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Patron: Kochar, Subhash

Journal Title: Communications in statistics. Stochastic models.

Volume: 5 (1989), no. 2, 235--245. **Issue:** 2
Month/Year: 1989**Pages:** 235-245

Article Author:

Article Title: Kochar; On extensions of DMRL and related partial orderings of life distributions. Comm. Statist. Stochastic Models

Imprint: New York, N.Y. ; Marcel Dekker, c1985-c2

ILL Number: 19657446

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Received: 11/9/1987

Accepted: 1/12/1989

Recommended by Dieter Konig, Editor

COMMUN. STATIST.-STOCHASTIC MODELS, 5(2), 235-245 (1989)

ON EXTENSIONS OF DMRL AND RELATED PARTIAL ORDERINGS OF LIFE DISTRIBUTIONS

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ABSTRACT

Some properties of recently proposed partial orderings of life distributions are studied. Relations between NBUE ordering and mean residual life orderings of two arbitrary life distributions are investigated. The relationship between HNBUE ordering and total time on test transforms is considered. It is observed that the HNBUE ordering is the same as the Lorenz ordering.

Key words: Mean residual life function, total time on test transform, Lorenz curve, dispersive ordering, HNBUE ordering.

1. INTRODUCTION

Recently Kochar and Wiens [14] introduced some new partial orderings of life distributions. These extend the concepts of *Decreasing Mean Residual Life* (*DMRL*), *New Better than Used in Expectation* (*NBUE*) and *Harmonic New Better than Used in Expectation* (*HNBUE*) to compare the aging properties of two life distributions.

In this paper some further consequences of these orderings are studied in terms of their mean residual life functions, total time on test transforms and Lorenz curves. Chandra and Singpurwala [8] observed some interesting relationships between the various concepts which are common to Reliability Theory and Economics. Here we extend and generalize some of their work. In particular, it is noted that the *HNBUE* ordering is equivalent to the Lorenz ordering.

Let $\mathcal{F}$ denote the class of distribution functions F on $[0,\infty)$ with $F(0) = 0$. We assume throughout that all distributions being considered in $\mathcal{F}$ have finite means and are strictly increasing on their supports. For F in $\mathcal{F}$, we shall denote by F^{-1} the inverse of F defined by $F^{-1}(u) = \inf\{x : F(x) \geq u\}$. Let X and Y be random variables with distribution functions F and G belonging to $\mathcal{F}$, with survival functions $\overline{F} = 1-F$ and $\overline{G} = 1-G$; and with means μ_F and μ_G, respectively. The mean residual life function of F is $\nu_F(x) = E[X-x \mid X>x] = \int_x^\infty \overline{F}(u)du/\overline{F}(x)$. Similarly, we denote by ν_G, the mean residual life function of G.

The equilibrium distributions corresponding to F and G are F_e and G_e defined by

$$F_e(x) = \int_0^x \frac{\overline{F}(t)}{\mu_F}\,dt \text{ and } G_e(x) = \int_0^x \frac{\overline{G}(t)}{\mu_G}\,dt \tag{1.1}$$

Kochar and Wiens [14] define the following order relationships between F and G belonging to $\mathcal{F}$.

Definition 1.1: F is said to have more *Decreasing Mean Residual Life* with respect to $G(F \overset{\text{DMRL}}{<} G)$ if and only if

$$\frac{\nu_F(F^{-1}(u))}{\nu_G(G^{-1}(u))} \textit{ is nonincreasing in } u \in [0,1) \tag{1.2}$$

or equivalently,

$$\frac{\overline{F}_e(x)}{\overline{G}_e[G^{-1}F(x)]} \textit{ is nonincreasing in } x \geq 0 \tag{1.3}$$

Definition 1.2: F is said to be more *New Better than Used in Expectation* than $G(F \overset{\text{NBUE}}{<} G)$ if and only if

$$\frac{\nu_F(F^{-1}(u))}{\nu_G(G^{-1}(u))} \leq \frac{\mu_F}{\mu_G}\ ,\ u \in [0,1), \tag{1.4}$$

or equivalently,

$$G_e^{-1}F_e(x) \geq G^{-1}F(x)\ ,\ x \geq 0 \tag{1.5}$$

Definition 1.3: F is said to be more *Harmonic New Better than Used in Expectation* than $G(F \overset{\text{HNBUE}}{<} G)$ if and only if

$$G_e^{-1}F_e(x) \geq \frac{\mu_G}{\mu_F}x\ ,\ x \geq 0$$

or equivalently,

$$\overline{F}_e(x\mu_F) \leq \overline{G}_e(x\mu_G)\ ,\ x \geq 0 \tag{1.6}$$

Taking G exponential in Definitions 1.1–1.3 reproduces the usual DMRL, NBUE, and HNBUE–aging properties. For details, see, Kochar and Wiens [14].

Definition 1.4: G is said to be more *dispersed* than $F(F \overset{disp}{<} G)$ if

$$G^{-1}F(x) - x \textit{ is nondecreasing in } x \tag{1.7}$$

For properties of this ordering, see, Ahmed, Alzaid, Bartoszewicz and Kochar [1].

RELATION WITH MEAN RESIDUAL LIFE ORDERING

orem 2.1: (a) *Let F(or G) be DMRL, then*

$$F_e \overset{disp}{<} G_e \Rightarrow \nu_F(x) \le \nu_G(x),\ x \ge 0.$$

(b) *Let F (or G) be IMRL, then*

$$\nu_F(x) \le \nu_G(x),\ x \ge 0\ \Rightarrow F_e \overset{disp}{<} G_e.$$

of: (a) $F_e \overset{disp}{<} G_e \Leftrightarrow G_e^{-1}F_e(x) - x$ *nondecreasing in* x

$$\Leftrightarrow \overline{F}(x) \ge \frac{\mu_F}{\mu_G} \cdot \overline{G}[G_e^{-1}F_e(x)]\ ,\ x \ge 0$$

(*On differentiating*)

$$\Leftrightarrow \frac{\overline{F}(x)}{\overline{F}_e(x)} \ge \frac{\mu_F}{\mu_G} \cdot \frac{\overline{G}[G_e^{-1}F_e(x)]}{\overline{G}_e[G_e^{-1}F_e(x)]},\ x \ge 0$$

$$\Leftrightarrow \nu_F(x) \le \nu_G[G_e^{-1}F_e(x)],\ x \ge 0$$

$$\le \nu_G(x)\ ,\ x \ge 0,$$

is DMRL as $F_e \overset{disp}{<} G_e \Rightarrow G_e^{-1}F_e(x) \ge x\ ,\ x \ge 0.$

Similarly, the result can be proved assuming F to be DMRL.

(b)

$$\nu_F(x) \le \nu_G(x)\ ,\ x \ge 0$$

$$\Leftrightarrow \frac{1}{\nu_F(x)} \ge \frac{1}{\nu_G(x)},\ x \ge 0$$

$$\Leftrightarrow -\frac{d}{dx}\log \overline{F}_e(x) \ge -\frac{d}{dx}\log \overline{G}_e(x),\ x \ge 0$$

$$\Leftrightarrow \frac{\overline{G}_e(x)}{\overline{F}_e(x)}\ \textit{nondecreasing in}\ x \ge 0$$

$$\Rightarrow G_e^{-1}F_e(x) \ge x,\ x \ge 0$$

Now

$$\nu_F(x) \le \nu_G(x)$$

$$\le \nu_G[G_e^{-1}F_e(x)],\ x \ge 0$$

if G is an IMRL distribution.

This is equivalent to

$$\mu_F \frac{\overline{F}_e(x)}{\overline{F}(x)} \le \mu_G \frac{\overline{G}_e[\overline{G}_e^{-1}\overline{F}_e(x)]}{\overline{G}[G_e^{-1}F_e(x)]},\ x \ge 0$$

$$\Leftrightarrow \overline{F}(x) \ge \frac{\mu_F}{\mu_G}\overline{G}[G_e^{-1}F_e(x)],\ x \ge 0$$

$$\Leftrightarrow F_e \overset{disp}{<} G_e.$$

Similarly, the result can be proved assuming that F is *IMRL*.

The following theorem gives sufficient condition under which *NBUE* ordering implies dispersive ordering.

Theorem 2.2: Let $\mu_F \le \mu_G$, then $F \overset{NBUE}{<} G \Rightarrow F_e \overset{disp}{<} G_e$.

Proof: Let $h(x) = G_e^{-1}F_e(x) - x\frac{\mu_G}{\mu_F}$. On differentiating, it is easy to see that

$$F \overset{NBUE}{<} G \Leftrightarrow h(x) \text{ is nondecreasing in } x, \text{ for } x \geq 0$$

$$\Rightarrow G_e^{-1}F_e(x) - x\frac{\mu_G}{\mu_F} + x(\frac{\mu_G}{\mu_F} - 1) \text{ nondecreasing in } x \text{ for } x \geq 0.$$

$$\Rightarrow G_e^{-1}F_e(x) - x \text{ nondecreasing in } x, \text{ for } x \geq 0.$$

Combining Theorem 2.1(a) and the above theorem, we get the following corollary.

Corollary 2.1: Let F (or G) be DMRL. Then $F \overset{NBUE}{<} G$ and $\mu_F \leq \mu_G \Rightarrow \nu_F(x) \leq \nu_G(x),\ x \geq 0$.

Results of the above type were obtained by Deshpande and Kochar [10], Bartoszewicz [7] and Bagai and Kochar [2] between star–ordering, dispersive ordering and failure rate ordering.

3. OTHER CONSEQUENCES

3.1 Relations with Lorenz Curve

The Lorenz curve of F is defined as

$$L_F(p) = \frac{1}{\mu_F}\int_0^p F^{-1}(s)ds\ ,\ 0 \leq p \leq 1 \tag{3.1}$$

and it is frequently used to describe income inequalities in Economics. If F denotes the distribution of incomes in a population, then $L_F(p)$ denotes the proportion of the total income received by $100\ p\%$ of the poorest in

the population. A Lorenz curve is a continuous function on $[0,1]$ with $L(0) = 0$ and $L(1) = 1$. It is nondecreasing and convex and thus will always lie below the 45^0 line joining $(0,0)$ to $(1,1)$.

Lorenz proposed ordering income distributions by the degree with which the Lorenz curve is bent. One associates a high level of inequality with a severely bent bow. The case of complete equality corresponds to the 45^0 line. We define the Lorenz partial order by

Definition 3.1.1: $X \underset{L}{\leq} Y$ (that is, X does not exhibit more inequality in the Lorenz sense than does Y) if $L_F(p) \geq L_G(p)$ for every $p \in [0,1]$.

Theorem 3.1.1: $F \overset{HNBUE}{<} G$ if and only if $X \underset{L}{\leq} Y$.

Proof: Via (1.6), $F \overset{HNBUE}{<} G \Leftrightarrow \tilde{X} \underset{St\ 2}{<} \tilde{Y}$, where $\tilde{X} = X/\mu_F$, $\tilde{Y} = Y/\mu_G$ and $St\,2$ denotes second order dominance (Daley [9]). The claim then follows from Theorem 4.1 of Taillie [16].

The above theorem weakens the IFR (Chandra and Singpurwala [8]), IFRA (Klefsjö [13]) conditions to *HNBUE*

As F is *HNBUE* if and only if $F \overset{HNBUE}{<} G$, where $\overline{G}(x) = e^{-x}$, it follows that we can use sample Lorenz curve

$$L_n(p) = \sum_{i=1}^{[np]} X_{(i)} / \sum_{i=1}^{n} X_{(i)}$$

to check for the aging properties like *HNBUE* of life distributions. If the sample Lorenz curve $L_n(p)$ of F lies below the Lorenz curve of G (which is $L_G(p) = p + (1-p)\ln(1-p)$), this will indicate that F is an *HNBUE* distribution.

Theil [17] has advocated the use of *redundancy* as a measure of income inequality. Chandra and Singpurwala [8] used the following

$$R_F = E_F[\frac{X}{\mu}\ln(\frac{X}{\mu})]$$

and proved that $F \overset{IFR}{<} G \Rightarrow R_F \leq R_G$. We have below a much stronger result.

Theorem 3.1.2: $F \overset{HNBUE}{<} G \Rightarrow R_F \leq R_G$.

Proof: As shown in Taillie [16], $\tilde{X} \underset{St\,2}{<} \tilde{Y}$ holds if and only if $E[\psi(\tilde{X})] \leq E[\psi(\tilde{Y})]$ for any convex function ψ with finite expectation. The required result follows since $x \ln x$ is convex.

Remark 3.1.1: Klefsjö [13] has proved a special case of Theorem 3.1.2 when G is negative exponential.

3.2 Relation with Total Time on Test Transforms

The scaled total time on test (TTT) transform of $F \in \mathcal{F}$ is defined as

$$\varphi_F(t) = \frac{1}{\mu_F}\int_0^{F^{-1}(t)} \overline{F}(s)ds \tag{3.3}$$

$$= F_e(F^{-1}(t)) \text{ , } 0 \leq t \leq 1$$

This transform has been studied by Barlow and Campo [4], Barlow [3], Klefsjö [12] and Sen [15] among others. It provides a powerful graphical method for preliminary data analysis. For details see Barlow [3].

Theorem 3.2.1: Let F and G belong to $\mathcal{F}$. Then

(i) $F \overset{DMRL}{<} G$ if and *only* if $\dfrac{1-\phi_F(t)}{1-\phi_G(t)}$ *is nonincreasing in* t,

$$0 < t < 1 \tag{3.4}$$

(ii) $F \overset{NBUE}{<} G$ if and *only* if $\phi_F(t) \geq \phi_G(t)$, $0 \leq t < 1$ (3.5)

(iii) $F \overset{HNBUE}{<} G$ if and *only* if $\phi_F(t) \geq \phi_G[G\{\frac{\mu_G}{\mu_F}F^{-1}(t)\}]$,

$$0 \leq t < 1 \tag{3.6}$$

Proof: Trivial. It follows from (1.3), (1.5) and (1.6).

3.3 Relations between the moments

We have the following extension of the result of Barlow and Proschan ([6], Ex. 13, p. 120) to *HNBUE* distributions.

Theorem 3.3.1: Let $\lambda_{F,r} = E_F[X^r]/\Gamma(r+1)$ and $\lambda_{G,r} = E_G[X^r]/\Gamma(r+1)$. Then

$$F \overset{HNBUE}{<} G \Rightarrow \frac{\lambda_{F,\, r+1}}{\mu_F^{r+1}} \leq \frac{\lambda_{G,\, r+1}}{\mu_G^{r+1}}, \ r \geq 1 \tag{3.7}$$

Proof: Using the necessary and sufficient condition in (1.6) implies, for $r \geq 1$

$$\int_0^\infty \frac{x^{r-1}}{\Gamma(r)}\overline{F}_e(x)dx \leq \int_0^\infty \frac{x^{r-1}}{\Gamma(r)} \cdot \overline{G}_e[\frac{\mu_G}{\mu_F}x]dx,$$

which simplifies on integration by parts to the desired conclusion.

Corollary 3.3.1: Let $F \overset{HNBUE}{<} G$. Then the coefficient of variation of F is less than that of G.

Proof. Take $r = 1$ in (3.7).

ACKNOWLEDGEMENTS

The author is grateful to the referee and the editor for making valuable suggestions which led to an improved version of the paper. This work was partly done at Northern Illinois University while visiting from Panjab University, India.

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On extensions of DMRL and related partial orderings of life distributions

Received: 2/4/1988

Revised: 12/10/1988

Accepted: 2/7/1989

Recommended by Martin Puterman, Editor