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On the 'triples test' for symmetry

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Abstract: Randles et al. (1980) and Davis and Quade (1978) independently proposed an asymptotically distribution-free test for testing symmetry about an unknown point. In this note, it has been shown that if X is convex ordered with respect to $-X$, then the one-sided test is consistent, thus solving an open problem of Randles et al. (1980).

Keywords: Symmetry, U-statistic, convex-ordering, nonparametric test.

1. Introduction

Let $X_1, \ldots, X_n$ be a random sample of size n from a continuous distribution with cumulative distribution function F and density function f. A random variable X is said to be symmetric about θ if and only if $X - \theta$ and $\theta - X$ have the same distribution.

Randles, Fligner, Policello and Wolfe (1980) and Davis and Quade (1978) independently proposed an asymptotically distribution-free test for testing the symmetry of a random variable about an unknown point. Their test, which is commonly known as the 'triples test', is based on the U-statistic estimator $\hat{\eta}$ of the parameter

$$\eta = \Pr[X_1 + X_2 - 2X_3 > 0] - \Pr[X_1 + X_2 - 2X_3 < 0] \qquad (1.1)$$

where X_1, X_2 and X_3 are three independent copies of X. Under the null hypothesis of symmetry (about an unknown point), $\eta = 0$.

The above authors proposed the 'triples test' purely on heuristic grounds. Randles et al. (1980) comment on page 169, "unfortunately, there is no apparent way to express simply the consistency condition that η be different from zero in terms of F". We attempt to answer this question in the next section and give a general class of distributions for which η is different from zero.

2. The main result

We shall use the concept of *convex ordering* of distributions as introduced by van Zwet (1964).

Definition 2.1. Let $X(Y)$ be a random variable with distribution function $F(G)$. Then X is said to be *convex ordered* with respect to Y if

$$G^{-1}F(x) \text{ is strictly convex on the support of } F. \qquad (2.1)$$

We write this as $X <^c Y$ or $F <^c G$. This partial ordering of distributions is location-scale invariant in the sense that $F =^c G$ if and only if $F(x) = G(\alpha x + \beta)$ for some constants α and β.

Ahmad and Kochar (1990) proved the following result:

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Theorem 2.2. *Let X_1', X_2' and X_3' (Y_1', Y_2', Y_3') be three independent copies of X' (Y') with distribution function F (G) and let λ be any real number such that $0 \leqslant \lambda \leqslant 1$. Then $F \prec^c G$ implies*

$$\Pr\left[X_1' - \lambda X_2' - \bar{\lambda} X_3' < 0\right] < \Pr\left[Y_1' - \lambda Y_2' - \bar{\lambda} Y_3' < 0\right],$$

where $\bar{\lambda} = 1 - \lambda$. □

Specializing this result to $X' = X - \theta$ and $Y' = \theta - X$, we get the following corollary.

Corollary 2.3. *Let $X - \theta \prec^c \theta - X$ (or equivalently, $X \prec^c -X$), then*

$$\Pr\left[X_1 - \lambda X_2 - \bar{\lambda} X_3 < 0\right] < \Pr\left[X_1 - \lambda X_2 - \bar{\lambda} X_3 > 0\right]. \qquad \square$$

In particular, by taking $\lambda = \bar{\lambda} = \frac{1}{2}$, we find that $X \prec^c -X$ implies $\eta < 0$.

Thus the test based on small values of $\hat{\eta}$ or its studentized version will be consistent for testing the null hypothesis

H_0: X is symmetric (about an unknown point θ)

against the one-sided alternative

H_A: $X - \theta \prec^c \theta - X$ (or equivalently, $X \prec^c -X$).

It can be seen that the alternative H_A is equivalent to

$$\frac{f\left[F^{-1}(1-u)\right]}{f\left[F^{-1}(u)\right]} \text{ nonincreasing in } u \quad \text{for } u \in (0, 1).$$

The following are some important classes of distributions for which H_A holds.

Example 2.4. Let Y be a random variable symmetric about 0 and let, for some $c > 0$,

$$X_\theta = \begin{cases} Y + \theta & \text{if } Y < \theta, \\ Y/c + \theta & \text{if } Y \geqslant \theta. \end{cases}$$

Then it can be seen that X_θ satisfies H_A for any real θ provided $c < 1$.

Example 2.5. Let $g(\cdot)$ be a density function on $[0, \infty)$. Define a new density function f by

$$f(x) = \begin{cases} \frac{1}{2} g(-x) & \text{if } x \leqslant 0, \\ (1/(2c)) g(x/c) & \text{if } x > 0, \end{cases}$$

where $0 < c < 1$. Then H_A holds for such a distribution f for any arbitrary density function g on $[0, \infty)$.

Remarks. (1) It should be noted that $X \prec^c -X$ is not a necessary condition for $\eta < 0$. Similarly, it can be shown that if $-X \prec^c X$ then $\eta > 0$ and the test based on large values of $\hat{\eta}$ will be consistent for testing H_0 against the alternative $-X \prec^c X$.

(2) By taking different values of λ between 0 and 1, one can obtain a class of asymptotically distribution-free tests for testing symmetry against H_A, thus generalizing the above test based on $\hat{\eta}$. However, we shall not pursue this matter further here.

References

Ahmad, I.A. and S.C. Kochar (1990), Testing for whether one distribution is more IFR than the other, *Metrika* **37**, 45–58.

Davis, C.E. and D. Quade (1978), U-statistics for skewness and symmetry, *Comm. Statist.—Theory Methods A* **7**, 413–418.

Randles, R.H., M.A. Fligner, G.E. Policello and D.A. Wolfe (1980), An asymptotically distribution-free test for symmetry versus asymmetry, *J. Amer. Statist. Assoc.* **75**, 168–172.

Zwet, W.R. van (1964), *Convex Transformations of Random Variables* (Math. Centrum, Amsterdam).