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SOME RESULTS ON WEIGHTED DISTRIBUTIONS FOR POSITIVE-VALUED RANDOM VARIABLES

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For nonnegative random variables, the weighted distributions have been compared with the original distributions with the help of partial orderings of probability distributions. Bounds on the moments of the weighted distributions have been obtained in terms of the moments of the original distributions for some nonparametric classes of aging distributions.

1. INTRODUCTION

The concepts of size-biased sampling and weighted distributions pertaining to observational studies and surveys of research related to forestry, ecology, bio-medicine, reliability, and several other areas have been widely studied in the literature. Rao [12] identified various situations that can be modeled by weighted distributions. These situations refer to instances where the recorded observations cannot be considered as a random sample from the original distribution. This may happen due to non-observability of some events or a dam-

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age caused to original observation resulting in a reduced value, or adoption of a sampling plan which gives unequal probabilities to the various units (see Patil and Rao [10,11] for details).

We are mainly interested in the original distributions and several questions do arise as to how the weight function affects the original distribution.

In this paper, we study the properties of the weighted distributions in comparison with those of the original distributions for positive-valued random variables. Such random variables and distributions arise naturally in life testing, reliability, and economics. For examples, see Blumenthal [3], Cox [4], Schaeffer [14], Mahfoud and Patil [9], and Gupta [7] and references therein.

In Section 2, we discuss some partial orderings of probability distributions for positive-valued random variables and use these results for comparing the univariate original and weighted distributions in the third section.

2. NOTATIONS, TERMINOLOGY, AND PRELIMINARIES

Let X be a nonnegative random variable with probability density function $f(x)$, distribution function $F(x)$, survival function $\bar{F}(x) = 1 - F(x)$, and mean μ_F. Let $\mathfrak{F}$ denote the class of all absolutely continuous distributions with $F(x) = 0$ for $x \le 0$. For $F \in \mathfrak{F}$, we have the following definitions:

i. The failure rate (or hazard rate) of X is defined as

$$r_F(x) = f(x)/\bar{F}(x), \qquad \text{for } x \ge 0 \text{ such that } \bar{F}(x) > 0.$$

ii. The mean residual life function (MRLF) of X is defined as

$$\begin{aligned} \mu_F(x) &= E(X - x \mid X > x) \\ &= \frac{1}{\bar{F}(x)} \int_x^\infty \bar{F}(t)\, dt, \qquad \text{for } x \ge 0 \text{ such that } \bar{F}(x) > 0. \end{aligned}$$

Clearly, $\mu_F(0) = \mu_F$. There is a one-to-one correspondence between $\mu_F(x)$, $r_F(x)$, and $F(x)$. Each one of these determines the other.

iii. F is said to be an increasing failure rate (IFR) distribution if $r_F(x)$ is nondecreasing in x.

iv. F is said to be a decreasing mean residual life (DMRL) distribution if $\mu_F(x)$ is nonincreasing in x.

v. F is said to be a harmonic new better than used in expectation (HNBUE) distribution if

$$\int_x^\infty \bar{F}(u)\, du \le \mu_F e^{-x/\mu_F}, \qquad \text{for } x \ge 0.$$

The duals of these are DFR, IMRL, and HNWUE and are defined analogously. For details, see Hollander and Proschan [8] and Deshpande, Kochar and Singh [6].

Some Partial Orderings of Univariate Distribution

Now we discuss some partial orderings for the pair of distributions, F and $G \in \mathfrak{F}$.

a. *Stochastic Ordering:* F is said to be stochastically ordered (smaller) with respect to $G (F \overset{st}{<} G)$ if

   $G(x) \leq F(x)$, for every x and with a strict inequality for some x.

b. *Tail Ordering (or Dispersive Ordering):* F is said to be tail-ordered with respect to $G (F \overset{t}{<} G)$ if

$$G^{-1}F(x) - x \text{ is nondecreasing in } x$$

   or, equivalently, if

$$G^{-1}(u) - G^{-1}(v) \geq F^{-1}(u) - F^{-1}(v), \text{ for } 1 \geq u \geq v \geq 0.$$

c. *Failure Rate Ordering:* F is said to be ordered with respect to G in the failure rate sense $(F \overset{r}{<} G)$ if

$$r_G(x) \leq r_F(x), \qquad \text{for every } x \geq 0$$

   or, equivalently, if

$$\bar{G}(x)/\bar{F}(x) \text{ is nondecreasing in } x.$$

d. *Likelihood Ratio Ordering:* F is said to be likelihood ratio ordered with respect to $G (F \overset{L.R.}{<} G)$ if

$$g(x)/f(x) \text{ is nondecreasing in } x.$$

We have the following chain of implications between the various partial orderings discussed above

$$\begin{array}{c} \qquad\qquad\qquad\qquad F \overset{t}{<} G \\ \qquad\qquad\qquad\qquad \Downarrow \\ F \overset{L.R.}{<} G \Rightarrow F \overset{r}{<} G \Rightarrow F \overset{st}{<} G. \end{array}$$

3. COMPARISON OF THE WEIGHTED DISTRIBUTION WITH THE ORIGINAL ONE

Consider a random variable X with distribution function F belonging to $\mathfrak{F}$. Let $W(x)$ be a nonnegative weight function. Denote a new probability density function

$$f_w(x) = \frac{w(x)f(x)}{E[w(X)]}, \tag{3.1}$$

and the corresponding random variable by x^w, which is called the weighted random variable corresponding to X.

When $w(x) = x^\alpha$, $\alpha > 0$, we say that X^w is size-biased of order α. Such a selection procedure is called size-biased sampling of order α. When $\alpha = 1$, X^w is simply called size-biased (or length-biased) and has probability density function

$$f_w(x) = \frac{xf(x)}{\mu_F}. \tag{3.2}$$

Gupta [7] has obtained some relations between the reliability measures of the original distribution and those of the length-biased distribution. In this paper, we generalize his results to more general weight functions.

THEOREM 3.1: *Let*

$$\begin{aligned} A(x) &= E[w(X) - w(x) | X > x] \\ &= \frac{1}{\bar{F}(x)} \int_x^\infty [w(t) - w(x)] f(t)\, dt. \end{aligned} \tag{3.3}$$

(i) *Then the survival function of* X^w *is*

$$\bar{F}_w(x) = \frac{\bar{F}(x)[w(x) + A(x)]}{E[w(X)]}. \tag{3.4}$$

(ii) *The failure rate of* X^w *is*

$$r_{F_w}(x) = \frac{w(x) r_F(x)}{w(x) + A(x)}. \tag{3.5}$$

PROOF. (i)

$$\begin{aligned} \bar{F}_w(x) &= \frac{1}{E[w(X)]} \int_x^\infty w(t) f(t)\, dt \\ &= \frac{1}{E[w(X)]} \left[\int_x^\infty \{w(t) - w(x) + w(x)\} f(t)\, dt \right] \\ &= \frac{1}{E[w(X)]} [A(x)\bar{F}(x) + w(x)\bar{F}(x)] \\ &= \frac{\bar{F}(x)}{E[w(X)]} [w(x) + A(x)]. \end{aligned}$$

(ii) From Eqs. (3.1) and (3.4)

$$r_{F_w}(x) = \frac{f_w(x)}{\bar{F}_w(x)} = \frac{w(x) r_F(x)}{w(x) + A(x)}. \qquad \blacksquare$$

THEOREM 3.2: *Let* $w(x)$ *be monotonically increasing, concave and differentiable. Then*

(i) *$A(x)$ is monotonically decreasing if F is* DMRL.
(ii) *The ratio of the failure rates $r_{F_w}(x)/r_F(x)$ is monotonically increasing if F is* DMRL.
(iii) *F_w is* IFR *if F is* IFR.

PROOF. (i) Integrating Eq. (3.3) by parts, gives

$$A(x) = \int_x^\infty \frac{\bar{F}(t)w'(t)\,dt}{\bar{F}(x)}. \tag{3.6}$$

It can be seen that F is DMRL, if and only if

$$\frac{\bar{F}(x)}{f(x)} \geq \int_x^\infty \frac{\bar{F}(t)\,dt}{\bar{F}(x)}, \qquad \forall x > 0 \text{ such that } \bar{F}(x) \neq 0. \tag{3.7}$$

On differentiating Eq. (3.6), we find that $A(x)$ is monotonically decreasing, if and only if

$$\frac{\bar{F}(x)}{f(x)} \geq \int_x^\infty \frac{w'(t)\bar{F}(t)\,dt}{w'(x)\bar{F}(x)}, \tag{3.8}$$

for all $x \geq 0$ such that $w'(x)F(x) \neq 0$. Since $w(u)$ is concave and differentiable, it follows that

$$\frac{w'(t)}{w'(x)} \leq 1, \qquad \text{for } x \leq t.$$

Hence, if F is DMRL, then

$$\begin{aligned}\frac{\bar{F}(x)}{f(x)} &\geq \int_x^\infty \frac{\bar{F}(t)\,dt}{\bar{F}(x)} \\ &\geq \int_x^\infty \frac{w'(t)\bar{F}(t)\,dt}{w'(x)\bar{F}(x)}.\end{aligned}$$

(ii)

$$\begin{aligned}\frac{r_{F_w}(x)}{r_F(x)} &= \frac{w(x)}{w(x) + A(x)}, \\ &= \frac{1}{1 + \dfrac{A(x)}{w(x)}},\end{aligned}$$

which is monotonically increasing since $A(x)$ is monotonically decreasing when F is DMRL and $w(x)$ is given to be monotonically increasing.

(iii) Equation (3.5) can be written as

$$r_{F_w}(x) = \frac{r_F(x)}{1 + \dfrac{A(w)}{w(x)}}.$$

Since IRF ⇒ DMRL ⇒ $A(x)$ nonincreasing under the conditions of the theorem, it follows that F_w will be IFR if F is IFR. ■

THEOREM 3.3: *If the weight function $w(x)$ is monotonically increasing, then*

(i) $$F \overset{L.R.}{<} F_w,$$

(ii) $$r_{F_w}(x) \leq r_F(x), \qquad \text{for } x \geq 0,$$

(iii) $$X \overset{st}{\leq} X_w,$$

(iv) $$E(X) \leq E(X_w).$$

PROOF. The proof follows immediately by observing that the likelihood ratio

$$\frac{f_w(x)}{f(x)} = \frac{w(x)}{E[w(X)]}$$

is monotonically increasing under the given conditions. The rest of the results follow immediately from the various implications between the partial orderings as discussed in Section 2. ■

THEOREM 3.4: *Let $w(x)$ be monotonically increasing and F be* DFR, *then* $F \overset{t}{<} F_w$ *and as a consequence* $\operatorname{var}(X) \leq \operatorname{var}(X_w)$.

PROOF. It follows from Theorem 2.1(b) of Bagai and Kochar [1] that $F \overset{t}{<} F_w$ since F is DFR and $r_{F_w}(x) \leq r_F(x)$ when $w(x)$ is monotonically increasing. The variance inequality follows from the remark following Theorem 2.1 of Shaked [15].

THEOREM 3.5: *In case of length-biased sampling,* $w(x) = x$,

$$\begin{aligned} E[X^w] &\leq 2E(X), \qquad \text{if } F \text{ is HNBUE}, \\ &\geq 2E(X), \qquad \text{if } F \text{ is HNWUE}. \end{aligned}$$

PROOF. From Eq. (3.1), we have

$$\begin{aligned} E[X^w] &= \frac{E[Xw(X)]}{E[w(X)]} \\ &= \frac{E[X^2]}{E[X]}, \qquad \text{since } w(x) = x \\ &= E(X) + \frac{\operatorname{var}(X)}{E(X)} \\ &= E(X) + \frac{\operatorname{var}(X)}{E^2(X)} \cdot E(X). \end{aligned}$$

Now since the coefficient of variation

$$\frac{\operatorname{var}(X)}{E^2(X)} \le 1, \qquad \text{if } F \text{ is HNBUE,}$$

$$\ge 1, \qquad \text{if } F \text{ is HNWUE,}$$

the required result follows. ■

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