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Monotonicity properties of the ordered ranks in the two-sample problem

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Abstract

Let $X_1, \dots, X_m$ and $Y_1, \dots, Y_n$ be independent random samples from two absolutely continuous distributions F and G, respectively. For $F = G$, Fligner and Wolfe (1976) established some interesting properties of the W_i's, the number of X-observations between the $(i-1)$th and ith order statistics of the Y-sample. In particular, it follows from their results that when $F = G$, the W_i's are identically distributed. In this note we study this problem when the X's are greater than the Y's according to likelihood ratio and hazard rate orderings. It is shown that in both these cases, the W_i's exhibit stochastic increasing trends of different types.

Key words: Likelihood ratio ordering; Hazard rate ordering; Stochastic ordering; P–P plots

1. Introduction

Let X and Y be two absolutely continuous random variables with distribution functions F and G and with probability density functions f and g, respectively. Let $X_1, \dots, X_m$ and $Y_1, \dots, Y_n$ be independent random samples from F and G, respectively. We denote by F_m and G_n the corresponding empirical distribution functions.

Let

$$V_{(j)} = F_m G_n^{-1}(j/n) = F_m(Y_{(j)}) = (R_{(j)} - j)/m, \tag{1.1}$$

where $Y_{(j)}$ is the jth order statistic of the Y-sample and $R_{(j)}$ is the rank of $Y_{(j)}$ in the combined increasing arrangement of X's and Y's.

Let

$$W_j = m[V_{(j)} - V_{(j-1)}] = R_{(j)} - R_{(j-1)} - 1 \tag{1.2}$$

and

$$W_{(n+1)} = n + m - mF_m(Y_{(n)}).$$

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Note that W_j is the number of X's in $(Y_{(j-1)}, Y_{(j)}]$, for $j = 2, \ldots, n$, W_1 is the number of X's less than or equal to $Y_{(1)}$ and W_{n+1} is the number of X's greater than $Y_{(n)}$. Also observe that $R_{(i)} = \sum_{j=1}^{i} W_j + i$, for $i = 1, \ldots, n$.

The plot of $FG^{-1}(y)$ against y is called a P–P plot and the process $\ell_N(y) := N^{1/2}[F_m G_n^{-1}(y) - FG^{-1}(y)]$, $0 \leqslant y \leqslant 1$, $N = m + n$, is known as the empirical P–P plot process. It is a powerful tool for exploratory data analysis (see Wilk and Gnanadesikan, 1968). A large number of nonparametric procedures in the literature are based on functions of $V_{(j)}$'s or, equivalently, on the *ordered ranks* $R_{(j)}$'s. Fligner and Wolfe (1976) discuss some of them. Other important references on this topic are the two-sample tests proposed by Sen and Govindarajulu (1966), Deshpandé (1972), Kochar (1981), Joe and Proschan (1984) and Aly (1988).

Fligner and Wolfe (1976) studied many interesting properties of the sample analogues $V_{(j)} \equiv F_m(Y_{(j)})$ of $F(Y_{(j)})$ under the hypothesis H_0: $F = G$. It follows from their Theorem 4.2 that under H_0, the random variables $V_{(l)} - V_{(k)}$ and $V_{(l-k)}$ are identically distributed, for $l > k$. In particular, it follows that $W_1, \ldots, W_{n+1}$ are all identically distributed (though they are dependent) when $F = G$. In fact, one can prove a more general result that under this hypothesis, the random variables $W_1, \ldots, W_{n+1}$ are exchangeable.

In this note, we study the stochastic relations between the W_i's under each of the following hypotheses:

(i) H_1: $f(x)/g(x)$ is nondecreasing in x, for all x, that is, X is stochastically greater than Y according to *likelihood ratio ordering* $X \overset{\text{lr}}{\geqslant} Y$.

(ii) H_2: $\bar{F}(x)/\bar{G}(x)$ is nondecreasing in x, for all x, that is, X is greater than Y according to *hazard rate ordering* and we write this as $X \overset{\text{hr}}{\geqslant} Y$. Here $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$.

(iii) H_3: $F(x)/G(x)$ is nondecreasing in x, for all x, that is, the *survival rate* $f(x)/F(x)$ of X is greater than that of Y for all x.

Note that H_1 implies H_2 and H_3 and these in turn imply that X is *stochastically greater* than Y.

Observe that $X \overset{\text{lr}}{\geqslant} Y$ if and only if $FG^{-1}(x)$ is convex in x. How will this convexity property be reflected in the sample? We should expect the increments $W_j\,(= m(V_{(j)} - V_{(j-1)}))$ to increase in some stochastic sense if H_1 holds. We study this probelm in the next section and show that W_i's do exhibit a very strong type of stochastic monotonicity. In particular, W_i's are shown to be stochastically increasing in this case. In Section 3 we show that $[W_1 + \cdots + W_j]/j$ is increasing in j in *expectation* under H_3. A similar result holds between the W_i's under H_2.

2. Stochastic monotonicity of the W_i's under likelihood ratio ordering

As observed earlier, the W_i's are dependent. Shanthikumar and Yao (1991) extended the concepts of *likelihood ratio ordering* to compare the components of a random vector. Let $\boldsymbol{x} = (x_1, \ldots, x_p)$ and $\boldsymbol{y} = (y_1, \ldots, y_p)$ be two p-dimensional vectors. We say that $\boldsymbol{x}$ is *better arranged than* $\boldsymbol{y}$ ($\boldsymbol{x} \geqslant_a \boldsymbol{y}$) if $\boldsymbol{x}$ can be obtained from $\boldsymbol{y}$ through successive pairwise interchanges of its components, with each interchange resulting in a decreasing order of the two interchanged components, e.g. $(4, 5, 3, 1) \geqslant_a (4, 3, 5, 1) \geqslant_a (4, 1, 5, 3)$. (Notice that $\boldsymbol{x}$ is necessarily a permutation of $\boldsymbol{y}$.) A function h: $\mathscr{R}^p \to \mathscr{R}$ that preserves the ordering $\geqslant_a$ is called an *arrangement increasing* function if $\boldsymbol{x} \geqslant_a \boldsymbol{y} \Rightarrow h(\boldsymbol{x}) \geqslant h(\boldsymbol{y})$. In this case we write $h \in \mathscr{AI}$.

Definition 2.1. Let $h(t_1, \ldots, t_p)$ denote the joint density of $\boldsymbol{T}$. Then

$$T_1 \overset{\text{lr:j}}{\geqslant} T_2 \overset{\text{lr:j}}{\geqslant} \cdots \overset{\text{lr:j}}{\geqslant} T_p \Leftrightarrow h \in \mathscr{AI}. \tag{2.1}$$

Shanthikumar and Yao (1991) have discussed many interesting properties of this ordering. We show in this section that under H_1, the W_i's are increasing according to *joint likelihood ratio ordering*. Let $p(w_1, \ldots, w_{n+1})$ denote the joint probability density function of $\boldsymbol{W}$.

Theorem 2.1. *Under* H_1,

$$p(w_1, \ldots, w_{i-1}, k-c, l+c, w_{i+2}, \ldots, w_{n+1}) \geqslant p(w_1, \ldots, w_{i-1}, k, l, w_{i+2}, \ldots, w_{n+1})$$

for $0 < c \leqslant k$.

Proof. It is sufficient to prove the result for $c = 1$. We shall use the following result due to Hoeffding (1951) (also see Hettmansperger, 1984, pp. 142–143):

$$\Pr[R_1 = r_1, \ldots, R_n = r_n] = \left(\binom{m+n}{m}\right)^{-1} \mathrm{E}\left[\prod_{i=1}^{n} \frac{g(X_{(r_i)})}{f(X_{(r_i)})}\right], \tag{2.2}$$

where $X_{(1)} < \cdots < X_{(m+n)}$ are the order statistics of a random sample of size $m+n$ from F. Let $w_{\cdot i} = \sum_{j=1}^{i} w_j$. Then

$$p(w_1, \ldots, w_{i-1}, k-1, l+1, w_{i+2}, \ldots, w_{n+1}) - p(w_1, \ldots, w_{i-1}, k, l, w_{i+2}, \ldots, w_{n+1})$$
$$= \left(\binom{m+n}{m}\right)^{-1} \mathrm{E}\left[\left\{\frac{g(X_{(w_{\cdot i-1}+k+i-1)})}{f(X_{(w_{\cdot i-1}+k+i-1)})} - \frac{g(X_{(w_{\cdot i-1}+k+i)})}{f(X_{(w_{\cdot i-1}+k+i)})}\right\} \prod_{j \neq 1}^{n} \frac{g(U_{(w_{\cdot j}+j)})}{f(U_{(w_{\cdot j}+j)})}\right] \geqslant 0,$$

as for $y > x$,

$$\frac{g(x)}{f(x)} - \frac{g(y)}{f(y)} \geqslant 0. \qquad \square$$

Note that the ordering between the W_i's as established in Theorem 2.1 is even stronger than the $\overset{\text{lr:j}}{\geqslant}$ ordering. For example, according to this ordering for $n = 1$ and $m = 4$, $p(1,3) \geqslant p(2,2) \geqslant p(3,1)$, whereas the arrangement increasing ordering is unable to make such refined comparisons. It only compares the arrangements (1, 3) and (3, 1) between themselves without worrying about the arrangement (2, 2). Also, as noted in Shanthikumar and Yao (1991), $T_1 \overset{\text{lr:j}}{\leqslant} T_2 \Rightarrow T_1 \overset{\text{st}}{\leqslant} T_2$, the stochastic ordering between the marginal distributions of T_1 and T_2. Note, however, that $T_1 \overset{\text{lr:j}}{\leqslant} T_2$ may not imply $T_1 \overset{\text{lr}}{\leqslant} T_2$. Using these results and the above theorem, we get the following corollary.

Corollary 2.1. *Under* H_1,

(a) $W_{n+1} \overset{\text{lr:j}}{\geqslant} W_n \overset{\text{lr:j}}{\geqslant} \cdots \overset{\text{lr:j}}{\geqslant} W_1$ *or, equivalently,* $R_{(j)} - R_{(j-1)} \overset{\text{lr:j}}{\geqslant} R_{(j-1)} - R_{(j-2)}$ *for* $j = 2, \ldots, n$;

(b) $W_{n+1} \overset{\text{st}}{\geqslant} W_n \overset{\text{st}}{\geqslant} \cdots \overset{\text{st}}{\geqslant} W_1$ *or, equivalently,* $R_{(j)} - R_{(j-1)} \overset{\text{st}}{\geqslant} R_{(j-1)} - R_{(j-2)}$ *for* $j = 2, \ldots, n$.

3. Relationships between the W_i's under H_2 and H_3

In this section we study the stochastic order relations between the W_i's under H_2 and H_3. Note that H_3 holds if and only if $FG^{-1}(x)$ is star shaped in the sense that the function $FG^{-1}(x)/x$ is nondecreasing in x. In this case we should expect

$$\frac{F_m G_n^{-1}(j/n)}{j/n} = nF_m(Y_{(j)})/j = \frac{n\{W_1 + \cdots + W_j\}}{mj} \tag{3.1}$$

$$= \frac{n}{m}\left[\frac{R_{(j)}}{j} - 1\right] \tag{3.2}$$

to increase in j in some stochastic sense. We establish such a result in this section. To prove this we shall need the following lemmas.

Lemma 3.1. $Y_{(j)} \overset{\text{lr}}{>} Y_{(j-1)}, j = 2, \dots, n.$

Proof. If we denote by $g_{(j)}(y)$ the density of $Y_{(j)}$, then

$$g_{(j)}(y)/g_{(j-1)}(y) = C(j, n)\, G(y)/\bar{G}(y)$$

is nondecreasing in y. Hence the result. □

Lemma 3.2. *Let $\alpha(x)$ and $\beta(x)$ be nonnegative functions such that $\alpha(x)/\beta(x)$ is nondecreasing in x; then $X \overset{\text{lr}}{\geqslant} Y$ implies*

$$\frac{E[\alpha(X)]}{E[\beta(X)]} \geqslant \frac{E[\alpha(Y)]}{E[\beta(Y)]}. \tag{3.3}$$

Proof. It follows from Lemma 2 of Bickel and Lehmann (1975). Also see Theorem 2.3 of Shanthikumar and Yao (1991). □

Theorem 3.1. (a) *Under* H_2,

$$\mathrm{E}\left[\sum_{i=j+1}^{n+1} W_i/(n-j+1)\right] \text{ is increasing in } j; \tag{3.4}$$

(b) *Under* H_3,

$$\mathrm{E}\left[\sum_{i=1}^{j} W_i/j\right] \text{ is increasing in } j, \tag{3.5}$$

$$\Leftrightarrow \mathrm{E}\,[R_{(j)}/j] \text{ is increasing in } j; \tag{3.6}$$

for $j = 2, \dots, n$.

Proof. We give the proof only for (b) as the proof for (a) is on the same lines. Since

$$\mathrm{E}\left[\sum_{i=1}^{j} W_i/j\right] = E[\text{number of } X\text{'s} \leqslant Y_{(j)}]/j = mP[X \leqslant Y_{(j)}]/j, \tag{3.7}$$

it is sufficient to show that under H_3, $P[X \leqslant Y_{(j)}]/j$ is nondecreasing in j.

Since under H_3, $F(x)/G(x)$ is nondecreasing in x, using Lemmas 3.1 and 3.2 with $\alpha(x) = F(x)$ and $\beta(x) = G(x)$, it follows that

$$\frac{E[F(Y_{(j)})]}{E[G(Y_{(j)})]} \geqslant \frac{E[F(Y_{(j-1)})]}{E[G(Y_{(j-1)})]}, \tag{3.8}$$

that is,

$$\frac{(n+1)}{j} P[X \leqslant Y_{(j)}] \geqslant \frac{(n+1)}{(j-1)} P[X \leqslant Y_{(j-1)}],$$

since $G(Y_{(j)})$ has the same distribution as $U_{(j)}$, the jth order statistic of a random sample of size n from the uniform distribution over (0, 1) and $E[U_{(j)}] = j/(n+1)$. □

One may wonder whether under H_3, the random variable $\sum_{i=1}^{j} W_i/j$ is *stochastically increasing* in j. The answer is no, since the random variables $\sum_{i=1}^{j} W_i/j$ and $\sum_{i=1}^{(j-1)} W_i/(j-1)$ may have overlapping supports with the second variable possibly taking greater values than the first one.

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