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Dispersive ordering – Some applications and examples

Jongwoo Jeon[1], Subhash Kochar[2], Chul Gyu Park3 *

[1] Department of Statistics, Seoul National University, Seoul, 151-742, Korea
[2] Department of Mathematics and Statistics, Portland State University, Portland, Oregon 97201, USA
[3] School of Mathematics and Statistics, Carleton University, Ottawa, Ontario, K1S 5B6, Canada

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Abstract A basic concept for comparing spread among probability distributions is that of dispersive ordering. Let X and Y be two random variables with distribution functions F and G, respectively. Let F^{-1} and G^{-1} be their right continuous inverses (quantile functions). We say that Y is less *dispersed* than X ($Y \leq_{disp} X$) if $G^{-1}(\beta) - G^{-1}(\alpha) \leq F^{-1}(\beta) - F^{-1}(\alpha)$, for all $0 < \alpha \leq \beta < 1$. This means that the difference between any two quantiles of G is smaller than the difference between the corresponding quantiles of F. A consequence of $Y \leq_{disp} X$ is that $|Y_1 - Y_2|$ is stochastically smaller than $|X_1 - X_2|$ and this in turn implies $var(Y) \leq var(X)$ as well as $E[|Y_1 - Y_2|] \leq E[|X_1 - X_2|]$, where X_1, X_2 (Y_1, Y_2) are two independent copies of X (Y). In this review paper, we give several examples and applications of dispersive ordering in statistics. Examples include those related to order statistics, spacings, convolution of non-identically distributed random variables and epoch times of non-homogeneous Poisson processes.

Key words : Exponential distribution, proportional hazard rates, hazard rate ordering, Schur functions, majorization and p-larger ordering, convolutions, parallel systems, gamma distribution, t-distribution.

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1 Introduction

Stochastic models are usually complex in nature. Obtaining bounds and approximations for some of their characteristics of interest is of practical importance. That is, the approximation of a stochastic model either by a simpler model or by a model with simple constituent components might lead to convenient bounds and approximations for some particular and desired characteristics of the model.

Beginning with the idea of stochastic ordering as introduced by Lehmann (1955), over the years several stochastic orders have been introduced in the literature for comparing different aspects of probability distributions. In this review paper we focus on dispersive ordering, a partial ordering useful for comparing spread among probability distributions. We give several examples of statistics that can be ordered according to dispersive ordering. These include order statistics, spacings and statistics which can be expressed as linear combinations of random variables. Order statistics play an important role in statistics, in general, and in reliability theory, in particular. The time to failure of a k-out-of-n system of n components corresponds to the $(n-k+1)$th order statistic. They have been studied extensively in the literature when the components are independent and identically distributed. But in real life, systems are usually made up of components with non-identically distributed lifetimes and often they are dependent as the components work in a common environment. Since their distribution theory is quite complicated, fewer results are available in the literature on their exact distributions.

We first review the various stochastic orders that will be useful in our discussion. Let us denote by f, F, $\overline{F}$ and r_F the density function, the distribution function, the survival function and the hazard rate of a random variable X, respectively. Similarly, let g, G, $\overline{G}$ and r_G denote these quantities for another random variable Y. Throughout this paper 'increasing' means nondecreasing and 'decreasing' means non increasing.

Definition 1 *(a) A random variable Y is said to be stochastically smaller than another random variable X (denoted by $Y \leq_{st} X$) if*

$$\overline{G}(x) \leq \overline{F}(x), \text{ for all } x .$$

(b) Y is said to be smaller than X in hazard rate ordering (denoted by $Y \leq_{hr} X$) if

$$\overline{F}(x)/\overline{G}(x) \text{ is increasing in } x.$$

(c) Y is said to be smaller than X in likelihood ratio ordering (denoted by $Y \leq_{lr} X$) if

$$f(x)/g(x) \text{ is increasing in } x.$$

(d) A random vector $\mathbf{Y} = (Y_1, \ldots, Y_n)$ is said to be smaller than another random vector $\mathbf{X} = (X_1, \ldots, X_n)$ in the multivariate stochastic order (denoted by $\mathbf{Y} \overset{st}{\preceq} \mathbf{X}$) if

$$\phi(\mathbf{Y}) \leq_{st} \phi(\mathbf{X}) \text{ for all increasing functions } \phi : R^n \to R.$$

Let X_t denote a random variable whose distribution is the same as that of $X - t$ given $X > t$. It is easy to show that $Y \leq_{hr} X$ if and only if $Y_t \leq_{st} X_t$ for all $t \geq 0$. In other words, the conditional distributions, given that the random variables are at least of a certain size, are all stochastically ordered (in the usual sense) in the same direction. In case the hazard rates exist, it is easy to see that $Y \leq_{hr} X$, if and only if $r_F(x) \leq r_G(x)$ for every x. The hazard rate ordering is also known as *uniform stochastic ordering* in the literature. When the supports of X and Y have a common finite left end-point, we have the following chain of implications among the above stochastic orders :

$$Y \leq_{lr} X \Rightarrow Y \leq_{hr} X \Rightarrow Y \leq_{st} X.$$

See Lehmann and Rojo (1992) and Shaked and Shanthikumar (1994) for further details.

We shall also be using the concept of majorization. Let $\{x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}\}$ denote the increasing arrangement of the components of a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Vector $\mathbf{x}$ is said to majorize another vector $\mathbf{y}$ (written $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$) if $\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}$ for$j = 1, \ldots, n-1$ and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$. Functions that preserve the majorization ordering are called Schur convex functions. See Marshall and Olkin (1979, Ch. 3) for more details. Vector $\mathbf{x}$ is said to majorize vector $\mathbf{y}$ weakly (written $\mathbf{x} \overset{w}{\succeq} \mathbf{y}$) if $\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}$ for $j = 1, \ldots, n$.

Recently Bon and Paltanea (1999) considered a new pre-order on $I\!R^{+^n}$, which they call *p-larger order*. A vector $\mathbf{x}$ in $I\!R^{+^n}$ is said to be p-larger than another vector $\mathbf{y}$, also in $I\!R^{+^n}$, (written $\mathbf{x} \overset{p}{\succeq} \mathbf{y}$) if $\log(\mathbf{x}) \overset{w}{\succeq} \log(\mathbf{y})$, where $\log(\mathbf{x})$ denotes the vector of the logarithms of the coordinates of $\mathbf{x}$. It is known that $\mathbf{x} \overset{m}{\succeq} \mathbf{y} \Longrightarrow (g(x_1), \ldots, g(x_n)) \overset{w}{\succeq} (g(y_1), \ldots, g(y_n))$ for all concave functions g (cf. Marshal and Olkin (1979), p. 115). Since log is a concave function, it follows that for $\mathbf{x}, \mathbf{y} \in I\!R^{+^n}$, $\mathbf{x} \overset{m}{\succeq} \mathbf{y} \Longrightarrow \mathbf{x} \overset{p}{\succeq} \mathbf{y}$. The converse is, however, not true. For example, $(0.2, 1, 5) \overset{p}{\succeq} (1, 2, 3)$ but majorization does not hold between these two vectors.

A basic concept for comparing spread among probability distributions is that of dispersive ordering as defined below.

Definition 2 *Y is said to be less dispersed than X (denoted by $Y \leq_{disp} X$) if*

$$G^{-1}(\beta) - G^{-1}(\alpha) \leq F^{-1}(\beta) - F^{-1}(\alpha), \quad \textit{whenever} \quad 0 < \alpha \leq \beta < 1. \quad (1)$$

Note that $Y \leq_{disp} X$ if and only if the following equivalent conditions hold :

(i) $F^{-1}G(x) - x$ increases in x;

(ii)
$$r_F(F^{-1}(p)) \leq r_G(G^{-1}(p)),\ \forall p \in (0,1), \tag{2}$$
if the densities exist;

(iii) $Y_{G^{-1}(p)} \leq_{st} X_{F^{-1}(p)} \quad \forall p \in (0,1)$;

(iv) $F(F^{-1}(p)+c) \leq G(G^{-1}(p)+c)$ for all $p \in (0,1)$ and $c > 0$;

(v) If F and G are strictly increasing and continuous, then $X =_{st} h(Y)$, where $h(x) = F^{-1}G(x)$ is such that $x - x' \leq h(x) - h(x')$ for all $x < x'$.

If (1) holds, Yanagimoto and Sibuya (1976) say that X is *statistically more spread out* than Y. Saunders and Moran (1978), Bickel and Lehmann (1979), Lewis and Thompson (1981) and Shaked (1982) systematically studied this partial ordering as a stochastic order for comparing spread among probability distributions. Doksum (1969), while studying the efficiencies of certain non-parametric tests, called this ordering *tail ordering.* Deshpandé and Kochar (1983) pointed out the equivalence between these concepts and established some connections between dispersive ordering and some other partial orders. Deshpandé and Kochar (1982), Deshpandé and Mehta (1982) and Capéraà (1988) used dispersive ordering in some inferential problems to obtain bounds on efficiencies of tests and probabilities of correct selections.

As pointed out by Bickel and Lehmann (1979) dispersive ordering compares spread between two distributions *locally.* We give an example where the variances of two random variables are ordered, but they are not ordered according to dispersive ordering.

Example 1 Let $F(x) = 1 - exp\left(-\frac{2}{\sqrt{\pi}}x\right)$, $x \geq 0$ and $G(x) = exp(-x^2)$, $x \geq 0$. Then
$$G^{-1}(p) = \sqrt{-\log(1-p)} \text{ and } F^{-1}(p) = -\tfrac{\sqrt{\pi}}{2}\log(1-p).$$
By plotting the function $G^{-1}(p) - F^{-1}(p)$ in the range $(0,1)$, we find that it is not monotione thus proving that $Y \not\leq_{disp} X$, but It is easy to see that $var(Y) = 1 - \pi/4 < \pi/4 = var(X)$.

Some important properties of dispersive ordering are :

P1. Dispersive ordering is location-invariant in the sense that $Y \leq_{disp} X \Leftrightarrow Y + c \leq_{disp} X$ for any real c.

P2. $X \leq_{disp} \sigma X$ whenever $\sigma > 1$.

P3. $Y \leq_{disp} X \Leftrightarrow -Y \leq_{disp} -X$.

P4. (Lewis and Thompson, 1981) Let Z be a random variable independent of X and Y and $Y \leq_{disp} X$. Then $Y + Z \leq_{disp} X + Z$ if and only if Z has a log-concave density.

P5. If X and Y are such that they have a common finite left end point of their supports, then $Y \leq_{disp} X \Rightarrow Y \leq_{st} X$.

P6. (Rojo and He, 1991) If $Y \leq_{disp} X$ and $Y \leq_{st} X$, then $\phi(Y) \leq_{disp} \phi(X)$ for all increasing convex and all decreasing concave functions ϕ.

P7. $Y \leq_{disp} X \Rightarrow E[\phi(Y - E(Y))] \leq E[\phi(X - E(X))]$ for every convex function ϕ, provided the expectations exist. In particular, $Y \leq_{disp} X$ implies $var(Y) \leq var(X)$ and $E|Y - E(Y)| \leq E|X - E(X)|$.

For further details, see Section 2.B of Shaked and Shanthikumar (1994).

As indicated by (2), there is an intimate connection between hazard rate ordering and dispersive ordering and which is made more explicit in the following result of Bagai and Kochar (1986).

Theorem 1 *Let X and Y be two nonnegative random variables.*

(a) If $Y \leq_{hr} X$ and either F or G is DFR (decreasing failure rate), then $Y \leq_{disp} X$;
(b) if $Y \leq_{disp} X$ and either F or G is IFR (increasing failure rate), then $Y \leq_{hr} X$.

Sometimes it is not easy to establish hazard rate ordering or dispersive ordering directly from the definitions and in those situations the above result can prove to be very useful. Here is an interesting example.

Example 2 Let X_γ denote a gamma random variable with shape parameter γ. Then for poistive integers γ_1 and γ_2 such that $1 \leq \gamma_1 \leq \gamma_2$, we show that

$$X_{\gamma_1} \leq_{disp} X_{\gamma_2} \text{ and } X_{\gamma_1} \leq_{hr} X_{\gamma_2}.$$

We can express X_{γ_2} as $X_{\gamma_1} + X_{\gamma_2 - \gamma_1}$, where $X_{\gamma_2 - \gamma_1}$ has gamma distribution with shape parameter $\gamma_2 - \gamma_1$, a positive integer and is independent of X_{γ_1}. Moreover X_{γ_1}, being the sum of γ_1 independent random variables, has log-concave density. It follows from property P4 that

$$X_{\gamma_1} \leq_{disp} X_{\gamma_2}. \tag{3}$$

Since X_{γ_1} is IFR for $\gamma_1 \geq 1$, it follows from Theorem 1(b) and (3) that $X_{\gamma_1} \leq_{hr} X_{\gamma_2}$.

Saunders and Moran (1978) and Shaked (1982) proved the above result for gamma random variables with arbitrary shape parameters using complicated analytic methods. The following technique given in Saunders and Moran (1978) is very useful in establishing dispersive ordering among members of a parametric family of probability distributions.

Theorem 2 *Let X_a be a random variable with distribution function F_a for each $a \in R$ such that*

(i) F_a is supported on some interval $(x_-^{(a)}, x_+^{(a)}) \subseteq (0, \infty)$ and has density f_a which does not vanish on any subinterval of $(x_-^{(a)}, x_+^{(a)})$,

(ii) derivative of F_a with respect to a exists and denoted by F'_a.

Then,

$$X_a \geq_{disp} X_{a^*} \quad \text{for} \quad a,\ a^* \in R \quad \text{and } a > a^*, \tag{4}$$

if and only if,

$$F'_a(x)/f_a(x) \quad \text{is decreasing in} \quad x. \tag{5}$$

Dispersive ordering is closely related to *star-ordering or more IFRA ordering*, a partial ordering used to compare skewness among probability distribution.

Definition 3 *Let X and Y be two non-negative random variables with distribution functions F and G, respectively. Y is said to be star-ordered with respect to X (written as $Y \leq_* X$ or $G \leq_* F$) if*

$$\frac{F^{-1}(u)}{G^{-1}(u)} \text{ is nondecreasing in } u \in (0,1).$$

Note that

$$Y \leq_* X \Leftrightarrow \log Y \leq_{disp} \log X. \tag{6}$$

Note that if X has exponential ditribution, then Y is an IFRA (increasing failure rate average) distribution. Deshpandé and Kochar (1983) proved the following connection between star-ordering and dispersive ordering.

Theorem 3 *Let F and G be absolutely continuous distributions such that $F(0) = G(0) = 0$ and let the corresponding density functions be such that $g(0) \geq f(0) > 0$. Then $G \leq_* F \Rightarrow G \leq_{disp} F$.*

Example 3 Let T_n denote a random variable which has central t-distribution with n degrees of freedom. We show below that for $n < m$, $T_m \leq_{disp} T_n$.

If $f_{T_n}(t)$ denotes the density of T_n, then the density function and the distribution functions of $|T_n|$ are

$$f_{|T_n|}(t) = \begin{cases} 2f_{T_n}(t) & \text{if } t > 0 \\ 0, & \text{otherwise;} \end{cases}$$

and

$$F_{|T_n|}(t) = 2F_n(t) - 1. \text{ for } t \geq 0.$$

Hence

$$F^{-1}_{|T_n|}(u) = F^{-1}_{T_n}\left(\frac{u+1}{2}\right) \text{ for all } u \in (0,1).$$

It is easy to see that for $n < m$, $f_{T_m}(0) \geq f_{T_n}(0) > 0$. It follows from Rivest (1982) that in this case $|T_m| \leq_* |T_n|$. Since $F_{|T_n|}(0) = F_{|T_m|}(0) = 0$ and $f_{|T_m|}(0) \geq f_{|T_n|}(0) > 0$, it follows from Theorem 3 that $|T_m| \leq_{disp} |T_n|$. Furthermore, it is easy to check from the properties of symmetric distributions (about 0) that $|T_m| \leq_{disp} |T_n|$ implies $T_m \leq_{disp} T_n$ for $n < m$.

Ahmed et al. (1986) established the following relations between super-additive (more NBU) ordering (which is implied by star-ordering) and dispersive ordering for nonnegative random variables. Recall that G is said to be super-additive with respect to F (or Y is more NBU than X) (written as $Y \leq_{su} X$ or $G <_{su} F$) if $F^{-1}G(x+y) \geq F^{-1}G(x) + F^{-1}G(y)$ for all x, y in the support of G.

Theorem 4 *If $G \leq_{su} F$ and either $G \leq_{st} F$ or $\lim_{x \to 0+} F^{-1}G(x)/x \geq 1$, then $G \leq_{disp} F$.*

Similar relations between dispersive ordering and other partial orderings for aging, like convex-ordering and star-ordering were earlier obtained by Sathe (1984), Bartoszewicz (1985 a, 1985 b) and Fernandez-Ponce et al. (1988). Pellerey and Shaked (1997) proved that a random variable X has an IFR distribution if and only if $X_t \leq_{disp} X_s$ for $s \leq t$, where X_t denotes a random variable whose distribution is the same as that of $X - t$ given $X > t$.

It is easy to prove that $Y \leq_{disp} X$ implies $|Y_1 - Y_2| \leq_{st} |X_1 - X_2|$ and which in turn implies $var(Y) \leq var(X)$ as well as $E[|Y_1 - Y_2|] \leq E[|X_1 - X_2|]$, where X_1, X_2 (Y_1, Y_2) are two independent copies of X (Y). Bartoszewicz (1986) extended this result to spacings of a random sample of size n. This is stated in the next theorem.

Theorem 5 *Let $X_{1:n}, \ldots, X_{n:n}$ denote the order statistics of a random sample $X_1, \ldots, X_n$ from a distribution with distribution function F. Similarly, let $Y_{1:n}, \ldots, Y_{n:n}$ denote the order statistics of a random sample $Y_1, \ldots, Y_n$ from a distribution with distribution function G. Let the corresponding spacings be denoted by $U_{i:n} \equiv X_{i:n} - X_{i-1:n}$ and $V_{i:n} \equiv Y_{i:n} - Y_{i-1:n}$, for $i = 1, \ldots, n$, where $X_{0:n} = Y_{0:n} \equiv 0$. Then*

$$Y \leq_{disp} X \Rightarrow \mathbf{V} \overset{st}{\preceq} \mathbf{U}.$$

This result leads to the following important consequences.

Corollary 1 *Under the conditions of Theorem* 5

(a) $Y_{j:n} - Y_{i:n} \leq_{st} X_{j:n} - X_{i:n}$ *for* $1 \leq i < j \leq n$.
In particular, $Y_{n:n} - Y_{i:n} \leq_{st} X_{n:n} - X_{1:n}$.

(b) $s_Y^2 \leq_{st} s_X^2$,
where s_X^2 and s_Y^2 are the sample variances of the two samples.

(c) $\eta_Y \leq_{st} \eta_X$,
where

$$\eta_X = \left[\binom{n}{2}\right]^{-1} \sum\sum\nolimits_{i<j} |X_{j:n} - X_{i:n}|$$

is the Gini's mean difference for the X-sample. Similarly we define η_Y.

PROOF :
(a) The result follows by adding the corresponding components of the random vectors **U** and **V** from $i+1$ to j and using the above theorem.

(b) Note that the sample variance can be expressed as

$$\begin{aligned} s_X^2 &= [n(n-1)]^{-1} \sum_{i<j}\sum (X_{j:n} - X_{i:n})^2 \\ &= [n(n-1)]^{-1} \sum_{i<j}\sum (U_{j:n} + U_{j-1:n} + \cdots + U_{i+1:n})^2 \end{aligned}$$

which is an increasing function of **U**. Since increasing functions of stochastically ordered random vectors are stochastically ordered, the required result follows from the above theorem.
(c) The proof follows from the previous theorem and the fact that, as in part (b), the Gini's mean difference can be expressed in the form of an increasing function of the vector of spacings.

■

In Section 2, we establish some dispersive ordering results between successive order statistics from a DFR (decreasing failure rate) distribution. Due to its special importance in reliability theory, Section 3 is exclusively devoted to the study of parallel systems with non-i.i.d. components. We examine how the changes in the parameters of the distributions affect the lifetimes of parallel systems in the sense of dispersive ordering and hazard rate ordering. These results are also extended to the proportional hazards rates (PHR) models. In Section 4, we study dispersive ordering among normalized spacings from some restricted families of distributions. Section 5 is devoted to convolutions of independent random variables differing in their scale parameters. The classes of distributions studied include gamma, uniform and normal distributions. We conclude the paper with the last section on some comments and some open problems.

2 Dispersive ordering among order statistics

In this section we give some results on dispersive ordering among order statistics from distribution with decreasing failure rates. For $i = 1, \ldots, n$, we shall denote by $X_{i:n}$, the *ith* order statistic of a set of n random variables $X_1, \ldots, X_n$. The X_i's need not be independent nor identically distributed. In case $X_1, \ldots, X_n$ is a random sample from a DFR distribution, David and Groenveld (1982) proved that $var(X_{i:n}) \le var(X_{j:n})$ for $1 \le i < j \le n$. Kochar (1996a) strengthened this result to prove that under the same condition, $X_{i:n} \le_{disp} X_{j:n}$ for $1 \le i \le j \le n$. We give below their technique for proving such results.

Theorem 6 *Let $X_1, \ldots, X_n$ be a random sample from a DFR distribution. Then*

$$X_{i:n} \leq_{disp} X_{j:m} \quad \text{for } i \leq j \text{ and } n-i \geq m-j.$$

To prove this theorem, we first prove it for the exponential distribution.

Lemma 1 *Let $X_{i:n}$ be the ith order statistic of a random sample of size n from an exponential distribution with parameter λ. Then*

$$X_{i:n} \leq_{disp} X_{j:m} \quad \text{for } i \leq j \text{ and } n-i \geq m-j. \tag{7}$$

PROOF : Suppose we have two independent random samples, $X_1, \ldots, X_n$ and $X_1', \ldots, X_m'$ of sizes n and m from an exponential distribution with hazard rate parameter λ. The *ith* order statistic, $X_{i:n}$ can be written as a convolution of sample spacings as

$$\begin{aligned} X_{i:n} &= (X_{i:n} - X_{i-1:n}) + \cdots + (X_{2:n} - X_{1:n}) + X_{1:n} \\ &\stackrel{dist}{=} \sum_{k=1}^{i} E_{n-i+k} \end{aligned}$$

where for $k = 1, \ldots, i$, E_{n-i+k} is an exponential random variable with hazard rate $(n-i+k)\lambda$. It is a well known fact that E_{n-i+k}'s are independent. Similarly we can express $X_{j:m}'$ as

$$X_{j:m}' \stackrel{dist}{=} \sum_{k=1}^{j} E_{m-j+k}'$$

where again for $k = 1, \ldots, j$, E_{m-j+k}' is an exponential random variable with hazard rate $(m-j+k)\lambda$ and E_{m-j+k}''s are independent. It is easy to verify that $E_{n-i+1} \leq_{disp} E_{m-j+1}'$ for $n-i \geq m-j$.

Since the class of distributions with log- concave densities is closed under convolutions (cf. Dharmadhiakri and Joag-dev, 1988, p. 17), it follows from the repeated applications of property P4 that

$$\sum_{k=1}^{i} E_{n-i+k} \leq_{disp} \sum_{k=1}^{i} E_{m-j+k}'. \tag{8}$$

Again since $\sum_{k=i+1}^{j} E_{m-j+k}'$, being the sum of independent exponential random variables has a log-concave density and since it is independent of $\sum_{k=1}^{i} E_{n-i+k}'$, it follows from property P4 that the R.H.S of (8) is less dispersed than $\sum_{k=1}^{j} E_{m-j+k}'$ for $i \leq j$. That is,

$$X_{i:n} \stackrel{dist}{=} \sum_{k=1}^{i} E_{n-i+k} \leq_{disp} \sum_{k=1}^{j} E_{m-j+k}' \stackrel{dist}{=} X_{j:m}'.$$

Since $X_{j:m}$ and $X'_{j:m}$ are stochastically equivalent, (7) follows from this.

■

Boland et al. (1998) proved a special case of the above result when the sample sizes are equal. To prove Theorem 6 we will need the next lemma due to Bartoszewicz (1987).

Lemma 2 *Let $\phi : R^+ \to R^+$ be a function such that $\phi(0) = 0$ and $\phi(x) - x$ is increasing. Then for every convex and strictly increasing function $\psi : R^+ \to R^+$ the function $\psi\phi\psi^{-1}(x) - x$ is increasing.*

Proof of Theorem 6 : The distribution function of $X_{j:m}$ is $F_{j:m}(x) = B_{j:m}F(x)$, where $B_{j:m}$ is beta distribution with parameters $(j, m-j+1)$.

Let G denote the distribution function of a unit mean exponential random variable. Then $H_{j:m}(x) = B_{j:m}G(x)$ is the distribution function of the jth order statistic in a random sample of size m from a unit mean exponential distribution. We can express $F_{j:m}$ as

$$\begin{aligned} F_{j:m}(x) &= B_{j:m}GG^{-1}F(x) \\ &= H_{j:m}G^{-1}F(x). \end{aligned}$$

To prove the required result, we have to show that for $i \leq j$ and $n - i \geq m - j$,

$$\begin{aligned} &F^{-1}_{j:m}F_{i:n}(x) - x \quad \text{is increasing in x} \\ \Leftrightarrow\ &F^{-1}GH^{-1}_{j:m}H_{i:n}G^{-1}F(x) - x \quad \text{is increasing in x.} \end{aligned}$$

By Lemma 1, $H^{-1}_{j:m}H_{i:n}(x) - x$ is increasing in x for $i \leq j$ and $n-i \geq m-j$. Also the function $\psi(x) = F^{-1}G(x)$ is strictly increasing and it is convex if F is DFR. The required result now follows from Lemma 2.

■

It follows from Theorem 6 that if $X_1, X_2, \ldots,$ is a sequence of i.i.d. observations from a DFR distribution, then

$$X_{i:n+1} \leq_{disp} X_{i:n} \leq_{disp} X_{i+1:n+1}, \qquad \text{for } i = 1, \ldots, n.$$

The DFR assumption is crucial for the above result to hold. As seen in Boland et al. (1998) in the case of a random sample of size 2 from a uniform distribution over $[0, 1]$, which is not DFR, $X_{1:2}$ is not less dispersed than $X_{2:2}$.

Another application of Lemma 1 is the following result on star-ordering among the order statistics of a uniform distribution.

Example 4 Let $U_{i:n}$ denote the ith order statistic of a random sample of size n from a uniform distribution over (0, 1). Note that $-\log U_{i:n}$ has the same distribution as the $(n-i+1)th$ order statistic of a random sample of size n from a standard exponential distribution. It follows from Lemma 1 and the relation (6) that

$$U_{j:m} \overset{*}{\preceq} U_{i:n} \quad \text{for } i \leq j \text{ and } n-i \geq m-j. \tag{9}$$

A consequence of (9) is that the if $U_1, U_2, \ldots$ is a sequence of uniform $(0,1)$ random variables, then the Lorenz curve of $U_{i,n}$ is smaller than that of $U_{j:m}$ for $i \leq j$ and $n-i \geq m-j$ since star-ordering implies Lorenz ordering (cf. Kochar, 1989), a partial ordering which is commonly used in economics to compare inequality among income distributions.

Khaledi and Kochar (2000a) established the following dispersive ordering result between order statistics when the random samples are drawn from different distributions.

Theorem 7 *Let $X_1, \ldots, X_n$ be a random sample of size n from a continuous distribution F and let $Y_1 \ldots, Y_m$ be a random sample of size m from another continuous distribution G. If either F or G is DFR, then*

$$X \leq_{disp} Y \Rightarrow X_{i:n} \leq_{disp} Y_{j:m} \quad \textit{for } i \leq j \textit{ and } n-i \geq m-j. \tag{10}$$

Since the property $X \leq_{hr} Y$ together with the condition that either F or G is DFR implies that $X \leq_{disp} Y$, we get the following result from the above theorem.

Corollary 2 *Let $X_1, \ldots, X_n$ be a random sample of size n from a continuous distribution F and $Y_1 \ldots, Y_m$ be a random sample of size m from another continuous distribution G. If either F or G is DFR, then*

$$X \leq_{hr} Y \Rightarrow X_{i:n} \leq_{disp} Y_{j:m} \textit{ for } i \leq j \textit{ and } n-i \geq m-j.$$

We get Theorem 6 as a special case of the above theorem by taking $F = G$ above.

Kochar (1996b) obtained similar results for the epoch times of a nonhomogeneous Poisson process (or equivalently for record values) with a decreasing intensity function. Using the technique used in the proof of Theorem 6, Hu and Zhuang (2004) extended the above type of results to generalized order statistics.

3 Dispersive ordering among parallel systems with heterogeneous components

The exponential distribution plays a very important role in statistics. Because of its non-aging property, it has many nice properties and it often gives very convenient bounds on survival probabilities and other characteristics of interest for systems with non-exponential components. Pledger and Proschan (1971) studied the problem of stochastically comparing the order statistics of non-identically distributed independent random variables with those corresponding to independent and identically distributed random variables. This topic has been followed up by many researchers including Proschan and Sethuraman (1976), Boland, El-Neweihi and Proschan (1994), Dykstra, Kochar and Rojo (1997), Boland, Shaked and Shanthikumar(1998), Bon and Paltanea (1999); and Khaledi and Kochar (2000a, 2000b), among others. In this section we compare parallel systems consisting of non-identical components in terms of dispersive ordering and hazard rate ordering. First we consider the case when the components have exponential distributions and then extend the results to proportional hazards rate family.

Pledger and Proschan (1971) proved the following result.

Theorem 8 *Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i, $i = 1, \ldots, n$. Let $X_1^*, \ldots, X_n^*$ be another set of independent exponential random variables with X_i^* having hazard rate λ_i^*. Then $\boldsymbol{\lambda} \overset{m}{\succeq} \boldsymbol{\lambda}^*$ implies*

$$X_{1:n} \overset{st}{=} X_{1:n}^* \text{ and } X_{i:n} \geq_{st} X_{i:n}^*, \quad i = 2, \ldots, n. \tag{11}$$

Proschan and Sethuraman (1976) strengthened this result to establish multivariate stochastic ordering between two vectors of order statistics. They proved that under the conditions of the above theorem,

$$(X_{1:n}, \ldots, X_{n:n}) \overset{st}{\succeq} (X_{1:n}^*, \ldots, X_{n:n}^*).$$

The question is to what extent this result can be extended. For the special case $n = 2$ and $i = 2$, Boland, El- Neweihi and Proschan (1994) partially strengthened the above result of Pledger and Proschan (1971) from stochastic ordering to hazard rate ordering. Their result is stated below.

Theorem 9 *Let $r_{\lambda_1, \lambda_2}(t)$ be the hazard rate of a parallel system of two components whose lifetimes are independent random variables with hazard rates λ_1 and λ_2, respectively. Then $r_{\lambda_1, \lambda_2}(t)$ is Schur-concave in (λ_1, λ_2). That is, $(\lambda_1, \lambda_2) \overset{m}{\succeq} (\lambda_1^*, \lambda_2^*)$ implies*

$$X_{2:2} \geq_{hr} X_{2:2}^*.$$

Boland, El-Neweihi and Proschan (1994) conclude that Theorem 8 cannot be generalized for arbitrary n. They show with the help of an example that the hazard rate of a parallel system of three components is *not* Schur concave in $\boldsymbol{\lambda}$. Dykstra, Kochar and Rojo (1997) proved that, however, the *reversed hazard rate* of $X_{n:n}$, the lifetime of a parallel system of n independent components, is Schur-convex in $\boldsymbol{\lambda}$.

The next natural problem is to compare $X_{n:n}$ with $Y_{n:n}$, where $Y_1, \ldots, Y_n$ is a random sample from an distribution with hazard rate $\overline{\lambda} = \sum_{i=1}^n \lambda_i / n$. Kochar and Rojo (1996) proved the following result.

Theorem 10 *Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i for $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be a random sample from the distribution with hazard rate $\bar{\lambda}$. Then*

$$Y_{n:n} \leq_{disp} X_{n:n} \text{ and } \quad Y_{n:n} \leq_{hr} X_{n:n} \tag{12}$$

These results give a lower bound for the variance of $X_{n:n}$ and an upper bound on the hazard rate of $X_{n:n}$ in terms of those of $Y_{n:n}$.

It will be interesting to know whether the above result can be extended to other order statistics. While we don't know the answer in general, we see from the next theorem that such a result is true for the second order statistic.

Theorem 11 *Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i for $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be a random sample from the distribution with hazard rate $\bar{\lambda}$. Then*

$$Y_{2:n} \leq_{disp} X_{2:n}.$$

PROOF: It follows from Theorem 3.7 of Kochar and Korwar (1996) that

$$Y_{2:n} - Y_{1:n} \leq_{disp} X_{2:n} - X_{1:n} \quad \text{and} \quad X_{1:n} \stackrel{st}{=} Y_{1:n} . \tag{13}$$

Since the distribution of $X_{1:n}$ $(Y_{1:n})$ is logconcave, it follows from property P4 of Section 1 that

$$Y_{2:n} = (Y_{2:n} - Y_{1:n}) + Y_{1:n} \leq_{disp} (X_{2:n} - X_{1:n}) + X_{1:n} = X_{2:n}, \tag{14}$$

since $X_{2:n} - X_{1:n}$ is independent of $X_{1:n}$ and $Y_{2:n} - Y_{1:n}$ is independent of $Y_{1:n}$. That is,

$$Y_{2:n} \leq_{disp} X_{2:n} .$$

∎

In the following theorem Khaledi and Kochar (2000b) improved upon the bounds of Dykstra, Kochar and Rojo (1997) by replacing $\bar{\lambda}$ with $\tilde{\lambda} = (\prod_{i=1}^{n} \lambda_i)^{1/n}$, the geometric mean of the λ's.

Theorem 12 *Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i, $i = 1, \ldots, n$. Let $Z_1, \ldots, Z_n$ be a random sample of size n from an distribution with common hazard rate $\tilde{\lambda} = (\prod_{i=1}^{n} \lambda_i)^{1/n}$. Then*

$$X_{n:n} \geq_{disp} Z_{n:n} \text{ and } \quad X_{n:n} \geq_{hr} Z_{n:n}.$$

Corollary 3 *Under the conditions of Theorem 12,*

(a) the hazard rate $r_{X_{n:n}}$ of $X_{n:n}$ satisfies

$$r_{X_{n:n}}(x; \boldsymbol{\lambda}) \leq \frac{n\tilde{\lambda}\left(1 - exp(-\tilde{\lambda}x)\right)^{n-1} exp(-\tilde{\lambda}x)}{1 - \left(1 - exp(-\tilde{\lambda}x)\right)^{n}},$$

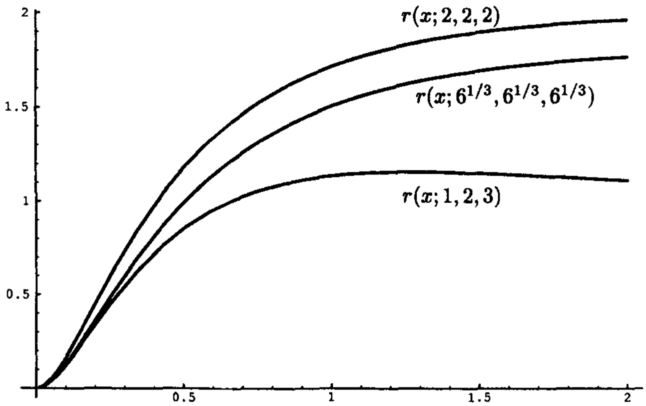

Fig. 1 Graphs of hazard rates of $X_{3:3}$

(b) $$var(X_{n:n}; \boldsymbol{\lambda}) \geq \frac{1}{\tilde{\lambda}^2} \sum_{i=1}^{n} \frac{1}{(n-i+1)^2}.$$

The new bounds given by Corollary 3 are better than those obtained by Dykstra, Kochar and Rojo (1997) since the hazard rate of $Y_{n:n}$ is a non-decreasing function of $\tilde{\lambda}$ and the fact that the geometric mean of λ_i's, is smaller than their arithmetic mean.

In Figures 1 and 2 we plot the hazard rates of parallel systems of three exponential components along with the upper bounds as given by Dykstra, Kochar and Rojo (1997) and the one's given by Corollary 4 (a). The vector of parameters in Figure 1 is $\boldsymbol{\lambda_1} = (1, 2, 3)$ and that in Figure 3.2 is $\boldsymbol{\lambda_2} = (0.2, 2, 3.8)$. Note that $\boldsymbol{\lambda_2} \overset{m}{\succeq} \boldsymbol{\lambda_1}$. It appears from these figures that the improvements on the bounds are relatively more if λ_i's are more dispersed in the sense of majorization. This is true because the geometric mean is Schur concave whereas the arithmetic mean is Schur constant and the hazard rate of a parallel system of i.i.d. exponential components with common parameter $\tilde{\lambda}$ is increasing in $\tilde{\lambda}$.

3.1 Extensions to the PHR model

Let $\overline{F}$ denote the survival function of a non-negative random variable X with hazard rate $h(\cdot)$. According to the proportional hazard rates (PHR) model, the independent random variables $X_1, \ldots, X_n$ are such that X_i has hazard rate $\lambda_i h(\cdot)$, $i = 1, \ldots, n$. Khaledi and Kochar (2002b) extended Theorem 12 to the PHR model as stated below.

Theorem 13 *Let $X_1, \ldots, X_n$ be independent random variables such that X_i has hazard rate $\lambda_i h(\cdot)$, $i = 1, \ldots, n$, where $h(\cdot)$ is the hazard rate of some non-negative random variable. Let $Y_1, \ldots, Y_n$ be a random sample from a distribution with common hazard rate $\tilde{\lambda} h(\cdot)$, where $\tilde{\lambda} = (\prod_{i=1}^{n} \lambda_i)^{1/n}$. Then*

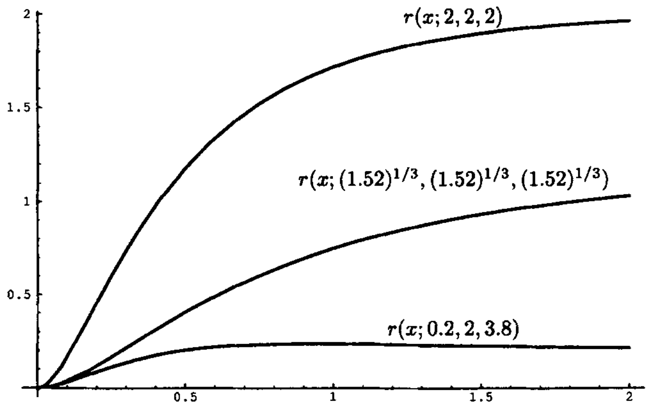

Fig. 2 Graphs of hazard rates of $X_{3:3}$

(a) $X_{n:n} \geq_{hr} Y_{n:n}$, *and*
(b) if F *is DFR, then* $X_{n:n} \geq_{disp} Y_{n:n}$.

4 Dispersive ordering among spacings

Let $X_1, \ldots, X_n$ be a random sample from a continuous distribution with cdf F and let $D_{i:n} = (n-i+1)(X_{i:n} - X_{i-1:n})$ denote the *ith* normalized spacing, $i = 1, \ldots, n$, with $X_{0:n} \equiv 0$. It is well known that $D_{1:n}, \ldots, D_{n:n}$ are independent and identically distributed if and only if F is exponential. Barlow and Proschan (1966) proved that if F is a DFR (IFR) distribution, then the successive normalized spacings are increasing (decreasing) stochastically. Kochar and Kirmani (1995) partially strengthened this result to the hazard rate ordering when the underlying random variables are DFR. This is stated below.

Theorem 14 *Let* $X_1, \ldots, X_n$ *be a random sample of size* n *from a DFR distribution. Then*

(a) $D_{i:n} \leq_{hr} D_{i+1:n}$ *for* $i = 1, \ldots, n-1$,
(b) $D_{i:n+1} \leq_{hr} D_{i:n}$, *for* $n \geq i$ *and for fixed* i.

Barlow and Proschan (1966) have shown that spacings of i.i.d. DFR random variables have also DFR distributions. The proof of the next theorem follows from Theorem 1 and the above theorem.

Theorem 15 *If* $X_1, \ldots, X_n$ *is a random sample from a DFR distribution, then*

(a) $D_{i:n} \leq_{disp} D_{i+1:n}$, *for* $i = 1, \ldots, n-1$,
(b) $D_{i:n+1} \leq_{disp} D_{i:n}$ *for* $n \geq i$ *and for fixed* i

Averous, Genest and Kochar (2004) used the technique of Lemma 1 to prove the following result on dispersive ordering between generalized spacings of order statistics of random samples from an exponential distribution.

Theorem 16 *Let $X_{1:n} \leq \cdots \leq X_{n:n}$ be the order statistics associated with a random sample of size n from an exponential distribution, and for $0 \leq i < j \leq n$, let*

$$V_{ij}^{(n)} = X_{j:n} - X_{i:n}$$

stand for the (i,j)th generalized spacing, with $X_{0:n} \equiv 0$. Then for $j - i \leq j' - i'$ and $n' - j' \leq n - j$, one has $V_{ij}^{(n)} \leq_{disp} V_{i'j'}^{(n')}$.

Kochar (1996c) obtained similar results for the inter-occurrence times of a non-homogeneous Poisson process. Franco, Ruiz and Ruiz (2002) and Hu and Zhuang (2004) extended these results to spacings of generalized order statistics.

Kochar and Korwar (1996) studied the problem of comparing spacings of independent exponential random variables with possibly different parameters. One of their results on dispersive ordering is stated below.

Theorem 17 *Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having exponential distribution with hazard rate λ_i, $i = 1, \ldots, n$ and let $D_{i:n}$ be the ith normalized spacing, $i = 1, \ldots, n$. Let $X_1^*, \ldots, X_n^*$ be a random sample of size n from an exponential distribution with common hazard rate $\overline{\lambda} = \sum_{i=1}^{n} \lambda_i / n$ and let $D_{i:n}^*$ be the corresponding ith normalized spacing. Then*

(a) $D_{i:n}^* \leq_{disp} D_{i:n}$, for $i = 2, \ldots, n$,
(b) $(\lambda_1, \lambda_2) \overset{m}{\succeq} (\lambda_1^*, \lambda_2^*) \Rightarrow D_{2:2}(\lambda_1^*, \lambda_2^*) \leq_{disp} D_{2:2}(\lambda_1, \lambda_2)$.

Kochar and Korwar (1996) conjectured that in the case of independent exponentials with different parameters, $D_{i:n} \leq_{hr} D_{i+1:n}$ for $i = 1, \ldots, n-1$. Khaledi and Kochar (2001) proved it for the special case when all except one of the parameters are equal. That is, they proved the above conjecture when $\lambda_1 = \cdots = \lambda_{n-1} = \lambda$ and $\lambda_n = \lambda^*$. Such a model is known as a single-outlier exponential model with parameters (λ, λ^*).

Theorem 18 *(Khaledi and Kochar, 2001) Let $X_1, \ldots, X_n$ follow the single-outlier exponential model. Then*

$$D_{i:n} \leq_{hr} D_{i+1:n} \text{ and } D_{i:n} \leq_{disp} D_{i+1:n} \quad i = 1, \ldots, n-1.$$

The next theorem for the two-sample problem is proved in (Khaledi and Kochar, 2001).

Theorem 19 *Let $X_1, \ldots, X_n$ follow the single-outlier exponential model with parameters (λ_1, λ_1^*) and let $Y_1, \ldots, Y_n$ be another set of random variables following the single-outlier exponential model with parameters (λ_2, λ_2^*). If*

$$\lambda_1^* < \lambda_2^* < \lambda_2 < \lambda_1 \text{ and } \lambda_1^* + (n-1)\lambda_1 = \lambda_2^* + (n-1)\lambda_2, \qquad (15)$$

then

$$D_{i:n}^{(1)} \geq_{hr} D_{i:n}^{(2)} \text{ and } D_{i:n}^{(1)} \geq_{disp} D_{i:n}^{(2)}, i = 1, \ldots n,$$

where $D_{i:n}^{(1)}$ *and* $D_{i:n}^{(2)}$, *respectively, are the ith spacings of single outlier exponential models with parameters* (λ_1, λ_1^*) *and* (λ_2, λ_2^*).

Remark : Note that under (15), $(\lambda_1^*, \lambda_1, \ldots, \lambda_1) \overset{m}{\succeq} (\lambda_2^*, \lambda_2, \ldots, \lambda_2)$.

5 Dispersive ordering among convolutions of random variables

In this section we study convolutions of independent random variables differing in their scale parameters and compare them according to dispersive ordering as the vectors of parameters vary. Boland, El-Neweihi and Proschan (1994) proved that a convolution of independent exponential random variables with unequal hazard rates is stochastically larger according to *likelihood ratio* ordering when the parameters of the distributions are more dispersed in the sense of *majorization.* Kochar and Ma (1999) established the following *dispersive ordering* result for a convolution of independent exponential random variables under the same conditions.

Theorem 20 *Let* $X_{\lambda_1}, \ldots, X_{\lambda_n}$ *be independent exponential random variables with respective hazard rates* $\lambda_1, \ldots, \lambda_n$, *respectively. Then* $\boldsymbol{\lambda} \overset{m}{\succeq} \boldsymbol{\lambda}^*$ *implies*

$$\sum_{i=1}^{n} X_{\lambda_i} \geq_{disp} \sum_{i=1}^{n} X_{\lambda_i^*}.$$

This result can be immediately extended to convolutions of independent Erlang random variables with different scale parameters but with a common shape parameter greater than 1. Korwar (2002) has generalized this result to convolutions of gamma random variables with an arbitrary common shape parameter greater than 1. Some related work on this problem is by Bock et al. (1987), Tong (1988 and 1994) Bon and Paltanea (1999) and Ma (2000), among others.

Khaledi and Kochar (2002a and 2004) pursued this problem further and obtained dispersive ordering results for convolutions of heterogeneous gamma, uniform and normal random variables under p-larger ordering, a partial ordering weaker than majorization. These results lead to better bounds on various quantities of interest associated with these statistics.

Theorem 21 *Let* $X_{\lambda_1}, \ldots, X_{\lambda_n}$ *be independent random variables such that* X_{λ_i} *has gamma distribution with shape parameter* $a \geq 1$ *and scale parameter* λ_i, *for* $i = 1, \ldots, n$. *Then,* $\boldsymbol{\lambda} \overset{p}{\succeq} \boldsymbol{\lambda}^*$ *implies* $S(\lambda_1, \ldots, \lambda_n) \geq_{disp} S(\lambda_1^*, \ldots, \lambda_n^*)$, *where* $S(\lambda_1, \ldots, \lambda_n) = \sum_{i=1}^{n} X_{\lambda_i}$.

A similar result holds for convolution of uniform and normal random variables. While the proof in the case of normal random variables is obvious, the proof in the case of uniform random variables is given in Khaledi and Kochar (2002a). It is stated below.

Theorem 22 *Let $X_{\lambda_1}, \ldots, X_{\lambda_n}$ be independent random variables such that X_{λ_i} has $U(0, 1/\lambda_i)$ distributions, for $i = 1, \ldots, n$. Then, $\boldsymbol{\lambda} \overset{p}{\succeq} \boldsymbol{\lambda}^*$ implies*

$$\sum_{i=1}^{n} X_{\lambda_i} \geq_{disp} \sum_{i=1}^{n} X_{\lambda_i^*}.$$

Korwar (2002) obtained a result similar to the above under the rather stronger restriction of majorization.

6 Concluding remarks

In this paper we have tried to review the literature on various results concerning dispersive ordering for nonnegative random variables putting more emphasis on its applications to order statistics, spacings and convolutions of independent random variables. As rightly pointed out by one of the referees, many of these results hold under a rather restrictive assumption that the underlying distributions are DFR. However, this is a technical requirement and it seems that this assumption can't be dispensed with. Our review is by no means exhaustive. For example, we have not touched upon the topic of dispersive ordering for multivariate distributions. The interested reader may look at the recent paper by Fernandez-Ponce and Suarez-Llorens (2003) on this problem and for other related references. Fernandez-Ponce et al. (1998) and Shaked and Shanthikumar (1998) introduced a weaker type of dispersive ordering called *right spread order* or *excess wealth order* and studied its properties. Also see Fagiuoli et al. (1999) for related results.

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