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Statistics & Probability Letters 43 (1999) 321–324

STATISTICS & PROBABILITY LETTERS

Dispersive ordering of convolutions of exponential random variables

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Received September 1998

Abstract

Let $X_{\lambda_1},\ldots,X_{\lambda_n}$ be independent random variables such that X_{λ_i} has exponential distribution with hazard rate λ_i, $i=1,\ldots,n$. It is shown that $\sum_{i=1}^{n} X_{\lambda_i}$ is more *dispersed* than $\sum_{i=1}^{n} X_{\lambda_i^*}$ if $(\lambda_1,\ldots,\lambda_n)$ *majorizes* $(\lambda_1^*,\ldots,\lambda_n^*)$. © 1999 Elsevier Science B.V. All rights reserved

MSC: 62G30; 60E15; 62N05; 62D05

Keywords: Convolution; Dispersive ordering; Likelihood ratio ordering; Majorization

1. Introduction

The exponential distribution plays a very important role in statistics. Because of its non-aging property, it has many nice properties and it often gives very convenient bounds on survival probabilities and other characteristics of interest for systems with non-exponential components. Boland et al. (1994) proved that a convolution of independent exponential random variables with unequal hazard rates is stochastically larger with respect to the likelihood ratio ordering when the parameters of the exponential distributions are more dispersed in the sense of *majorization*. We pursue this problem further in this note and obtain some dispersive ordering results for convolutions of heterogeneous exponential random variables.

Let X and Y be two random variables with distribution functions F and G, respectively. Let F^{-1} and G^{-1} be their right continuous inverses. The distribution of the random variable X is less dispersed than that of Y ($X \overset{\text{disp}}{\leqslant} Y$) if

$$F^{-1}(v) - F^{-1}(u) \leqslant G^{-1}(v) - G^{-1}(u) \quad \text{for} \leqslant u \leqslant v \leqslant 1.$$

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[1] Supported by a NSERC Canada grant.

PII: S0167-7152(98)00279-X

This means that the difference between any two quantiles of F is smaller than the difference between the corresponding quantiles of G. A consequence of $X \overset{\text{disp}}{\leqslant} Y$ is that $|X_1 - X_2| \leqslant_{\text{st}} |Y_1 - Y_2|$ and which in turn implies $\text{var}(X) \leqslant \text{var}(Y)$ as well as $E[|X_1 - X_2|] \leqslant E[|Y_1 - Y_2|]$, where $X_1, X_2 (Y_1, Y_2)$ are two independent copies of $X(Y)$, and st represents the usual stochastic order. For details, see Section 2.B of Shaked and Shanthikumar (1994). A related concept is that of star-ordering. X is said to be smaller than Y in the *star order* (denoted by $X \overset{*}{\leqslant} Y$) if $G^{-1}F(x)/x$ is increasing in x. It is easy to see that $X \overset{\text{disp}}{\leqslant} Y \Leftrightarrow e^X \overset{*}{\leqslant} e^Y$. It is well known that a distribution F is IFRA (increasing failure rate average) if and only if it is star-ordered with respect to the exponential distribution. Also $X \overset{*}{\leqslant} Y$ implies that the Lorenz curve of F is uniformly greater than that of G (cf. Kochar, 1989).

2. Main results

The concept of majorization of vectors will be needed in this work. For reference see Marshall and Olkin (1979). Let $\{x_{(1)} \leqslant x_{(2)} \leqslant \cdots \leqslant x_{(n)}\}$ denote the increasing arrangement of the components of a vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$. $\boldsymbol{x}$ is said to *majorize* another vector $\boldsymbol{y}$ (written as $\boldsymbol{y} \overset{m}{\leqslant} \boldsymbol{x}$) if $\sum_{1=1}^{j} x_{(i)} \leqslant \sum_{i=1}^{j} y_{(i)},\ j=1,\ldots,n-1$ and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$. In terms of majorization of parameter vectors, we now compare two convolutions of exponential random variables with respect to the dispersive ordering.

Theorem 2.1. *Let $X_{\lambda_1}, \ldots, X_{\lambda_n}$ be independent exponential random variables with respective hazard rates $\lambda_1, \ldots, \lambda_n$, respectively. Then $\boldsymbol{\lambda} \overset{m}{\geqslant} \boldsymbol{\lambda}^*$ implies*

$$\sum_{i=1}^{n} X_{\lambda_i} \overset{\text{disp}}{\geqslant} \sum_{i=1}^{n} X_{\lambda_i^*}. \tag{2.1}$$

Proof. By the nature of majorization, it suffices to prove the result when (λ_1, λ_2) majorizes $(\lambda_1^*, \lambda_2^*)$ and $\lambda_i = \lambda_i^*$ $(i = 3, \ldots, n)$. Let $X_{1:2} \leqslant X_{2:2}$ $(X_{1:2}^* \leqslant X_{2:2}^*)$ be order statistics corresponding to $X_{\lambda_1}, X_{\lambda_2}$ $(X_{\lambda_1^*}^*, (X_{\lambda_2^*}^*)$. It follows from Theorem 3.7(b) of Kochar and Korwar (1996) that

$$X_{2:2} - X_{1:2} \overset{\text{disp}}{\geqslant} X_{2:2}^* - X_{1:2}^*, \tag{2.2}$$

and from Theorem 2.1 of Kochar and Korwar (1996) that $X_{2:2} - X_{1:2}$ $(X_{2:2}^* - X_{1:2}^*)$ and $X_{1:2}$ $(X_{1:2}^*)$ are independent. Moreover, $X_{1:2} \overset{\text{dist}}{=} X_{1:2}^*$ has exponential distribution with hazard rate $\lambda_1 + \lambda_2 = \lambda_1^* + \lambda_2^*$. It follows from (2.2) and Theorem 6 of Lewis and Thompson (1981) that

$$\sum_{i=1}^{2} X_{\lambda_i} = (X_{2:2} - X_{1:2}) + 2X_{1:2} \overset{\text{disp}}{\geqslant} (X_{2:2}^* - X_{1:2}^*) + 2X_{1:2}^* = \sum_{i=1}^{2} X_{\lambda_i^*}.$$

Repeatedly using the above result of Lewis and Thompson (1981), one obtains

$$\sum_{i=1}^{2} X_{\lambda_i} + \sum_{i=3}^{n} X_{\lambda_i} \overset{\text{disp}}{\geqslant} \sum_{i=1}^{2} X_{\lambda_i^*} + \sum_{i=3}^{n} X_{\lambda_i}. \qquad \square$$

Remark. Under the conditions of the above theorem, Boland et al. (1994) proved that

$$\sum_{i=1}^{n} X_{\lambda_i} \overset{\text{lr}}{\geqslant} \sum_{i=1}^{n} X_{\lambda_i^*}, \tag{2.3}$$

where lr stands for the likelihood ratio ordering, which implies the usual stochastic ordering. To the best of our knowledge we are not aware of any results on dispersive ordering of convolutions of exponential random variables.

Theorem 2.1 can be easily extended to convolutions of Erlang distributions (gamma distributions with integer valued shape parameters) as follows.

Corollary 2.1. *Let $X_{\lambda_1},\dots,X_{\lambda_n}$ be independent random variables such that X_{λ_i} has Erlang distribution with scale parameter λ_i and shape parameter r_i for $i=1,\dots,n$. Then $\boldsymbol{\lambda} \overset{m}{\geqslant} \boldsymbol{\lambda}^*$ implies*

$$\sum_{i=1}^{n} X_{\lambda_i} \overset{\text{disp}}{\geqslant} \sum_{i=1}^{n} X_{\lambda_i^*}.$$

Proof. It follows immediately from Theorem 2.1 since X_{λ_i} can be expressed as a sum of r_i independent exponential random variables each with hazard rate λ_i, $i=1,\dots,n$. □

It consequently provides a simple lower bound for the variance of $\sum_{i=1}^{n} X_{\lambda_i}$,

$$\operatorname{var}\left(\sum_{i=1}^{n} X_{\lambda_i}\right) \geqslant \frac{\sum_{i=1}^{n} r_1}{(\bar{\lambda})^2},$$

where $\bar{\lambda} = \sum_{i=1}^{n} r_i\lambda_i / \sum_{i=1}^{n} r_i$, which is obtained by noting that $\boldsymbol{\lambda} \overset{m}{\geqslant} (\bar{\lambda},\dots,\bar{\lambda})$.

The Pareto distribution of the first kind with parameter λ,

$$F(x) = 1 - x^{-\lambda} \quad x \geqslant 1,$$

is widely used as an income distribution. By exploiting the relation between dispersive ordering and star ordering, we get the following result from Theorem 2.1.

Corollary 2.2. *Let $X_{\lambda_1},\dots,X_{\lambda_n}$ be independent random variables such that X_{λ_i} has Pareto distribution of the first kind with parameter λ_i, $i=1,\dots,n$. Then $\boldsymbol{\lambda} \overset{m}{\geqslant} \boldsymbol{\lambda}^*$ implies*

$$\prod_{i=1}^{n} X_{\lambda_i} \overset{*}{\geqslant} \prod_{i=1}^{n} X_{\lambda_i^*}.$$

Acknowledgements

This joint work was done while S. Kochar was visiting the University of British Columbia, who wishes to sincerely thank Professor Harry Joe for arranging this visit.

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