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Dispersive ordering of order statistics

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Abstract

Let $X_1,\ldots,X_n$ be a random sample of size n from a DFR distribution and let $X_{(i:n)}$ denote its ith order statistic. It is shown that for $i<j$, $X_{(i:n)}$ is *less dispersed* than $X_{(j:n)}$. Also, if X_i's are independent DFR random variables, but not necessarily identical, then $X_{(1:n)}$ is less dispersed than $X_{(j:n)}$, for $j>1$.

Keywords: Hazard rate ordering; Likelihood ratio ordering; k-out-of-n system; DFR distribution

1. Introduction

k-out-of-n systems play an important role in reliability theory and these have been discussed extensively in the literature. The system consisting of n components works as long as at least k components are working. The parallel and series systems are 1-out-of-n and n-out-of-n systems, respectively. The survival function of a k-out-of-n system is the same as that of the $(n-k+1)$th order statistic $X_{(n-k+1:n)}$ of a set of n random variables. Thus, the study of k-out-of-n systems is equivalent to the study of order statistics. Needless to say, it is important and also of interest to compare the stochastic properties of order statistics. In the literature, many interesting results have been obtained in this area. The recent book by Shaked and Shanthikumar (1994) is a good reference for the various kinds of stochastic orders and their inter relationships.

In this paper, we shall be assuming that all the random variables under consideration are nonnegative and with their underlying distribution functions strictly increasing on their supports. We shall use "increasing" ("decreasing") to mean "nondecreasing" ("nonincreasing").

While designing systems from components, engineers are very much concerned about the variability in the lifetimes of their products. There are many criteria for comparing the variability or dispersion in two probability distributions. One of them is the partial ordering known as *dispersive ordering*. Let X and Y be two random variables with distribution functions F_X and F_Y, respectively. Let F_X^{-1} and F_Y^{-1} be their right continuous inverses (quantile functions). We say that X is less *dispersed* than Y ($X \overset{\text{disp}}{\preccurlyeq} Y$) if $F_X^{-1}(\beta) - F_X^{-1}(\alpha) \leqslant F_Y^{-1}(\beta) - F_Y^{-1}(\alpha)$, for all $0 \leqslant \alpha \leqslant \beta \leqslant 1$. It requires the difference of any two quantiles of X to be smaller than the difference of the corresponding quantiles of Y for $X \overset{\text{disp}}{\preccurlyeq} Y$. An equivalent definition of $X \overset{\text{disp}}{\preccurlyeq} Y$ is to require that $F_Y^{-1}F_X(x)-x$ to be increasing in x. Let r_X and r_Y denote the hazard (failure) rates of X and Y, respectively. Then it can be seen that $X \overset{\text{disp}}{\preccurlyeq} Y \Leftrightarrow r_X(F_X^{-1}(u)) \geqslant r_Y(F_Y^{-1}(u))$ for all $0 \leqslant u \leqslant 1$. A consequence

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of $X \overset{\text{disp}}{\preccurlyeq} Y$ is that $\text{var}(X) \leqslant \text{var}(Y)$. Also $X \overset{\text{disp}}{\preccurlyeq} Y$ implies $E[|X_1 - X_2|] \leqslant E[|Y_1 - Y_2|]$, where $X_1, X_2(Y_1, Y_2)$ are two independent copies of $X(Y)$.

In the next section, we show that if $X_1, \ldots, X_n$ is a random sample from a decreasing failure rate (DFR) distribution, then for $i < j$, $X_{(i:n)} \overset{\text{disp}}{\preccurlyeq} X_{(j:n)}$. In Section 3, we consider the case of nonidentical independent DFR random variables and show that in this case a series system of n components is less dispersed than any other k-out-of-n system of such components.

2. The case of iid DFR random variables

It is easy to prove that if $X_1, \ldots, X_n$ is a random sample from any arbitrary distribution, then for $i < j$, $X_{(i:n)}$ is smaller than $X_{(j:n)}$ according to likelihood ratio ordering sense. It is a very strong type of ordering and it implies that the hazard rate of $X_{(j:n)}$ is uniformly smaller than that of $X_{(i:n)}$.

Boland et al. (1995) proved the following result on dispersive ordering between order statistics from an exponential distribution.

Lemma 2.1. *Let $X_1, \ldots, X_n$ be a random sample from an exponential distribution. Then for $i < j$,*

$$X_{(i:n)} \overset{\text{disp}}{\preccurlyeq} X_{(j:n)}.$$

We shall need the following result (see Bartoszewicz, 1987) to extend Lemma 2.1 from exponential distributions to DFR distributions.

Lemma 2.2. *Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that $\phi(0) = 0$ and $\phi(x) - x$ is increasing. Then for every convex and strictly increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ the function $\psi \phi \psi^{-1}(x) - x$ is increasing.*

Now we prove the main result of this paper.

Theorem 2.1. *Let $X_1, \ldots, X_n$ be a random sample from a DFR distribution. Then for $i < j$,*

$$X_{(i:n)} \overset{\text{disp}}{\preccurlyeq} X_{(j:n)}.$$

Proof. Let F denote the distribution function of the identically distributed X's. Then the distribution function of $X_{(j:n)}$ is $F_{(j:n)}(x) = B_{(j:n)}F(x)$ for every x, where $B_{(j:n)}$ is the distribution function of the beta distribution with parameters $(j, n-j+1)$.

Let G denote the distribution function of a unit mean exponential random variable. Then $H_{(j:n)}(x) = B_{(j:n)}G(x)$ is the distribution function of the jth-order statistic in a random sample of size n from a unit mean exponential distribution. We can express $F_{(j:n)}$ as

$$F_{(j:n)}(x) = B_{(j:n)}GG^{-1}F(x) \tag{2.1}$$

$$= H_{(j:n)}G^{-1}F(x). \tag{2.2}$$

To prove our result, we have to show that for $i < j$,

$$F^{-1}_{(j:n)}F_{(i:n)}(x) - x \quad \text{is increasing in } x$$

$$\Leftrightarrow F^{-1}GH^{-1}_{(j:n)}H_{(i:n)}G^{-1}F(x) - x \quad \text{is increasing in } x. \tag{2.3}$$

By Lemma 2.1, $H^{-1}_{(j:n)}H_{(i:n)}(x)-x$ is increasing in x. Also the function $\psi(x) = F^{-1}G(x)$ is strictly increasing and convex if F is DFR. The required result now follows from Lemma 2.2. □

The DFR assumption is very crucial for the above result to hold. Boland et al. (1995) have shown that in the case of a random sample of size 2 from uniform distribution over [0, 1], which is not DFR, $X_{(1:2)}$ is not less dispersed than $X_{(2:2)}$.

3. The case of independent but non identically distributed DFR random variables

Now we consider the problem of comparing order statistics when the parent observations are independent but not necessarily identically distributed.

Boland et al. (1994) have shown that if $X_1,\ldots,X_n$ are independent random variables, then for $i<j$, $X_{(j:n)}$ has smaller hazard (failure) rate than $X_{(i:n)}$. We denote this relation by $X_{(i:n)} \overset{\text{hr}}{\preccurlyeq} X_{(j:n)}$. Note that a random variable with smaller hazard rate survives longer *stochastically* than one with larger hazard rate. Bapat and Kochar (1994) strengthened the above result from hazard rate ordering to *likelihood ratio* ordering provided the parent random variables are themselves likelihood ratio ordered.

To prove our main result on dispersive ordering between two order statistics from a heterogeneous sample, we shall use the following lemma of Bagai and Kochar (1986).

Lemma 3.1. *Let X and Y be two nonnegative random variables.*

If $X \overset{\text{hr}}{\preccurlyeq} Y$ *and X or Y is DFR, then* $X \overset{\text{disp}}{\preccurlyeq} Y$.

Using the above result of Boland et al. (1994) together with Lemma 3.1, we get the following theorem.

Theorem 3.1. *Let* $X_1,\ldots,X_n$ *be independent nonnegative random variables; then for* $i<j$,

$$X_{(i:n)} \overset{\text{disp}}{\preccurlyeq} X_{(j:n)} \quad \text{provided } X_{(i:n)} \text{ is DFR.}$$

Even if we sample from a DFR distribution, it may not be true that $X_{(i:n)}$ is DFR for arbitrary i. But the smallest order statistics $X_{(1:n)}$ will always be DFR in this case. This follows from the fact that the hazard rate of a series system of independent components is the sum of the hazard rates of the components. So if each component of the series system has decreasing hazard (failure) rate, the system will have DFR property. This leads us to the following result.

Corollary 3.1. *Let* $X_1,\ldots,X_n$ *be independent DFR random variables, then for* $j>1$,

$$X_{(1:n)} \overset{\text{disp}}{\preccurlyeq} X_{(j:n)}.$$

It follows from the above discussion that if we make k-out-of n systems out of n independent DFR components, then among them the series system will be least dispersed but with greatest hazard rate.

We end this section with the following result on dispersive ordering between series systems of independent DFR components based on different number of components.

Theorem 3.2. *Let* $X_1,\ldots,X_{n+1}$ *be independent DFR random variables. Then*

$$X_{(1:n+1)} \overset{\text{disp}}{\preccurlyeq} X_{(1:n)}.$$

Proof. Since the hazard rate of $X_{(1:n)}$ is smaller than that of $X_{(1:n+1)}$,

$$X_{(1:n+1)} \overset{\text{hr}}{\leqslant} X_{(1:n)}.$$

The required result follows from Lemma 3.1 since $X_{(1:n)}$ has DFR distribution under the assumed conditions. □

For stochastic relations between normalized spacings from DFR distributions, see Kochar and Kirmani (1995).

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