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On extensions of DMRL & related partial orderings.
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Partial Orderings of Life Distributions with Respect to their Aging Properties

Subhash C. Kochar*
Department of Statistics, Panjab University, Chandigarh, India

Douglas P. Wiens†
Department of Mathematics, Statistics and Computing Science,
Dalhousie University, Halifax, Nova Scotia, Canada B3H 3J5

New partial orderings of life distributions are given. The concepts of decreasing mean residual life, new better than used in expectation, harmonic new better than used in expectation, new better than used in failure rate, and new better than used in failure rate average are generalized, so as to compare the aging properties of two arbitrary life distributions.

1. INTRODUCTION

By the aging of a mechanical unit, component, or some other physical or biological system, we mean the phenomenon by which an older system has a shorter remaining lifetime, in some stochastic sense, than a newer or younger one. Many criteria of aging have been developed in the literature. See, for example, Bryson and Siddiqui [7], Barlow and Proschan [4], Klefsjö [2], and Hollander and Proschan [11].

Let X be a random variable representing the lifetime of a system. Let F be the cumulative distribution function of X, $\overline{F} = 1 - F$ the survival function, and μ_F the mean lifetime. We assume throughout that all distributions being considered have finite means, and are strictly increasing on their supports. In addition, if F is absolutely continuous, with density f, then the failure rate, or hazard rate, is defined by $r_F(x) = f(x)/\overline{F}(x), 0 \le x < \overline{F}^{-1}(0)$. The mean residual life function is

$$\mu_F(x) = E[X - x | X > x] = \int_x^\infty \overline{F}(t)\, dt / \overline{F}(x)$$

with $\mu_F(0) = \mu_F$.

We briefly discuss below some of the well-known criteria of aging.

(i) F is said to be an *increasing failure rate* (IFR) distribution if $-\log \overline{F}(x)$ is convex. If the density exists, this is equivalent to saying that $r_F(x)$ is nondecreasing.

(ii) F is said to be an *increasing failure rate average* (IFRA) distribution if $-\log \bar{F}(x)$ is a star-shaped function, i.e., if $-\log \bar{F}(\lambda x) \leq -\lambda \log \bar{F}(x)$ for $0 \leq \lambda \leq 1$ and $x \geq 0$. When the failure rate exists, this is equivalent to saying that $\int_0^x r_F(t)\, dt/x$ is nondecreasing.

(iii) F is said to be a *new better than used* (NBU) distribution if $-\log \bar{F}(x)$ is superadditive, i.e., if $-\log \bar{F}(x + y) \geq -\log \bar{F}(x) - \log \bar{F}(y)$; $x, y \geq 0$. This is equivalent to the statement $P[X > x + y | X > x] \leq P[X > y]$.

(iv) F is said to be a *decreasing mean residual life* (DMRL) distribution if $\mu_F(x)$ is nonincreasing.

(v) F is said to be a *new better than used in expectation* (NBUE) distribution if $\mu_F(x) \leq \mu_F(0)$, $x \geq 0$.

(vi) F is said to be a *harmonic new better than used in expectation* (HNBUE) distribution if $\int_x^\infty \bar{F}(t)\, dt \leq \mu_F \exp(-x/\mu_F)$, $x \geq 0$.

(vii)   If F is absolutely continuous, with failure rate $r_F(x)$, we say that F is a *new better than used in failure rate* (NBUFR) distribution if $r_F(x) \geq r_F(0)$, $x \geq 0$.

(viii)   If F is absolutely continuous, we say that F is a *new better than used in failure rate average* distribution if

$$r_F(0) \leq \frac{1}{x} \int_0^x r_F(t)\, dt = \frac{-\log \bar{F}(x)}{x}.$$

These last two criteria of aging have been discussed in Loh [14] and Deshpandé, Kochar, and Singh [9]. In the last reference, a unified theory of positive aging has been discussed in terms of various types of stochastic dominances.

The following implications, and no others, hold between the criteria discussed above.

$$\begin{array}{ccccccc}
\text{IFR} & \Rightarrow & \text{IFRA} & \Rightarrow & \text{NBU} & \Longrightarrow & \text{NBUFR} \Rightarrow \text{NBUFRA} \\
\Downarrow & & & & \Downarrow & & \\
\text{DMRL} & & \Longrightarrow & & \text{NBUE} & \Rightarrow & \text{HNBUE}
\end{array} \tag{1}$$

In all these criteria, we are essentially comparing F with the negative exponential distribution, whose memoryless property ensures that it belongs to each age class, with equality in each defining inequality. The IFR, IFRA, and NBU criteria have previously been generalized in the literature to the comparison of two arbitrary life distributions.

We say that F is more IFR than G ($F \overset{\text{IFR}}{<} G$, also written $F \overset{c}{<} G$ in the literature) if $G^{-1} \circ F(x)$ is convex. If the failure rates exist, an equivalent formulation is

$$\frac{r_F(F^{-1}(u))}{r_G(G^{-1}(u))} \text{ is nondecreasing in } u \in [0,1]. \tag{2}$$

We say that F is more IFRA than G ($F \overset{\text{IFRA}}{<} G$, also written $F \overset{*}{<} G$) if $G^{-1} \circ F(x)$ is star shaped. We say that F is more NBU than G ($F \overset{\text{NBU}}{<} G$, or $F \overset{su}{<} G$) if $G^{-1} \circ F(x)$ is superadditive.

For properties of the IFR ordering, see Bickel and Doksum [6], Barlow and Doksum [3], and Barlow and Proschan [4]. For the IFRA ordering, see Doksum [10], Shaked [16], Deshpandé and Kochar [8], Sathé [15], and Bartoszewicz [5].

For the NBU ordering see Hollander and Proschan [11], Barlow and Proschan [4], and Ahmed, Alzaid, Bartoszewicz, and Kochar [1].

All of the above orderings are scale invariant, and have the property that if $E(x) = 1 - e^{-x}$ is the negative exponential distribution, then

$$F \overset{\mathcal{P}}{<} E \text{ iff } F \text{ has aging property } \mathcal{P} \qquad (3)$$

for $\mathcal{P} \in \{\text{IFR,IFRA,NBU}\}$. As well, the implications

$$F \overset{\text{IFR}}{<} G \Rightarrow F \overset{\text{IFRA}}{<} G \Rightarrow F \overset{\text{NBU}}{<} G$$

are maintained.

In this article a similar program is carried out for the remaining five aging criteria discussed above. In each case (3) holds. With one exception—from NBU to NBUE—the implications (1) are also maintained.

2. THE DMRL ORDERING

Denote by $\mathcal{F}$ the class of distribution functions F on $[0,\infty)$, with $F(0) = 0$. Let $F,G \in \mathcal{F}$ have mean residual life functions $\mu_F(x)$, $\mu_G(x)$ and equilibrium survival functions

$$\overline{F}_e(x) = \int_x^\infty \frac{\overline{F}(t)}{\mu_F} \, dt, \qquad \overline{G}_e(x) = \int_x^\infty \frac{\overline{G}(t)}{\mu_G} \, dt.$$

Let $\tilde{r}_F(x) = \overline{F}(x)/\mu_F \overline{F}_e(x)$, $\tilde{r}_G(x) = \overline{G}(x)/\mu_G \overline{G}_e(x)$ be the hazard rates of F_e and G_e, and note that $\tilde{r}_F(x) = (\mu_F(x))^{-1}$, $\tilde{r}_G(x) = (\mu_G(x))^{-1}$. Define

$$W_F(u) = \overline{F} \circ \overline{F}_e^{-1}(u), \qquad W_G(u) = \overline{G} \circ \overline{G}_e^{-1}(u), \qquad 0 \le u \le 1.$$

Note that W_F and W_G are proper distribution functions on $[0,1]$. They are related to the scaled total time on test transform $H_F^{-1}(u) = F_e \circ F^{-1}(u)$ studied by Barlow and Doksum [3], Barlow [2], and Klefsjö [13] through the relationship $\overline{W}_F(u) = H_F(1 - u)$. Define also

$$\alpha(x) = G^{-1} \circ F(x) = \overline{G}^{-1} \circ \overline{F}(x), \qquad \beta(x) = G_e^{-1} \circ F_e(x) = \overline{G}_e^{-1} \circ \overline{F}_e(x).$$

Proposition 2.1 below is now an immediate consequence of the preceding definitions.

PROPOSITION 2.1:   The following are equivalent:

(a) $\dfrac{\mu_F(F^{-1}(u))}{\mu_G(G^{-1}(u))}$ is nonincreasing in $u \in [0,1]$.

(b) $\dfrac{\tilde{r}_F(F^{-1}(u))}{\tilde{r}_G(G^{-1}(u))}$ is nondecreasing in $u \in [0,1]$.

(c) $\dfrac{\overline{G}_e \circ \beta(x)}{\overline{G}_e \circ \alpha(x)}$ is nonincreasing in $x \ge 0$.

(d) $W_F^{-1} \circ W_G(u)$ is star-shaped, $u \in [0,1]$.

DEFINITION 2.1:   If the equivalent conditions of Proposition 2.1 hold, we say that F is *more decreasing in mean residual life* than G, and write $F \overset{\text{DMRL}}{<} G$.

REMARK: · This ordering, and the others discussed in this article, are orderings of equivalence classes of $\mathfrak{F}$, where two members F, G of $\mathfrak{F}$ are equivalent $(F \sim G)$ iff they differ by at most a positive scale factor. Such an ordering is then a partial ordering if it possesses the properties of

(a) *Reflexivity:* $F \sim G \Rightarrow F < G$.
(b) *Antisymmetry:* $F < G$ and $G < F \Rightarrow F \sim G$.
(c) *Transitivity:* $F < G$ and $G < H \Rightarrow F < H$.

THEOREM 2.1: (i) the relationship $F \overset{\text{DMRL}}{<} G$ is a partial ordering of the equivalence classes of $\mathfrak{F}$. (ii) If $\overline{G}(x) = e^{-x}$, then $F \overset{\text{DMRL}}{<} G$ iff F is a DMRL distribution. (iii) If $F \overset{\text{IFR}}{<} G$, then $F \overset{\text{DMRL}}{<} G$.

PROOF: (i) (a) if $F \sim G$, then $\overline{F}(x) = \overline{G}(\theta x)$ for some $\theta > 0$. Then $\theta = \mu_G / \mu_F$, so that $\overline{F}_e(x) = \overline{G}_e(\theta x)$ and $\alpha(x) = \beta(x) = \theta x$. Using (c) of Proposition 2.1, $F \overset{\text{DMRL}}{<} G$.

(b) Note that $\beta(x)$ is absolutely continuous, with

$$\beta'(x) = \frac{\mu_G}{\mu_F} \frac{\overline{G} \circ \alpha(x)}{\overline{G} \circ \beta(x)} = \beta'(0) \frac{\overline{G} \circ \alpha(x)}{\overline{G} \circ \beta(x)}. \tag{4}$$

If $F \overset{\text{DMRL}}{<} G$ and $G \overset{\text{DMRL}}{<} F$, then by part (a) of Proposition 2.1, $\mu_F(F^{-1}(u))$ is a constant multiple of $\mu_G(G^{-1}(u))$. It follows that $\overline{F}_e(x) = \overline{G}_e \circ \beta(x)$ is proportional to $\overline{G}_e \circ \alpha(x)$; putting $x = 0$ shows that the constant of proportionality is unity. Thus $\alpha(x) = \beta(x)$, $x \geq 0$. Now (4) and $\beta(0) = 0$ imply that $\alpha(x) = \beta(x) = \theta x$ for some $\theta > 0$, hence $F \sim G$.

(c) Transitivity is an immediate consequence of condition (a) in Proposition 2.1.

(ii) If $\overline{G}(x) = e^{-x}$, then $W_G(u) = u$. Condition (d) of Proposition 2.1 is then equivalent to the statement that F is DMRL.

(iii) If F and G have positive densities f and g, then $\alpha(x)$ and $\gamma(u) = W_F^{-1} \circ W_G(u)$ are differentiable, with

$$\gamma'(u) = \frac{\mu_G}{\mu_F} \frac{d}{dx} \alpha^{-1}(x) \Big|_{x = G_e^{-1}(u)}.$$

Then $F \overset{\text{IFR}}{<} G$ iff $\alpha(x)$ is convex iff $\gamma(u)$ is convex, hence $\gamma(u)$ is as well star shaped if $F \overset{\text{IFR}}{<} G$. The general result follows from the remark that the distributions with positive densities are dense in $\mathfrak{F}$.

3. THE NBUE AND HNBUE ORDERINGS

Proposition 3.1 below follows from the definitions, and (4).

PROPOSITION 3.1: The following are equivalent:

(a) $\dfrac{\mu_F(F^{-1}(u))}{\mu_G(G^{-1}(u))} \leq \dfrac{\mu_F}{\mu_G}$, $u \in [0,1]$.

(b) $\dfrac{\tilde{r}_F(F^{-1}(u))}{\tilde{r}_G(G^{-1}(u))} \geq \dfrac{\mu_G}{\mu_F}, \qquad u \in [0,1].$

(c) $\dfrac{\tilde{r}_F(F_e^{-1}(u))}{\tilde{r}_G(G_e^{-1}(u))} \geq \dfrac{\mu_G}{\mu_F}, \qquad u \in [0,1].$

(d) $\dfrac{\overline{G}_e \circ \beta(x)}{\overline{G}_e \circ \alpha(x)} \leq 1, \qquad x \geq 0.$

(e) $\beta(x) \geq \alpha(x), \qquad x \geq 0.$

(f) $\beta'(x) \geq \beta'(0) = \mu_G / \mu_F.$

DEFINITION 3.1:   If the equivalent conditions of Proposition 3.1 hold, we say that F is *more new better than used in expectation* than G, and write $F \overset{\text{NBUE}}{<} G$.

THEOREM 3.1:   (i) The relationship $F \overset{\text{NBUE}}{<} G$ is a partial ordering of the equivalence classes of $\mathcal{F}$. (ii) If $\overline{G}(x) = e^{-x}$, then $F \overset{\text{NBUE}}{<} G$ iff F is a NBUE distribution. (iii) If $F \overset{\text{DMRL}}{<} G$, then $F \overset{\text{NBUE}}{<} G$. (iv) If $F \overset{\text{IFRA}}{<} G$, then $F \overset{\text{NBUE}}{<} G$.

PROOF: (i) Reflexivity is established as in Theorem 2(i). For antisymmetry, note that $G^{-1} \circ F(x) \leq G_e^{-1} \circ F_e(x)$ and $F^{-1} \circ G(x) \leq F_e^{-1} \circ G_e(x)$, for all $x \geq 0$, together imply that $G^{-1} \circ F(x) \equiv G_e^{-1} \circ F_e(x)$, i.e., $\alpha(x) \equiv \beta(x)$. Now proceed as in Theorem 2.1(i). Transitivity follows from the remark that $H^{-1} \circ F(x) = H^{-1} \circ G(G^{-1} \circ F(x))$. (ii) is an immediate consequence of the definitions. For (iii) and (iv), define

$$q_{F,G}(x) = \int_x^\infty \overline{F}(y) \, d\alpha \, (y) \bigg/ \int_x^\infty \overline{F}(y) \, dy, \qquad x \geq 0$$

and note that $q_{F,G}(x) = \mu_G \overline{G}_e \circ \alpha(x) / \mu_F \overline{G}_e \circ \beta(x)$. Then $F \overset{\text{NBUE}}{<} G$ iff $q_{F,G}(x) \geq q_{F,G}(0)$, $x \geq 0$. Now (iii) is a consequence of part (c) of Proposition 2.1. Part (iv) is established by arguing in a manner very similar to that in the proof of Lemma 2.3 of Barlow [2]. The inequality $q_{F,G}(x) \geq q_{F,G}(0)$ is easily seen to hold if $\alpha(y) = yI(y \geq y_0)$ for some $y_0 > 0$. But if $F \overset{\text{IFRA}}{<} G$, then $\alpha(y)/y$ is nondecreasing and hence is the limit of an increasing sequence of positive linear combinations of indicator functions $I(y \geq y_i)$. The result then follows from the monotone convergence theorem.

REMARK: If $\overline{G}(x) = e^{-x}$, then $F \overset{\text{NBU}}{<} G \Rightarrow F \overset{\text{NBUE}}{<} G$. We have not been able to characterize the class of distributions for which this implication holds. The following example, communicated orally by Rolf Clack and Tony Thompson, shows that it does not hold generally.

Put $\alpha(x) = \{x, 3x - 2, 2x + 2, 4x - 9\}$ on $[0,1]$, $[1,4]$, $[4,5.5]$, $[5.5,\infty)$, respectively. Then $\alpha(x)$ is strictly increasing, and can be seen to be superadditive by a case-by-case inspection. For $\epsilon \in (0, 2/9)$, define

$$\overline{F}(x) = \left\{ 1 - \epsilon x, \frac{11 - 18\epsilon}{9} - \left(\frac{2 - 9\epsilon}{9}\right) x, \frac{7}{2} \epsilon \, e^{-(x - 5.5)} \right\}$$

on [0,1], [1,5.5], [5.5,∞), respectively. Then $\overline{F}$ is a strictly decreasing survival function. Define $\overline{G}(x) = \overline{F} \circ \alpha^{-1}(x)$. Then $F \overset{\text{NBU}}{<} G$ but $F \overset{\text{NBUE}}{\not<} G$, since

$$q_{F,G}(4) - q_{F,G}(0) = \frac{4 + 178\epsilon}{2 + 61\epsilon} - \frac{60 + 210\epsilon}{26 + 69\epsilon}$$

is negative for all sufficiently small ϵ, say for $\epsilon = .01$.

DEFINITION 3.2: We say that F is *more harmonic new better than used in expectation* than G, and write

$$F \overset{\text{HNBUE}}{<} G, \quad \text{if } \frac{G_e^{-1} \circ F_e(x)}{x} \geq \frac{d}{dx} G_e^{-1} \circ F_e(x)\big|_{x=0} = \frac{\mu_G}{\mu_F}, \quad x \geq 0.$$

THEOREM 3.2: (i) the relationship $F \overset{\text{HNBUE}}{<} G$ is a partial ordering of the equivalence classes of $\mathcal{F}$. (ii) If $\overline{G}(x) = e^{-x}$, then $F \overset{\text{HNBUE}}{<} G$ iff F is an HNBUE distribution. (iii) If $F \overset{\text{NBUE}}{<} G$, then $F \overset{\text{HNBUE}}{<} G$.

PROOF: The proofs of parts (i) and (ii) are very similar to those in Theorems 2.1 and 3.1. For (iii), integrate both sides of inequality (f) in Proposition 3.1, using $\beta(0) = 0$.

4. THE NBUFR AND NBUFRA ORDERINGS

Assume that F and G are absolutely continuous, so that $\alpha(x) = G^{-1} \circ F(x)$ is differentiable.

DEFINITION 4.1: We say that F is *more new better than used in failure rate (in failure rate average)* and write $F \overset{\text{NBUFR}}{<} G$ ($F \overset{\text{NBUFRA}}{<} G$) if $\alpha'(x) \geq \alpha'(0)$, $[\alpha(x) \geq x\alpha'(0)]$, $x \geq 0$.

THEOREM 4.1: (i) The relationships $F \overset{\text{NBUFR}}{<} G$ and $G \overset{\text{NBUFRA}}{<} G$ are partial orderings of the equivalence classes of $\mathcal{F}$. (ii) If $\overline{G}(x) = e^{-x}$, then $F \overset{\text{NBUFR}}{<} G$ iff F is NBUFR, and $F \overset{\text{NBUFRA}}{<} G$ iff F is NBUFRA. (iii) $F \overset{\text{NBU}}{<} G \Rightarrow F \overset{\text{NBUFR}}{<} G \Rightarrow F \overset{\text{NBUFRA}}{<} G$.

PROOF: (i) and (ii) are straightforward, and (iii) is the statement that $\alpha(x)$ superadditive, $\alpha(0) = 0 \Rightarrow \alpha'(x) \geq \alpha'(0) \Rightarrow \alpha(x) \geq x\alpha'(0)$.

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