$

for the two-sample scale problem. Mielke (1972) has investigated the asymptotic behaviour of these tests. He has identified the families of probability distributions for which the tests based on the statistics A_k are asymptotically optimal in the Pitman asmyptotic relative efficiency (ARE) sense. In this paper, we consider an alternative family of distribution-free tests for the above problem.

Define

$$h_{c,d}(x_1, \ldots, x_c; y_1, \ldots, y_d) = \begin{cases} 1, & \text{if } \max(y_1, \ldots, y_d) \leqslant \max(x_1, \ldots, x_c); \\ 0, & \text{otherwise,} \end{cases} \quad \ldots (1.3)$$

for some fixed positive integers c and d such that $1 \leqslant c \leqslant n$ and $1 \leqslant d \leqslant m$. Let $U_{c,\,d}$ be the corresponding generalized U-statistic defined by

$$U_{c,\,d} = \left[\binom{n}{c}\binom{m}{d}\right]^{-1} \Sigma_C h_{c,\,d}\left(X_{i_1}, \ldots, X_{i_c}; Y_{j_1}, \ldots, Y_{j_d}\right) \quad \ldots (1.4)$$

where Σ_C denotes summation extended over all combinations $(i_1, \ldots, i_c)$ of c integers chosen from $(1, \ldots, n)$ and all combinations $(j_1, \ldots, j_d)$ of d integers chosen from $(1, \ldots, m)$.

Under the alternative hypothesis H_A, the X observations are expected to be large compared to the Y observations. Hence intuitively it seems that large values of the statistic $U_{c,\,d}$ shall indicate the alternative hypothesis. In fact $U_{1,\,1}$ is the familiar Mann-Whitney statistic and $U_{c,\,c}$ are the statistics proposed by Kochar (1978b) specifically for testing the slippage alternative in the context of nonnegative random variables. The $U_{c,\,d}$ statistics are generalizations of these statistics. By proper choice of c and d it is expected that one can match statistics to various types of random variables and alternatives belonging to H_A leading to high efficiency.

The statistic $U_{c,\,d}$ may be seen to be a function of the ranks of the ordered observations. Let $X_{(i)}$ be the i-th order statistic in X-sample $(i = 1, \ldots, n)$ and let $R_{(i)}$ be its rank in the combined increasing arrangement of X's and Y's. Then it can be seen that

$$U_{c,\,d} = \left[\binom{n}{c}\binom{m}{d}\right]^{-1} \sum_{i=1}^{n} \binom{i-1}{c-1}\binom{R_{(i)}-1}{d}. \quad \ldots (1.5)$$

The statistic $U_{k-1,\,1}$ $(2 \leqslant k < n)$ belongs to the class of 'Linear Ordered Rank Statistics' studied by Deshpandé (1972). Sen (1963) calls the tests based on such type of statistics 'Weighted Rank Sum Tests'.

In Section 2 the distributions of the test statistics are derived. We compare these tests with those of Tamura in Section 3 and find that the tests based on $U_{c,\,k-c}$, $c = 1, \ldots, k-1$ have Pitman efficiency 1 with respect to the test based on A_k. We also compare the approximate Bahadur slopes of the new statistics as well as those of the statistics A_k. Some interesting results in terms of asymptotic Bahadur efficiency are obtained. These are discussed in Section 4.

2. Distributions of the test statistics

The expected value of the statistic $U_{c,d}$ is

$$E[U_{c,d}] = E[h_{c,d}(X_1, \ldots, X_c; Y_1, \ldots, Y_d)]$$

$$= P[\max(Y_1, \ldots, Y_d) \leqslant \max(X_1, \ldots, X_c)]$$

$$= c \int_{-\infty}^{\infty} G^d(x) F^{c-1}(x)\, dF(x)$$

$$= \mu_{c,d} \text{ (say)}. \qquad \ldots \quad (2.1)$$

Under H_0, $\mu_{c,d} = c/(c+d)$ but under H_A, $\mu_{c,d} > c/(c+d)$.

Adopting the methods of Hájek and Šidák (1967) the following recurrence relation for generating the exact significant points has been developed.

Let $p_{n,m}(k)$ be the probability under H_0 that $U_{c,d}$ equals $k \left[\binom{n}{c}\binom{m}{d}\right]^{-1}$ when the sample sizes are n and m, respectively, for $k = 1, 2, \ldots, \binom{n}{c}\binom{m}{d}$.

Then

$$(n+m)p_{n,m}(k) = mp_{n,m-1}(k) + np_{n-1,m}\left(k - \binom{m}{d}\binom{n-1}{c-1}\right), \qquad \ldots \quad (2.2)$$

subject to the following conditions

$$p_{i,j}(k) = 0 \text{ if either } k < 0 \text{ or } k > \binom{n}{c}\binom{m}{d} \text{ for } i \geqslant c+1,\ j \geqslant d+1.$$

The following theorem gives the asymptotic distribution of $U_{c,d}$. It can be easily proved using the well-known properties of the generalized U-statistics (see, e.g., Lehmann, 1963).

Theorem 2.1 : *The limiting distribution of* $N^{1/2}[U_{c,d} - \mu_{c,d}]$ *as* $N \to \infty$ *in such a way that* $n/N \to p$, $0 < p < 1$, *is normal with mean zero and variance* σ^2. *Under* H_0, $\mu_{c,d} = c/(c+d)$ *and*

$$\sigma^2 = \frac{c^2 d^2}{pq(2k-1)k^2}, \qquad \ldots \quad (2.3)$$

where $k = c+d$ *and* $q = 1-p$.

3. Asymptotic efficiencies

In order to calculate Pitman ARE's, we parametrize the problem in the following way. Let $F(x) = F_0(x)$ and $G(x) = F_\theta(x)$, where θ is a real positive number such that $F_0(x) \leqslant F_\theta(x)$ for every real x and with strict inequality over a set of nonzero probability whenever $\theta > 0$. We have the following theorem :

Theorem 3.1 : *For testing* $H_0 : \theta = 0$ *against* $H_A : \theta > 0$, *the Pitman ARE of the test based on* $U_{c,\,d}$ *is one w.r.t. the test based on the statistic* A_k, $k = c+d$, *provided* $E_\theta[U_{c,\,d}]$ *and* $E_\theta[A_k]$ *allow differentiation w.r.t.* θ *under the integral sign.*

Proof : Let

$$\bar{f}(x) = \frac{\partial}{\partial\theta} F_\theta(x) \Big|_{\theta=0}. \qquad \ldots \quad (3.1)$$

Then,

$$\frac{\partial}{\partial\theta} E_\theta[U_{c,\,d}] \Big|_{\theta=0} = cd \int_{-\infty}^{\infty} \bar{f}(x) F^{k-2}(x)\, dF(x), \qquad \ldots \quad (3.2)$$

as differentiation under the integral sign is permissible.

From Theorem 2.1, the asymptotic null variance of $N^{1/2}U_{c,\,d}$ is

$$\sigma^2_{c,d} = \frac{c^2d^2}{pq(2k-1)k^2}.$$

Therefore, the limiting efficacy of $U_{c,d}$ is

$$e(U_{c,d}) = pq(2k-1)k^2 \Big[\int_{-\infty}^{\infty} \bar{f}(x) F^{k-2}(x)\, dF(x) \Big]^2. \quad \ldots \quad (3.3)$$

Now we obtain the limiting efficacy of the test based on A_k. We know from Chernoff and Savage (1958) that

$$E_\theta[A_k] = \int_{-\infty}^{\infty} [pF(x)+qF_\theta(x)]^{k-1}\, dF(x) \qquad \ldots \quad (3.4)$$

and under H_0, the variance of the limiting normal distribution of $N^{1/2}A_k$ is

$$\sigma_k^2 = \frac{q}{p} \Big[\int_0^1 u^{2k-2}\, du - \Big\{ \int_0^1 u^{k-1}\, du \Big\}^2 \Big]$$

$$= \frac{q(k-1)^2}{p(2k-1)k^2}. \qquad \ldots \quad (3.5)$$

Now,

$$\frac{\partial}{\partial \theta} E_\theta[A_k]\Big|_{\theta=0} = q(k-1) \int_{-\infty}^{\infty} \bar{f}(x)F^{k-2}(x)\, dF(x). \qquad \ldots \quad (3.6)$$

Therefore, the limiting efficacy of A_k is

$$e(A_k) = pq(2k-1)k^2 \Big[\int_{-\infty}^{\infty} \bar{f}(x)F^{k-2}(x)\, dF(x) \Big]^2.$$

This is the same as $e(U_{c,d})$. Hence the Pitman ARE of $U_{c,d}$ w.r.t. A_k is one, where $k = c+d$.

It is clear from (3.3) that the efficacy of $U_{c,\,d}$ depends upon c and d only through the sum $k = c+d$. Thus for each fixed k, $2 \leqslant k < \min(n, m)$, the tests $U_{c,\,k-c}$, $1 \leqslant c \leqslant k-1$, are equally efficient in the Pitman ARE sense. In the following theorem, we identify a class of Lehmann type alternatives for which the $U_{c,d}$ and A_k type tests are asymptotically optimal in the Pitman ARE sense.

Theorem 3.2 : *The test based on the statistic* $U_{c,\,k-c}$, $2 \leqslant k < \min(n, m)$, $1 \leqslant c \leqslant k-1$, *is asymptotically optimal (in the Pitman ARE sense) against the Lehmann type alternative*

$$F_\theta(x) = F(x)+\theta[1-F^{k-1}(x)]F(x),\ \theta > 0. \qquad \ldots \quad (3.7)$$

Proof: The linear rank statistic A_k is based on the approximate scores (see, Hájek and Sidák, 1967, Chapter II) of the locally most powerful rank test for testing H_0 against H_A with F_θ given by (3.7). The required result follows from Theorem 3.1.

4. Remarks

(i) It can be easily seen that the tests based on statistics $U_{c,\,d}$ have the properties of consistency and unbiasedness for testing H_0 against H_A.

(ii) The approximate Bahadur slopes of the statistics $U_{c,\,d}$ and A_k have been obtained in Kochar (1978a). Some comments are made below on the basis of these results.

For any fixed $\theta > 0$ such that $F_0(x) \leqslant F_\theta(x)$ for every real x, let $B(T)$ denote the approximate Bahadur slope of a statistic T. Then

(a) $B(U_{c-1,\,d+1}) \geqslant B(U_{c,\,d})$ for $2 \leqslant c < n$, $1 \leqslant d < m$;

(b) $B(U_{1,\,k-1}) \geqslant B(A_k) \geqslant B(U_{k-1,\,1})$;

(c) When $k = 4$ and $p = q = 1/2$,

$$B(U_{3,\,1}) \leqslant B(U_{2,\,2}) \leqslant B(A_4) \leqslant B(U_{1,\,3}).$$

However, as pointed out by Bahadur (1967) conclusions based on approximate Bahadur slopes may not coincide with those based on exact slopes of the statistics, especially, when the alternatives are far away from the null hypothesis. Nonetheless our results do indicate that the statistics $U_{c,\,k-c}$, $c = 1, \ldots, k-1$ and A_k will have nonidentical properties in terms of Bahadur efficiency. Further insight in this regard will become possible when results on probabilities of large deviations of U-statistics become available.

(iii) Mielké (1972) has discussed the choice of the statistics A_k for various alternatives. Similar studies can be made. Our Theorem 3.2 gives some hints regarding the choice of appropriate $U_{c,d}$. Further studies on this aspect can also be carried out if exact Bahadur slopes become available.

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