ARTICLE IN PRESS

Available online at www.sciencedirect.com

SCIENCE DIRECT®

ELSEVIER

Statistics & Probability Letters ▮ (▮▮▮▮) ▮▮▮–▮▮▮

STATISTICS & PROBABILITY LETTERS

www.elsevier.com/locate/stapro

Lorenz ordering of order statistics

Subhash Kochar*

Department of Mathematics and Statistics, Portland State University, Portland, OR 97201, USA

Received 15 February 2006; received in revised form 7 April 2006; accepted 24 April 2006

Abstract

Let $X_{i:n}$ denote the *i*th order statistic of a random sample of size *n* from a continuous distribution with cdf *F*. Sufficient conditions are obtained on *F* so that $X_{j:m} \leqslant_{\star} X_{i:n}$ (hence $X_{j:m} \leqslant_{\text{Lorenz}} X_{i:n}$) for $i \leqslant j$ and $n-i \geqslant m-j$.

Keywords: Star ordering; Dispersive ordering; Hazard rate; Reverse hazard rate

1. Introduction

Both in reliability theory and economics we deal with probability distributions which are basically skewed on the positive part of the real line. To compare the relative aging of two components or systems, many concepts have been introduced in the reliability literature which partially order the various probability distributions. There seems to be a close connection between these partial orders in the reliability theory with those used in the economics literature to compare income inequalities. Some of the reliability concepts have direct interpretations in the economics context, but others need further investigation.

A component or system with exponential life distribution does not age with time in the sense that a used component is as good as a new one irrespective of its age. In the reliability literature exponential distribution is taken as bench mark and the relative aging of any component or system is compared with it. Several partial orderings, with varying degree of strength, have been proposed in the literature to compare the relative aging of two arbitrary life distributions when none of them is necessarily exponential. See Kochar and Weins (1987) and Kochar (1989) for details. In this note we consider star-ordering (also known as more IFRA ordering).

Let X and Y be two nonnegative random variables with survival function $\bar{F}$ and $\bar{G}$, and distribution functions F and G, respectively. Let the corresponding density functions be denoted by f and g, and their failure (hazard) rates be denoted by $r_F = f/\bar{F}$ and $r_G = g/\bar{G}$, respectively.

*Fax: +1 503 725 3661.
E-mail address: kochar@pdx.edu.

0167-7152/$ - see front matter
doi:10.1016/j.spl.2006.04.032

Definition 1.1. F is said to be more IFRA than G or F star-ordered with respect to G (written as $F \leqslant_{\star} G$ or $X \leqslant_{\star} Y$) if

$$\begin{aligned} &G^{-1}F(x) \text{ is star-shaped} \\ &\Leftrightarrow \frac{G^{-1}F(x)}{x} \uparrow x \\ &\Leftrightarrow \frac{G^{-1}(u)}{F^{-1}(u)} \uparrow u \in (0,1). \end{aligned} \tag{1.1}$$

The average failure of F at x is

$$\bar{r}_F(x) = \frac{1}{x}\int_0^x r_F(t)\,\mathrm{d}t = \frac{-\ln \bar{F}(x)}{x}.$$

Thus $F \leqslant_{\star} G$ can be interpreted in terms of average failure rates as

$$\frac{\bar{r}_F(F^{-1}(u))}{\bar{r}_G(G^{-1}(u))} \uparrow u \in (0,1).$$

Note that F is an IFRA distribution only if F is star-ordered with respect to exponential distribution.

One of the most frequently used devices to describe and compare inequality in income or wealth distribution is the Lorenz curve. Let the random variable X denote the income of an individual in a population with distribution function F and mean μ_F. The Lorenz curve of X is defined as

$$L_F(p) = \frac{\int_0^p F^{-1}(u)\,\mathrm{d}u}{\mu_F}, \quad p \in [0,1]. \tag{1.2}$$

A Lorenz curve $L_F(p)$ denotes the fraction of the total income that the poorest p proportion of the population posses. If every one in the population has the same income, the Lorenz curve would be the diagonal line $y = x$ in the unit square. The further down the Lorenz curve is from the diagonal line, the more is the disparity (inequality) among the incomes. Lorenz curves are also used to compare the amount of inequality (in incomes) among different populations.

Definition 1.2. F is said to be smaller than G in the *Lorenz* order (denoted by $F \leqslant_{\text{Lorenz}} G$) if

$$L_F(p) \geqslant L_G(p) \quad \text{for all } p \in [0,1]. \tag{1.3}$$

If $F \leqslant_{\text{Lorenz}} G$, then F exhibits less inequality than G. It can be shown that

$$F \leqslant_{\text{Lorenz}} G \Rightarrow \left(\frac{\sigma_F}{\mu_F}\right)^2 \leqslant \left(\frac{\sigma_G}{\mu_G}\right)^2.$$

Thus Lorenz ordering provides a scale invariant variability ordering among nonnegative random variables. For a comprehensive discussion on Lorenz ordering, see Arnold (1987). Assuming that the two populations have equal means, Chandra and Singapurwalla (1981) proved that

$$F \leqslant_{\star} G \Rightarrow F \leqslant_{\text{Lorenz}} G.$$

Klefsjo (1984) proved this connection between star ordering and Lorenz ordering without making any restriction on the means of the two distributions. In fact, he proved that $F \leqslant_{\star} G$ implies

$$\frac{L_G(p)}{L_F(p)} \quad \text{increasing in } p \in [0,1], \tag{1.4}$$

which in turn implies $L_G(p) \leqslant L_F(p)$ for $p \in [0,1]$.

Order statistics arise naturally in many branches of statistics. In the reliability theory, they appear as lifetimes of k-out-of-n systems. It is but natural to study their distributional properties. Many authors have studied various types of variability orders among order statistics. Whereas Kim and David (1990) established that the variances of the successive order statistics of a random sample from a decreasing failure rate (DFR) distribution are increasing, Kochar (1996) strengthened this result to dispersive ordering (defined in the next section). Khaledi and Kochar (2000) further extended this work to order statistics from different samples and distributions. Arnold and Villasenor (1989), Arnold and Nagaraja (1991), Wilfling (1996), and Kleiber (2002), among others, studied Lorenz order relations between order statistics from uniform and other distributions. In particular, Arnold and Villasenor (1989) proved the following result.

Theorem 1.1. *Let $U_{i:n}$ denote the ith order statistic of a random sample of size n from a uniform distribution over* $(0, 1)$, $i = 1, \ldots, n$. *Then*

(a) $U_{i+1:n} \leqslant_{\text{Lorenz}} U_{i:n}$, *for all* $i \leqslant n-1$,
(b) $U_{i:n} \leqslant_{\text{Lorenz}} U_{i:n+1}$, *for all* $i \leqslant n+1$,
(c) $U_{n-i+1:n+1} \leqslant_{\text{Lorenz}} U_{n-i:n}$, *for all* $i \leqslant n$,
(d) $U_{n+2:2n+3} \leqslant_{\text{Lorenz}} U_{n+1:2n+1}$, *for all* n.

The last inequality may be described as ''sample medians exhibit less variability as sample size increases''. Arnold and Villasenor (1989) wonder about the conditions on i, j, m and n under which

$$U_{j:m} \leqslant_{\text{Lorenz}} U_{i:n}$$

holds.

We answer this question in Theorem 2.2 in the next section where we find sufficient conditions on the parent distribution F under which $X_{j:m} \leqslant_{\star} X_{i:n}$ holds. Many of the previously known results follow from this general result as particular cases.

2. Main results

As mentioned in Shaked and Shanthikumar (1994), there is an intimate relation between star ordering and dispersive ordering, a basic concept for comparing variability or spread between two probability distributions. X is said to be less dispersed than Y, written as $X \leqslant_{\text{disp}} Y$, if

$$F^{-1}(\beta) - F^{-1}(\alpha) \leqslant G^{-1}(\beta) - G^{-1}(\alpha) \quad \text{for all } 0 < \alpha \leqslant \beta < 1.$$

It is easy to prove that for nonnegative random variables X and Y,

$$X \leqslant_{\star} Y \quad \Leftrightarrow \quad \ln(X) \leqslant_{\text{disp}} \ln(Y).$$

Jeon et al. (2006) proved the following result.

Theorem 2.1. *Let $U_{i:n}$ denote the ith order statistic of a random sample of size n from a uniform distribution over* $(0, 1)$, $i = 1, \ldots, n$. *Then*

$$U_{j:m} \leqslant_{\star} U_{i:n} \quad \text{for } i \leqslant j \text{ and } n - i \geqslant m - j. \tag{2.1}$$

Can Theorem 2.1 be extended to other distributions? To answer this question we use the following lemma.

Lemma 2.1. *Let ϕ be a star-shaped function on* $[0, \infty)$ *such that $\phi(x) \leqslant x$ for all $x \geqslant 0$. Let ψ be a differentiable function from* $[0, \infty)$ *to* $[0, 1]$ *such that*

$$x \frac{\psi'(x)}{\psi(x)} \quad \text{is decreasing in } x. \tag{2.2}$$

Then the function

$$\psi \phi \psi^{-1}(u) \quad \text{is also star-shaped}.$$

Proof. Note that ϕ is star-shaped if and only if

$$\phi'(x) \geqslant \frac{\phi(x)}{x} \quad \text{for all } x > 0. \tag{2.3}$$

We have to prove that under the given conditions,

$$\frac{\mathrm{d}}{\mathrm{d}u}\psi\phi\psi^{-1}(u) \geqslant \frac{\psi\phi\psi^{-1}(u)}{u} \quad \text{for all } u \geqslant 0.$$

That is,

$$\frac{\psi'\phi\psi^{-1}(u)\phi'\psi^{-1}(u)}{\psi'\psi^{-1}(u)} \geqslant \frac{\psi\phi\psi^{-1}(u)}{u}. \tag{2.4}$$

Now the left-hand side of (2.4) is at least

$$\frac{\psi'\phi\psi^{-1}(u)}{\psi'\psi^{-1}(u)} \cdot \frac{\phi\psi^{-1}(u)}{\psi^{-1}(u)},$$

because of (2.3).

Thus to establish the required result, it is sufficient to prove that

$$\frac{\psi'\phi\psi^{-1}(u)}{\psi'\psi^{-1}(u)} \cdot \frac{\phi\psi^{-1}(u)}{\psi^{-1}(u)} \geqslant \frac{\psi\phi\psi^{-1}(u)}{u}. \tag{2.5}$$

Since by assumption (2.2),

$$\frac{\psi'(x)}{\psi(x)}x \quad \text{is decreasing in } x \text{ and } \phi(x) \leqslant x,$$

it follows that

$$\frac{\psi'\phi\psi^{-1}(u)}{\psi\phi\psi^{-1}(u)} \cdot \phi\psi^{-1}(u) \geqslant \frac{\psi'\psi^{-1}(u)}{u} \cdot \psi^{-1}(u),$$

or

$$\frac{\psi'\phi\psi^{-1}(u)}{\psi'\psi^{-1}(u)} \cdot \frac{\phi\psi^{-1}(u)}{\psi^{-1}(u)} \geqslant \frac{\psi\phi\psi^{-1}(u)}{u}.$$

This proves the required inequality (2.5) and hence the result. □

The reverse hazard rate $\tilde{r}_F$ of a random variable X with pdf f and cdf F is defined as $\tilde{r}_F(x) = f(x)/F(x)$. In the next theorem we extend the results of Theorem 2.1 to distributions satisfying condition (2.6) below.

Theorem 2.2. *For $i = 1, \ldots, n$, let $X_{i:n}$ denote the ith order statistic of a random sample of size n from a distribution with reverse hazard rate $\tilde{r}_F$. If*

$$x\tilde{r}_F(x) \quad \text{is increasing in } x, \tag{2.6}$$

then for $i \leqslant j$ and $n - i \geqslant m - j$,

$$X_{j:m} \leqslant_{\star} X_{i:n}. \tag{2.7}$$

Proof. The distribution function of $X_{i:n}$ is $F_{i:n}(x) = B_{i:n}F(x)$, where

$$B_{i:n}(p) = n\binom{n-1}{i-1}\int_0^p u^{i-1}(1-u)^{n-i}\,\mathrm{d}u.$$

We have to prove that under condition (2.6),

$$F_{i:n}^{-1}F_{j:m} = F^{-1}\left(B_{i:n}^{-1}B_{j:m}\right)F^{-1} \quad \text{is star-shaped.}$$

By Theorem 2.1, for $i \leqslant j$ and $n-i \geqslant m-j$, the function $\phi = B_{i:n}^{-1} B_{j:m}$ is star-shaped. By Lemma 2.1, a sufficient condition for (2.7) is that the function $\psi(u) = F^{-1}(u)$ satisfies condition (2.2). That is,

$$u\frac{(\mathrm{d}/\mathrm{d}u)F^{-1}(u)}{F^{-1}(u)} = \frac{u}{f\left(F^{-1}(u)\right)F^{-1}(u)} = \left[F^{-1}(u)\tilde{r}_F(F^{-1}(u)\right]^{-1} \quad \text{is decreasing in } u,$$

which is equivalent to (2.6). □

Remark. Arnold and Villasenor (1989) mention (2.6) as a sufficient condition for the relation

$$X_{i+1:n} \leqslant_{\text{Lorenz}} X_{i:n}$$

to hold. We have a more general result.

Theorem 2.3. *For $i = 1, \ldots, n$, let $X_{i:n}$ denote the ith order statistic of a random sample of size n from a distribution with its hazard rate $r_F(x) = f(x)/\bar{F}(x)$ satisfying the condition,*

$$x r_F(x) \quad \text{is decreasing in } x. \tag{2.8}$$

Then

$$X_{i:n} \leqslant_{\star} X_{j:m} \quad \text{for } i \leqslant j \text{ and } n-i \geqslant m-j. \tag{2.9}$$

Proof. Khaledi and Kochar (2000) proved that a sufficient condition for $X_{i:n} \leqslant_{\text{disp}} X_{j:m}$ for $i \leqslant j$ and $n-i \geqslant m-j$ is that $r_F(x)$ is decreasing. Since $X \leqslant_{\star} Y \Leftrightarrow \ln(X) \leqslant_{\text{disp}} \ln(Y)$, it follows that a sufficient condition for (2.9) is that the random variable $\ln(X)$ has a DFR. It is easy to verify that the hazard rate of $\ln(X)$ is

$$\mathrm{e}^{y}\frac{f(\mathrm{e}^{y})}{\bar{F}(\mathrm{e}^{y})},$$

which is decreasing in y if and only if condition (2.8) is satisfied. □

The above theorems immediately lead to the following result because of the relation between star-ordering and Lorenz ordering.

Corollary 2.1. *If for $i = 1, \ldots, n$, $X_{i:n}$ denotes the ith order statistic of a random sample of size n from a distribution satisfying:*

(a) *condition* (2.6), *then*

$$X_{j:m} \leqslant_{\text{Lorenz}} X_{i:n}$$

for $i \leqslant j$ and $n-i \geqslant m-j$.

(b) *condition* (2.8), *then*

$$X_{j:m} \geqslant_{\text{Lorenz}} X_{i:n}$$

for $i \leqslant j$ and $n-i \geqslant m-j$.

Example 2.1. (2.6) is satisfied by the power function distribution with distribution function, $F_X(x) = [x/c]^{\gamma}$, $o \leqslant x \leqslant c$, $\gamma > 0$. Therefore, the conclusions of Theorem 2.2 and Corollary 2.1(a) hold for this distribution. Arnold and Villasenor (1989) also conjectured that for this distribution,

$$X_{n-j+1:n+1} \leqslant_{\text{Lorenz}} X_{n-j:n} \quad \text{for every } 1 \leqslant j \leqslant n.$$

Its proof immediately follows from Theorem 2.2 and Corollary 2.1.

Example 2.2. Let X has Pareto distribution with $F(x) = 1-(x/c)^{-\alpha}$, $x > c$, $\alpha > 1$. Condition (2.8) is satisfied by this distribution. Hence the conclusions of Theorem 2.3 and Corollary 2.1(b) hold for this distribution.

Acknowledgement

The author is thankful to an anonymous referee for his comments and suggestions which have led to an improved version of the paper.

References

Arnold, B.C., 1987. Majorization and its Lorenz Order: a Brief Introduction. Lecture Notes in Statistics, vol. 43. Springer, Berlin.
Arnold, B.C., Nagaraja, H.N., 1991. Lorenz ordering of exponential order statistics. Stat. Probab. Lett. 11, 485–490.
Arnold, B.C., Villasenor, J.A., 1989. Lorenz ordering of order statistics. Stochastic orders and decision under risk (Hamburg, 1989), IMS Lecture Notes Monograph. Series, vol. 19. Institute of Mathematics and Statistics, Hayward, CA, 1991, pp. 38–47.
Chandra, M., Singapurwalla, N., 1981. Relationships between some notions, which are common to reliability and economics. Math. Oper. Res. 6, 113–121.
Jeon, J., Kochar, S., Park, C.G., 2006. Dispersive ordering—some applications and examples. Stat. Papers 47, 227–247.
Khaledi, B., Kochar, S.C., 2000. On dispersive ordering between order statistics in one-sample and two-sample problems. Stat. Probab. Lett. 46, 257–261.
Kim, S.H., David, H.A., 1990. On the dependence structure of order statistics and concomitants of order statistics. J. Stat. Plann. Inference 24, 363–368.
Klefsjo, B., 1984. Reliability interpretations of some concepts from economics. Naval Res. Logistics Quart. 31, 301–308.
Kleiber, C., 2002. Variability ordering of heavy-tailed distributions with applications to order statistics. Stat. Probab. Lett. 58, 381–388.
Kochar, S., 1989. On extensions of DMRL and related partial orderings of life distributions. Stochastic Models 5, 235–246.
Kochar, S., 1996. Dispersive ordering of order statistics. Stat. Probab. Lett. 27, 271–274.
Kochar, S., Weins, D., 1987. Partial orderings of life distributions with respect to their aging properties. Naval Res. Logistic 34, 823–829.
Shaked, M., Shanthikumar, J.G., 1994. Stochastic Orders and Their Applications. Academic Press, Boston.
Wilfling, B., 1996. Lorenz ordering of power-function order statistics. Stat. Probab. Lett. 30, 313–319.