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# Connections among various variability orderings

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**Abstract**

Recently, a new variability ordering, called *right spread ordering* or *excess wealth* ordering has been introduced. This new ordering is weaker than dispersive ordering. We show in this note that if $X$ is less variable than $Y$ in the sense of right spread ordering or convex ordering, then it implies that $|X_1 - X_2|$ is less variable than $|Y_1 - Y_2|$ according to increasing convex ordering. Here $X_1$ and $X_2$ ($Y_1$ and $Y_2$) are two independent copies of $X$ ($Y$). An application of the right spread ordering in the study of spacings from an increasing mean residual life distribution is given. © 1997 Elsevier Science B.V.

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## 1. Introduction

Recently, Fernandez-Ponce et al. (1995) and independently, Shaked and Shanthikumar (1996) have introduced a new partial ordering called *right spread ordering* to compare two probability distributions in terms of their variability. In this note we further study the properties of this new ordering and develop some new connections among several variability orderings. First, we review some of the definitions.

Let $X$ and $Y$ be two random variables with distribution functions $F$ and $G$ and survival functions $\overline{F}$ and $\overline{G}$, respectively. The random variables are not necessarily restricted to be positive valued. Let $F^{-1}$ and $G^{-1}$ be the right continuous inverses of $F$ and $G$, defined by $F^{-1}(u) = \sup\{x : F(x) \leqslant u\}$ and $G^{-1}(u) = \sup\{x : G(x) \leqslant u\}$, $u \in [0, 1]$. Throughout this paper the term *increasing* is used for *monotone nondecreasing* and *decreasing* for *monotone nonincreasing*.

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**Definition 1.1.** $X$ is said to be stochastically smaller than $Y$ (denoted by $X \leqslant_{\mathrm{st}} Y$) if

$$E[\phi(X)] \leqslant E[\phi(Y)] \quad \text{for all increasing functions } \phi : \mathscr{R} \rightarrow \mathscr{R}, \tag{1.1}$$

for which the expectations exist.

**Definition 1.2.** $X$ is said to be smaller than $Y$ in the increasing convex order (denoted by $X \leqslant_{\mathrm{icx}} Y$) if

$$E[\phi(X)] \leqslant E[\phi(Y)] \quad \text{for all increasing convex functions } \phi : \mathscr{R} \rightarrow \mathscr{R}, \tag{1.2}$$

for which the expectations exist.

It is known (see Shaked and Shanthikumar, 1994, Section 3.A) that $X \leqslant_{\mathrm{icx}} Y$ if and only if

$$\int_x^\infty \overline{F}(u)\,\mathrm{d}u \leqslant \int_x^\infty \overline{G}(u)\,\mathrm{d}u, \tag{1.3}$$

for all $x$ for which the integrals exist.

**Definition 1.3.** $X$ is said to be smaller than $Y$ in the convex order (denoted by $X \leqslant_{\mathrm{cx}} Y$) if

$$E[\phi(X)] \leqslant E[\phi(Y)] \quad \text{for all convex functions } \phi : \mathscr{R} \rightarrow \mathscr{R}, \tag{1.4}$$

for which the expectations exist.

It is well known that $X$ is smaller than $Y$ in the usual convex order if and only if $E[X] = E[Y]$ and (1.3) holds.

**Definition 1.4.** $X$ is less dispersed than $Y$ ($X \overset{\mathrm{disp}}{\preccurlyeq} Y$) if

$$F^{-1}(v) - F^{-1}(u) \leqslant G^{-1}(v) - G^{-1}(u), \quad \forall 0 < u \leqslant v < 1. \tag{1.5}$$

This means that the difference between any two quantiles of $F$ is smaller than the difference between the corresponding quantiles of $G$. The above partial orderings of distributions have been extensively studied in the literature. See Shaked and Shanthikumar, 1994, Chs. 2 and 3 for details.

Muñoz-Perez (1990) has shown that $X \overset{\mathrm{disp}}{\preccurlyeq} Y$ if and only if the random variable $(X - F^{-1}(u))^+$ is stochastically smaller than the random variable $(Y - G^{-1}(u))^+$ for every $u \in (0,1)$, where $(Z)^+ = \max\{Z, 0\}$. Based on this observation, Fernandez-Ponce et al. (1995) proposed the following new variability ordering, which they call as *right spread ordering*,

**Definition 1.5.** $X$ is less right spread out than $Y$ ($X \overset{\mathrm{RS}}{\preccurlyeq} Y$) if

$$E[(X - F^{-1}(u))^+] \leqslant E[(Y - G^{-1}(u))^+], \ \forall u \in (0,1), \tag{1.6}$$

provided the expectations exist.

Or equivalently, if

$$\int_{F^{-1}(u)}^\infty \overline{F}(t)\,\mathrm{d}t \leqslant \int_{G^{-1}(u)}^\infty \overline{G}(t)\,\mathrm{d}t, \quad \forall u \in (0,1). \tag{1.7}$$

Since $F^{-1}(u)=\overline{F}^{-1}(1-u)$ and $G^{-1}(u)=\overline{G}^{-1}(1-u)$, we see that

$$X \overset{\text{RS}}{\leqslant} Y \Leftrightarrow \int_{\overline{F}^{-1}(u)}^{\infty} \overline{F}(t)\,\mathrm{d}t \leqslant \int_{\overline{G}^{-1}(u)}^{\infty} \overline{G}(t)\,\mathrm{d}t, \quad \forall u \in (0,1).$$

The reason for calling this ordering as the *right spread* (RS) ordering is that the function $E[(X-F^{-1}(u))^+]$ is known as the right spread function of $X$. Shaked and Shanthikumar (1996) call this ordering as the "excess wealth" ordering and write $X \overset{\text{ew}}{\leqslant} Y$ if (1.7) holds.

It is clear from the definition of RS ordering that it is weaker than dispersive ordering. The RS ordering has some nice properties and these have been discussed in Fernandez-Ponce et al. (1995) and Shaked and Shanthikumar (1996). For example, it is location-free in the sense that

$$X \overset{\text{RS}}{\leqslant} Y \Rightarrow X + c \overset{\text{RS}}{\leqslant} Y \quad \text{for any } c \in \mathscr{R}.$$

Also $X \overset{\text{RS}}{\leqslant} Y$ implies $\mathrm{var}(X) \leqslant \mathrm{var}(Y)$ as well as $E[|X_1-X_2|] \leqslant E[|Y_1-Y_2|]$, where $X_1$ and $X_2$ ($Y_1$ and $Y_2$) are two independent copies of $X$ ($Y$). Note that one can express $\mathrm{var}(X)=\frac{1}{2}E[|X_1-X_2|^2]$. This led us to consider whether the following holds:

$$X \overset{\text{RS}}{\leqslant} Y \Rightarrow |X_1-X_2| \leqslant_{\text{icx}} |Y_1-Y_2|. \tag{1.8}$$

We prove in this note that our conjecture (1.8) is, in fact, true. Another motivation for considering the above relationship is the following result on dispersive ordering (see Shaked and Shanthikumar, 1994, Theorem 2.B.16).

**Theorem 1.1.** *Let $X_1$ and $X_2$ ($Y_1$ and $Y_2$) be two independent copies of $X$ ($Y$), then*

$$X \overset{\text{disp}}{\leqslant} Y \Rightarrow |X_1-X_2| \leqslant_{\text{st}} |Y_1-Y_2|. \tag{1.9}$$

Note that dispersive ordering implies RS ordering while the usual stochastic order implies the increasing convex order, so that (1.8) seems natural compared to (1.9).

We shall be assuming throughout this note that all the random variables under consideration have finite expectations. We prove the various results in this note assuming that the random variables are absolutely continuous although we feel that some of the results may continue to hold without this restriction. The main results of this note are presented in Section 2. In Section 3 we give an interesting application of the RS ordering in the study of spacings from increasing mean residual life (IMRL) distributions.

## 2. The main results

First, we show that two random variables are equivalent in terms of RS ordering if and only if either they are identically distributed or they differ by a location parameter.

**Theorem 2.1.** $X \overset{\text{RS}}{=} Y \Leftrightarrow X \overset{\text{st}}{=} Y + c$ *for some real constant c.* (2.1)

**Proof.** Since the RS function is location-free, for any real $c$,

$$X \overset{\text{st}}{=} Y + c \Rightarrow X \overset{\text{RS}}{=} Y.$$

Conversely, suppose that $X \overset{\text{RS}}{=} Y$. That is, for all $u \in (0,1)$,

$$\int_{F^{-1}(u)}^{\infty} \overline{F}(t)\,\mathrm{d}t = \int_{G^{-1}(u)}^{\infty} \overline{G}(t)\,\mathrm{d}t. \tag{2.2}$$

Since we are assuming that the random variables under consideration are absolutely continuous, it follows that the quantile functions $F^{-1}$ and $G^{-1}$ are differentiable. Differentiating both sides of (2.2) with respect to $u$ and cancelling out the common factor $(1-u)$, we get

$$\frac{\mathrm{d}}{\mathrm{d}u}F^{-1}(u) = \frac{\mathrm{d}}{\mathrm{d}u}G^{-1}(u) \quad \text{for all } u \in (0,1).$$

The solution of this differential equation leads to

$$F^{-1}(u) = G^{-1}(u) + x_0 \quad \text{for some real } x_0 \quad \text{and} \quad \text{for all } u \text{ in } (0,1). \tag{2.3}$$

It is easy to see that (2.3) will hold if and only if $F$ and $G$ differ by a location parameter. Hence the result follows. □

To prove our conjecture (1.8), we shall need the following result on convex ordering which is also of independent interest.

**Lemma 2.1.** *Let the random variables $X$ and $Y$ be symmetric about the origin. Then*

$$X \leqslant_{\text{cx}} Y \Leftrightarrow |X| \leqslant_{\text{icx}} |Y|. \tag{2.4}$$

**Proof.** Suppose that $X \leqslant_{\text{cx}} Y$. Since the random variables $X$ and $Y$ are symmetric about the origin, the survival function of $|X|$ is

$$\overline{H}_{|X|}(x) = 2\overline{F}(x) \quad \text{for } x \geqslant 0, \tag{2.5}$$

and that of $|Y|$ is

$$\overline{H}_{|Y|}(x) = 2\overline{G}(x) \quad \text{for } x \geqslant 0. \tag{2.6}$$

It immediately follows from (2.5), (2.6) and (1.3) that $X \leqslant_{\text{cx}} Y$ implies $|X| \leqslant_{\text{icx}} |Y|$.

Conversely, suppose that $|X| \leqslant_{\text{icx}} |Y|$. That is,

$$\int_{x}^{\infty} \overline{F}(t)\,\mathrm{d}t \leqslant \int_{x}^{\infty} \overline{G}(t)\,\mathrm{d}t, \tag{2.7}$$

for $x \geqslant 0$.

To prove that $X \leqslant_{\text{cx}} Y$, it remains to show that (2.7) holds for negative values of $x$ also. Let $x<0$. Then $x=-y$ for some $y>0$. Now

$$\begin{aligned}\int_{-y}^{\infty} \overline{F}(t)\,\mathrm{d}t &= \int_{-y}^{0} \overline{F}(t)\,\mathrm{d}t + \int_{0}^{y} \overline{F}(t)\,\mathrm{d}t + \int_{y}^{\infty} \overline{F}(t)\,\mathrm{d}t \\ &= \int_{0}^{y} \overline{F}(-t)\,\mathrm{d}t + \int_{0}^{y} \overline{F}(t)\,\mathrm{d}t + \int_{y}^{\infty} \overline{F}(t)\,\mathrm{d}t\end{aligned}$$

(by a change of variables in the first integral)

$$= \int_0^y [F(t) + \overline{F}(t)]\,\mathrm{d}t + \int_y^\infty \overline{F}(t)\,\mathrm{d}t$$

$$= y + \int_y^\infty \overline{F}(t)\,\mathrm{d}t.$$

Therefore, for $y>0$,

$$\int_{-y}^\infty \overline{G}(t)\,\mathrm{d}t - \int_{-y}^\infty \overline{F}(t)\,\mathrm{d}t$$
$$= \int_y^\infty \overline{G}(t)\,\mathrm{d}t - \int_y^\infty \overline{F}(t)\,\mathrm{d}t$$
$$\geqslant 0 \tag{2.8}$$

by (2.7). This proves the required result. □

**Lemma 2.2.** *Let $X_1$ and $X_2$ ($Y_1$ and $Y_2$) be two independent copies of $X$ ($Y$), then*

$$X \leqslant_{\mathrm{cx}} Y \Rightarrow X_1 - X_2 \leqslant_{\mathrm{cx}} Y_1 - Y_2 \tag{2.9}$$

$$\Leftrightarrow |X_1 - X_2| \leqslant_{\mathrm{icx}} |Y_1 - Y_2|. \tag{2.10}$$

**Proof.** The proof of (2.9) follows by noting the closure of the convex ordering under convolution (see Shaked and Shanthikumar, 1994, Theorem 2.A.6(d)) and the fact that $X \leqslant_{\mathrm{cx}} Y$ if and only if $-X \leqslant_{\mathrm{cx}} -Y$ (see Shaked and Shanthikumar, 1994, Theorem 2.A.6(a)).

Since $X_1 - X_2$ and $Y_1 - Y_2$ are symmetric about the origin, the proof of (2.10) follows from the previous lemma. □

Shaked and Shanthikumar (1996) have studied the relationship between the RS ordering and the increasing convex ordering. Assuming that the left endpoints $l_X$ and $l_Y$ of the supports of $X$ and $Y$ are finite and equal, they prove that $X \overset{\mathrm{RS}}{\preccurlyeq} Y \Rightarrow X \leqslant_{\mathrm{icx}} Y$. Such restrictions are needed because the increasing convex ordering does not own the location-free property whereas the right spread ordering does. Their proof is quite involved and lengthy based on geometric considerations. We give below an alternative short proof of the above result.

**Theorem 2.2.** *Let $X$ and $Y$ be two random variables with finite means and with 0 as the common left endpoint of their supports. If $X \overset{\mathrm{RS}}{\preccurlyeq} Y$, then $X \leqslant_{\mathrm{icx}} Y$.*

**Proof.** Let $F$ and $G$ be the distribution functions of $X$ and $Y$, respectively. Let $R_F(x) = \int_x^\infty \overline{F}(t)\,\mathrm{d}t$, $R_G(x) = \int_x^\infty \overline{G}(t)\,\mathrm{d}t$, $\beta(x) = R_G^{-1} R_F(x)$ and $\alpha(x) = G^{-1}F(x)$.

Note that

$$\beta'(x) = \frac{\overline{F}(x)}{\overline{G}(R_G^{-1}R_F(x))} = \frac{\overline{G}(\alpha(x))}{\overline{G}(\beta(x))}. \tag{2.11}$$

Now

$$\begin{aligned} X \overset{\text{RS}}{\leqslant} Y &\Leftrightarrow R_F(x) \leqslant R_G(G^{-1}F(x)) \quad \text{for } x \geqslant 0 \\ &\Leftrightarrow R_G^{-1}R_F(x) \geqslant G^{-1}F(x) \quad \text{for } x \geqslant 0 \text{ as } R_G^{-1} \text{ is nonincreasing} \\ &\Leftrightarrow \overline{G}(\beta(x)) \leqslant \overline{G}(\alpha(x)) \quad \text{for } x \geqslant 0 \text{ as } \overline{G} \text{ is nonincreasing} \\ &\Leftrightarrow \frac{\overline{G}(\alpha(x))}{\overline{G}(\beta(x))} \geqslant 1 \quad \text{for } x \geqslant 0 \\ &\Leftrightarrow \beta'(x) \geqslant 1 \quad \text{for } x \geqslant 0. \end{aligned} \tag{2.12}$$

Integrating both sides of (2.12) with respect to $x$ from 0 to $y$, we find that $X \overset{\text{RS}}{\leqslant} Y$ implies $\beta(y) \geqslant y + \beta(0) \geqslant y$, as $\beta(0) = R_G^{-1}R_F(0) = R_G^{-1}(\mu_F) \geqslant R_G^{-1}(\mu_G) = 0$, since $R_G$ is nonincreasing and $X \overset{\text{RS}}{\leqslant} Y$ implies $\mu_F \leqslant \mu_G$, where $\mu_F$ and $\mu_G$ are the means of $X$ and $Y$, respectively.

Hence $X \overset{\text{RS}}{\leqslant} Y$ implies $\beta(y) \geqslant y$ for $y \geqslant 0$. That is, $R_F(y) \leqslant R_G(y)$, $y \geqslant 0$. Or, $X \leqslant_{\text{icx}} Y$. □

We are now ready to prove (1.8). We do assume in the next theorem that the left endpoints of the supports of the random variables are finite, but they need not be equal.

**Theorem 2.3.** *Let $X$ and $Y$ be two random variables with finite means and with finite left endpoints of their supports. Then*

$$\begin{aligned} X \overset{\text{RS}}{\leqslant} Y &\Rightarrow X_1 - X_2 \leqslant_{\text{cx}} Y_1 - Y_2 \\ &\Leftrightarrow |X_1 - X_2| \leqslant_{\text{icx}} |Y_1 - Y_2|, \end{aligned}$$

*where $X_1$ and $X_2$ ($Y_1$ and $Y_2$) are two independent copies of $X$ ($Y$).*

**Proof.** Let $l_X$ and $l_Y$ denote the finite left endpoints of the supports of $X$ and $Y$, respectively. We can write

$$X_1 - X_2 = X_1^\star - X_2^\star \quad \text{and} \quad Y_1 - Y_2 = Y_1^\star - Y_2^\star,$$

where

$$X_i^\star = X_i - l_X \quad \text{and} \quad Y_i^\star = Y_i - l_Y, \quad i = 1, 2.$$

Since RS ordering is shift invariant,

$$X \overset{\text{RS}}{\leqslant} Y \Leftrightarrow X^\star \overset{\text{RS}}{\leqslant} Y^\star. \tag{2.13}$$

Also note that the left endpoints of the supports of $X^\star$ and $Y^\star$ are equal to 0. The required result now follows from Theorem 2.2 and Lemma 2.2. □

## 3. Application

Let $X$ be a nonnegative random variable with finite mean and with mean residual life function $\mu_F(t) \equiv E[X - t \mid X > t] = \int_t^\infty \overline{F}(x)\,\mathrm{d}x / \overline{F}(t)$. We say that $F$ has the *increasing mean residual life* (IMRL) property if $\mu_F(t)$ is nondecreasing in $t \geqslant 0$.

Let $Y$ be another nonnegative random variable with mean residual life function $\mu_G(t)$. We say that $X$ *is smaller than* $Y$ *in the mean residual life order* ($X \leqslant_{\mathrm{mrl}} Y$) if

$$\mu_F(t) \leqslant \mu_G(t) \quad \text{for all } t. \tag{3.1}$$

Shaked and Shanthikumar (1996) have established the following relation between mean residual life ordering and RS ordering.

**Theorem 3.1.** *Let $X$ and $Y$ be nonnegative random variables with finite means and with 0 as the common left end point of their supports. If $X \leqslant_{\mathrm{mrl}} Y$ and if either $X$ or $Y$ or both are IMRL, then $X \overset{\mathrm{RS}}{\preccurlyeq} Y$.*

A variant of this result is also given in Fernandez-Ponce et al. (1995). In this section we discuss an interesting application of the above result in the study of sample spacings. Let $X_1,\ldots,X_n$ be a random sample from a continuous distribution $F$ and let $X_{1:n} \leqslant \cdots \leqslant X_{n:n}$ be the corresponding order statistics. The random variables $D_{i:n} = X_{i:n} - X_{i-1:n}$, $i = 1,\ldots,n$; $X_{0:n} \equiv 0$; are called the sample spacings from the distribution $F$.

**Theorem 3.2.** *Let $X_1,\ldots,X_n$ be a random sample from an absolutely continuous IMRL distribution $F$. Then*

$$D_{i:n} \overset{\mathrm{RS}}{\preccurlyeq} D_{i+1:n} \quad \text{for } i = 1,\ldots,n-1. \tag{3.2}$$

**Proof.** Kirmani (1996) has proved that if $F$ is IMRL, then each $D_{i:n}$ is IMRL and $D_{i:n} \leqslant_{\mathrm{mrl}} D_{i+1:n}$ for $i = 1,\ldots,n-1$. The required proof then follows from Theorem 3.1. □

A consequence of this result is that $\mathrm{var}(D_{i:n}) \leqslant \mathrm{var}(D_{i+1:n})$, for $i = 1,\ldots,n-1$, a result proved in Kirmani (1996) also. But (3.2) is a much stronger result.

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## Note added in proof

After this manuscript had gone to the press, it was discovered that the Lemma 1(a) of Kirmani (1996) is incorrect. This may invalidate some of his other results in that paper and also Theorem 3.2 above.