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Testing Whether F is More IFR than G

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Abstract: A distribution F is said to be "more IFR" than another distribution G if $G^{-1}F$ is convex. When $F(0) = G(0) = 0$, the problem of testing $H_0 : F(x) = G(\theta x)$ for some $\theta > 0$ and $x \geqslant 0$, against the alternative $H_A : F$ is more IFR than G, is considered in this paper. Both cases, when G is completely specified (one-sample case) and when it is not specified but a random sample form it is available (two-sample case) are considered. The proposed tests are based on U-statistics. The asymptotic relative efficiency of the tests are compared with several other tests and the test statistics remain asymptotically normal under certain dependency assumptions.

Key Words and Phrases: U-statistic, increasing failure rate, unbiasedness, asymptotic relative efficiency, robustness, strong mixing processes.

1 Introduction

Let F be the family of absolutely continuous distributions such that $F(0) = 0$. For any $F \in F$, let $\bar{F} = 1 - F$ denote the corresponding survival function and let f denote the probability density function (*pdf*) corresponding to F. The failure rate function of F is $r_F(x) = f(x)/\bar{F}(x)$, $\bar{F}(x) > 0$. F is said to be an increasing failure rate (IFR) distribution if r_F is nondecreasing. The exponential distribution $\bar{G}(x) = e^{-x}$, $x \geqslant 0$ has the property that it has a constant failure rate, that is, it does not age (deteriorate) with time. Thus in the IFR sense, we compare the aging of a distribution F with that of the exponential. This leads to the comparison of the aging of two arbitrary life distributions F and

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G in F via a partial ordering called the convex ordering. We give this comparison in the next definition.

Definition 1.1: F is said to be convex-ordered with respect to G, written as $F \overset{c}{<} G$ if $G^{-1}F(x)$ is a convex function for $x \in [0, \infty)$.

Denoting the densities of F and G by f and g, we find that $F \overset{c}{<} G$ implies that the generalized failure rate

$$h(x) = \frac{d}{dx} G^{-1}F(x) = \frac{f(x)}{g[G^{-1}F(x)]},$$

is nondecreasing in x. Or equivalently, $r_F(F^{-1}(u))/r_G(G^{-1}(u))$, the ratio of failure rates at quantiles of the same order u is nondecreasing in u. If either of the above happens, we say that F is "more IFR than G".

Note that F is IFR if and only if F is convex ordered with respect to the negative exponential distribution $\bar{G}(x) = e^{-x}$, $x \geqslant 0$. In general, if F and G are related by the relation $F(x) = G(x^{\alpha})$, for some $\alpha \geqslant 1$ and for all $x \geqslant 0$, then $G^{-1}F(x) = x^{\alpha}$ is convex in x for $\alpha \geqslant 1$ and hence $F \overset{c}{<} G$. This relation implies, for examples, that the members of the Weibull family of distributions are ordered in the above sense according to the shape parameter for the same value of the scale parameter.

It is easy to see that $F \overset{c}{=} G$ if and only if $F(x) = G(\theta x)$ for some $\theta > 0$. Thus this partial ordering is scale invariant.

This concept of convex ordering was introduced by Van Zwet (1964) where moment inequalities were developed and applications to Statistics were shown. Barlow and Proschan (1981) also contains some results on this ordering. Chandra and Singpurwala (1981) have shown that $F \overset{c}{<} G$ implies that the Lorenz curve of G dominates the Lorenz curve of F. In Economics, it is important, to compare the concentration of two distributions.

The purpose of this investigation is to provide a test statistic for testing $H_0 : F \overset{c}{=} G$ versus $H_A : F \overset{c}{<} G$. We present test statistics in both the situations when G is known and when G is unknown and a random sample is available from it. We shall call these two cases the one- and the two-sample cases. The one-sample problem was considered by Barlow and Doksum (1972) for general G. While when G is exponential, this problem was considered by Bickel and Doksum (1969), Proschan and Pyke (1967), Ahmad (1975), and Deshpande and Kochar (1983) among others. The two-sample problem has not been discussed to the best of our knowledge. However, tests for a somewhat weaker ordering namely "NBU ordering " have been proposed by Hollander, Park and Proschan (1982) and Gerlach (1986) for the two-sample problem.

The following lemma sets the basis for defining our test statistics.

Lemma 1.1: Let X_1, X_2, and X_3 (Y_1, Y_2, and Y_3) be three independent copies of a random variable with distribution function $F(G)$. If $F \overset{c}{<} G$ then for any $\alpha \in (0, 1)$,

$$P[X_1 > \alpha X_2 + (1-\alpha)X_3] \geqslant P[Y_1 > \alpha Y_2 + (1-\alpha)Y_3]. \tag{1.1}$$

Proof: Note that $F \overset{c}{<} G$ is equivalent to

$$\bar{F}(\alpha x + (1-\alpha)y) \geqslant \bar{G}[\alpha \bar{G}^{-1}\bar{F}(x) + (1-\alpha)\bar{G}^{-1}\bar{F}(y)], \tag{1.2}$$

for all $\alpha \in (0, 1)$ and all $x, y \geqslant 0$. But (1.2) implies that

$$\begin{aligned} &\int_0^\infty \int_0^\infty \bar{F}(\alpha x + (1-\alpha)y)dF(x)dF(y) \\ &\qquad \geqslant \int_0^\infty \int_0^\infty \bar{G}[\alpha \bar{G}^{-1}\bar{F}(x) + (1-\alpha)\bar{G}^{-1}\bar{F}(y)]dF(x)dF(y) \\ &\qquad = \int_0^\infty \int_0^\infty \bar{G}(\alpha u + (1-\alpha)v)dG(u)dG(v), \end{aligned} \tag{1.3}$$

for all $\alpha \in (0, 1)$. Now (1.3) is equivalent to (1.1). □

From the above lemma, we take as a measure of departure from H_0 the following

$$\begin{aligned} \gamma(F, G) = &\int_0^\infty \int_0^\infty \bar{F}(\alpha x + (1-\alpha)y)dF(x)dF(y) \\ &- \int_0^\infty \int_0^\infty \bar{G}(\alpha x + (1-\alpha)y)dG(x)dG(y). \end{aligned} \tag{1.4}$$

Note that under H_0, $\gamma(F, G) = 0$ and under H_A, $\gamma(F, G) > 0$.

In Section 2, we deal with the one-sample case (G specified except for the scale parameter). A test statistic is introduced and the value of null asymptotic variance is obtained when G is exponential. The test is shown to be unbiased and consistent. In Section 3, we treat the two-sample problem. A consistent estimator of the asymptotic null variance of the test statistic is given. Section 4 is devoted to asymptotic relative efficiency (ARE) comparisons. The newly proposed tests have been compared with the existing tests. Finally, in Section 5 the results concerning the limiting distribution of the test statistic are extended to the case when there is dependence of strong mixing type within the samples.

2 The One-Sample Problem

Let $X_1, \ldots, X_m$ be a random sample from a distribution F. We wish to test $H_0 : F \overset{c}{=} G$ vs. $H_A : F \underset{c}{<} G$ where G is a specified distribution up to the scale parameter. Recall that

$$\gamma(F, G) = \delta_{\alpha,F} - \delta_{\alpha,G}, \tag{2.1}$$

where

$$\delta_{\alpha,F} = P_F[X_1 > \alpha X_2 + (1-\alpha)X_3]$$

and

$$\delta_{\alpha,G} = P_G[Y_1 > \alpha Y_2 + (1-\alpha)Y_3)] = \delta_0^* \quad \text{(a known quantity).}$$

We estimate $\delta_{\alpha,F}$. Let

$$\phi^*(x_1, x_2, x_3) = \begin{cases} 1 & \text{if } x_1 > \alpha x_2 + (1-\alpha)x_3 \\ 0 & \text{otherwise} \end{cases}$$

Set the symmetric version of ϕ^* as follows:

$$\phi(x_1, x_2, x_3) = \frac{1}{6} \sum_p \phi^*(x_i, x_j, x_k), \tag{2.2}$$

where the summation is taken over all permutations of the integers 1, 2 and 3. Thus we can estimate $\delta_{\alpha,F}$ unbiasedly by the U-statistic:

$$U_{\alpha,m}(\underline{X}) = \binom{m}{3}^{-1} \sum_{C} \phi(X_{i_1}, X_{i_2}, X_{i_3}), \tag{2.3}$$

where the summation is over all combinations of the integers (i_1, i_2, i_3) out of the set of integers $(1, 2, \ldots, m)$. Large values of $U_{\alpha,m}(\underline{X})$ are significant for testing H_0 against H_A. The proof of the following theorem follows from the general theory of U-statistics, see Hoeffding (1948) or Puri and Sen (1971).

Theorem 2.1: The asymptotic distribution of $\sqrt{m}\,(U_{\alpha,m}(\underline{X}) - \delta_{\alpha,G})$ as $m \to \infty$, is normal with mean $\gamma(F, G)$ and variance $\sigma^2_{\alpha,F} = 9\,\xi_1(F)$, where

$$\xi_1(F) = E_F(\Psi_1^2(X)) - \delta^2_{\alpha,F} \tag{2.4}$$

with

$$\Psi_1(x_1) = E_F[\phi(x_1, X_2, X_3)] \tag{2.5}$$

Note that under H_0, $\delta_{\alpha,F} = \delta_{\alpha,G}$ and $\sigma^2_{\alpha,F} = \sigma^2_{\alpha,G}$ but under H_A, $\delta_{\alpha,F} > \delta_{\alpha,G}$. In the case $\bar{G}(x) = e^{-\lambda x}$, $x > 0, \lambda > 0$, it can be shown that

$$\delta_{\alpha,G} = [(\alpha + 1)(\bar{\alpha} + 1)]^{-1}, \tag{2.6}$$

where $\bar{\alpha} = 1 - \alpha$ and

$$\Psi_1(x) = \begin{cases} \dfrac{1}{3}\left[1 + \dfrac{\bar{\alpha}e^{-x/\bar{\alpha}}}{2\alpha - 1} + \dfrac{\alpha}{2\bar{\alpha} - 1}e^{-x/\alpha} + \dfrac{e^{-\alpha x}}{\bar{\alpha} + 1} + \dfrac{e^{-\bar{\alpha}x}}{\alpha + 1}\right] & \text{if } \alpha \neq \dfrac{1}{2} \\[2ex] \dfrac{1}{3}\left[1 - e^{-2x} - 2xe^{-2x} + \dfrac{4}{3}e^{-x/2}\right] & \text{if } \alpha = \dfrac{1}{2} \end{cases} \tag{2.7}$$

Thus

$$\sigma^2_{\alpha,G} = \begin{cases} 1 + \dfrac{\alpha^4 - 2\alpha^3 - 4\alpha^2 + 5\alpha + 2}{(2+\bar{\alpha})(2+\alpha)(\alpha\bar{\alpha}+1)} - \dfrac{2(\alpha\bar{\alpha}^2 + \alpha\bar{\alpha} + 1)}{(\alpha+1)(\bar{\alpha}^2 + \bar{\alpha} + 1)(\alpha\bar{\alpha} + \alpha + 1)} \\ - \dfrac{2(\bar{\alpha}\alpha^2 + \alpha\bar{\alpha} + 1)}{(\bar{\alpha}+1)(\alpha^2+\alpha+1)(\alpha\bar{\alpha}+\bar{\alpha}+1)} - \dfrac{8\alpha^3\bar{\alpha}^3 + 10\alpha^2\bar{\alpha}^2 - \alpha^2 - \bar{\alpha}^2 + 5\alpha\bar{\alpha} + 3}{(\alpha+1)^2(\bar{\alpha}+1)^2(2\alpha+1)(2\bar{\alpha}+1)} & \text{if } \alpha \neq \dfrac{1}{2} \\ \dfrac{248}{55125}, \quad \text{if } \alpha = \dfrac{1}{2} \end{cases}$$

Clearly, $\sigma^2_{\alpha,G} = \sigma^2_{\bar{\alpha},G}$. The following table gives the values of $\sigma^2_{\alpha,G}$ for some values of α:

α	0.01	0.10	0.20	0.30	0.40	0.50
$\sigma^2_{\alpha,G}$	$(2175)10^{-8}$	$(1268)10^{-6}$	$(2864)10^{-6}$	$(3868)10^{-6}$	$(4356)10^{-6}$	$(4499)10^{-6}$

Thus for testing the null hypothesis that F is negative exponential versus that it is IFR, the critical values of the test statistic $\sqrt{m}\,\{U_{\alpha,m}(\underline{X}) - [(\alpha+1)(\bar{\alpha}+1)]^{-1}\}/\sigma_{\alpha,G}$ can be approximated by the variates of the standard normal distribution for large samples.

Note also that the above procedure can be used in problems not handled before such as testing F is Weibull versus that F is more IFR than the Weibull. We show below that these proposed tests are unbiased for testing H_0 against H_A.

Theorem 2.2: Let $F \overset{c}{<} G$. Then $U_{\alpha,m}(\underline{X}) \overset{st}{\geqslant} U_{\alpha,m}(\underline{Y})$ where $\underline{X}$ is a random sample from F and $\underline{Y}$ is a random sample from G of the same size.

Proof: Let $Y_i^* = G^{-1}F(X_i)$, $i = 1, \ldots, m$. Then $(Y_1^*, \ldots, Y_m^*) \overset{st}{=} (Y_1, \ldots, Y_m)$. Next, suppose that $X_1 \leqslant \alpha X_2 + (1-\alpha)X_3$, then

$$G^{-1}F(X_1) \leqslant G^{-1}F(\alpha X_2 + (1-\alpha)X_3) \leqslant \alpha G^{-1}F(X_1) + (1-\alpha)G^{-1}F(X_2). \quad (2.9)$$

Thus $X_1 \leqslant \alpha X_2 + (1-\alpha)X_3$ implies that $Y_1^* \leqslant \alpha Y_2^* + (1-\alpha)Y_3^*$, and hence $\phi^*(x_1, x_2, x_3) = 0$ implies that $\phi^*(y_1^*, y_2^*, y_3^*) = 0$, that is, $\phi^*(x_1, x_2, x_3) \underset{st}{\geqslant} \phi(y_1^*, y_2^*, y_3^*)$. Since, $U_{\alpha,m}(\underline{X})$ is an average of ϕ^*'s, it follows that $U_{\alpha,m}(\underline{X}) \overset{st}{\geqslant} U_{\alpha,m}(\underline{Y})$. □

It follows immediately from the above result that the test statistic based on $U_{\alpha,m}(\underline{X})$ is unbiased for testing H_0 versus H_A. In particular the IFR test is unbiased.

Since $\delta_{\alpha,F} > \delta_{\alpha,G}$ under H_A and $\delta_{\alpha,F} = \delta_{\alpha,G}$ under H_0, it follows that (c.f. Theorem 2.1 above) the $U_{\alpha,m}(\underline{X})$ test is consistent for the above problem.

3 The Two-Sample Problem

We now consider the two-sample problem when G is completely unknown and independent random samples $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ are available from F and, G, respectively. Let $N = m + n$. The test we propose is based on the U-statistic estimator $T_{\alpha,N} = U_{\alpha,m}(\underline{X}) - U_{\alpha,n}(\underline{Y})$ of $\delta_{\alpha,F} - \delta_{\alpha,G}$. Large values of $T_{\alpha,N}$ being significant. The proof of the following theorem is immediate form the theory of U-statistics.

Theorem 3.1: The asymptotic distribution of $\sqrt{N}\,[T_{\alpha,N} - \delta_{\alpha,F} + \delta_{\alpha,G}]$ as $N \to \infty$ in such a way that $\frac{m}{N} \to p$, $0 < p < 1$, is normal with mean 0 and variance

$$\sigma_\alpha^2 = 9[\xi_1(F)/p + \xi_1(G)/(1-p)], \tag{3.1}$$

where ξ_1 as given in Theorem 2.1.

Under H_0, the limiting distribution of $\sqrt{N}\,T_{\alpha,N}$ is normal with mean zero and variance $9\xi_1(F_0)/p(1-p)$, where F_0 is the common (but unspecified) distribution. Note that while the asymptotic null mean is 0 independent of F_0, the asymptotic null variance does depend on F_0 via $\xi_1(F_0)$ and thus must be estimated. We will do the more general situation and estimate σ_α^2 consistently and unbiasedly. Since $\xi_1(F) =$ Var $[\Psi_1^2(X)]$ we estimate it by

$$\hat{\xi}_1(F) = (m-1)^{-1} \sum_{i=1}^{m} [\hat{\Psi}_1(X_i) - U_{\alpha,m}(\underline{X})]^2, \tag{3.2}$$

where

$$\hat{\Psi}_1(X_i) = \binom{m-1}{2}^{-1} \sum_{\substack{j<k \\ j,k \neq i}}\sum \phi(X_i, X_j, X_k). \tag{3.3}$$

Similarly we estimate $\xi_1(G)$ by $\hat{\xi}_1(G)$.

Clearly, from the general theory of U-statistics, $\hat{\xi}_1(F)$ and $\hat{\xi}_1(G)$ are consistent estimators of $\xi_1(F)$ and $\xi_1(G)$ respectively (see Puri and Sen 1971). Thus a consistent (and unbiased) estimate of σ_α^2 is given by

$$\hat{\sigma}^2_{\alpha,N} = 9N[\hat{\xi}_1(F)/m + \hat{\xi}_1(G)/n]. \tag{3.4}$$

Thus by Slutsky's lemma, it follows that $\sqrt{N}\,[T_{\alpha,N} - \gamma(F, G)]/\hat{\sigma}_{\alpha,N}$ is asymptotically standard normal. Under H_0, $\gamma(F, G) = 0$. Thus we reject H_0 in favor of H_A if $\sqrt{N}\, T_{\alpha,N}/\hat{\sigma}_{\alpha,N} > z_\alpha$. From this and the fact that $\delta_{\alpha,F} > \delta_{\alpha,G}$ under H_A, it follows that the test is consistent for testing H_0 against H_A.

Unlike the one-sample case the two-sample test is only asymptotically unbiased as shown in the next theorem.

Theorem 3.2: The two-sample test $\sqrt{N}\, T_{\alpha,N}/\hat{\sigma}_{\alpha,N} > z$ is asymptotically unbiased for testing $H_0 : F \overset{c}{=} G$ against $H_A : F \overset{c}{<} G$.

Proof: Observe that

$$P[N^{1/2} T_{\alpha,N} \hat{\sigma}^{-1}_{\alpha,N} \geqslant z] = P[N^{1/2}\{T_{\alpha,N} - \delta_{\alpha,F} + \delta_{\alpha,G}\} \geqslant \hat{\sigma}_{\alpha,N} z - N^{1/2}\{\delta_{\alpha,F} - \delta_{\alpha,G}\}]. \tag{3.5}$$

Also observe that $\delta_{\alpha,F} > \delta_{\alpha,G}$ under the alternative H_A and that $\hat{\sigma}_{\alpha,N} \to \sigma_\alpha < \infty$ in probability as $N \to \infty$. Under H_0, (3.5) is equal to the significance level and under H_A it is at least equal to the significance level. □

4 Asymptotic Relative Efficiencies

One-Sample-Problem: For the purpose of Pitman asymptotic relative efficiency comparisons, we consider a sequence of alternatives $\{F_{\theta_k}\}$, where $\theta_k = \theta_0 + \frac{c}{\sqrt{m_k}}$, $c > 0$ is an arbitrary constant and θ_0 corresponds to G, under H_0.

We consider the following special families of distributions:

1. The Makeham distribution:

$$F_1(x, \theta) = 1 - \exp[-\{x + \theta(x + e^{-x} - 1)\}], \quad x \geqslant 0$$

2. The linear failure rate distribution,

$$F_2(x, \theta) = 1 - \exp\left[-\left\{x + \frac{\theta x^2}{2}\right\}\right], \quad x \geqslant 0$$

The case $\theta = \theta_0 = 0$, corresponds to the null hypothesis with $G(x) = 1 - \exp(-x)$.

For this one-sample for general G, a class of tests was proposed by Barlow and Doksum (1972). Let

$$H_m^{-1}\left(\frac{i}{m}\right) = \sum_{j=1}^{i} gG^{-1}\left(\frac{j-1}{m}\right) \cdot (X_{j,m} - X_{j-1,m}),$$

$$W_{i,m} = H_m^{-1}\left(\frac{i}{m}\right) / H_m^{-1}(1),$$

where $X_{j,m}$ is the j-th order statistic in a sample of size m from F and $X_{0,m} \equiv 0$. If $G(x) = 1 - \exp(-x)$ for $x \geqslant 0$, then

$$W_{i,m} = m^{-1} \sum_{j=1}^{i} (m-j+1)(X_{j,m} - X_{j-1,m}) / \sum_{j=1}^{m} X_{j,m},$$

is the "scaled time on test" until the i-th ordered observation.

They have proposed tests based on statistics of the form

$$B_m(J) = m^{-1} \sum_{i=1}^{m} J[W_{i,m}]$$

where J is an increasing function. We consider here a member of this class corresponding to $J(x) = x$ and when $G(x) = 1 - \exp(-x)$; that is we consider the "scaled cumulative time on test statistic"

$$K_m = \sum_{j=1}^{m} (m-j+1)(X_{j,m} - X_{j-1,m}) / m\bar{X}$$

for efficiency comparisons. This test is asymptotically optimal for testing exponentiallity against Makeham distribution F_1.

Table 1. ARE of $U_{\alpha,m}$ w.r.t. K_m test

	0.5	0.9	0.99
F_1	0.7401	0.8665	0.9830
F_2	0.7227	0.7610	0.8150

Table 1 gives the asymptotic relative efficiencies of the $U_{\alpha,m}$ test against the scaled cumulative total on test statistic for the above mentioned distributions for $\alpha = 0.5, 0.9$ and 0.99.

Observe that $U_{\alpha,m} = U_{\bar{\alpha},m}$. It is clear from the above table that $U_{\alpha,m}$ statistics with α values near one (or zero) are more efficient.

Two-Sample Problem: We consider a sequence of sample sizes m_k and n_k such that $m_k/N_k \to p$ and $n_k/N_k \to 1-p$ as $k \to \infty$. We are not aware of any other test for this problem. Hence for efficiency comparisons, we compare $T_{\alpha,N}$ test with the one proposed by Hollander, Park and Proschan (1982) for testing whether F is more NBU than G. Observe that the set of distributions contained in this new alternative contains distributions which are more IFR since $F \overset{c}{<} G \Rightarrow F \overset{\text{NBU}}{<} G$. Their test is based on the studentized version of the statistic

$$S_N = J_m(\underline{X}) - J_n(\underline{Y})$$

where $J_m(\underline{X})$ is the U-statistic associated with the kernel $h(x_1, x_2, x_3) = 1$ if $x_1 > x_2 + x_3$ and 0 otherwise. Remember that $J_m(\underline{X})$ is nothing other than the Hollander and Proschan (1972) test statistic for testing exponentiality against NBU.

Here also, $T_{\alpha,N} = T_{\bar{\alpha},N}$. Table 2 gives the Pitman asymptotic relative efficiencies of the $T_{\alpha,N}$ test w.r.t. S_N test.

Table 2. ARE's of $T_{\alpha,N}$ w.r.t. S_N

	0.5	0.9	0.99
F_1	0.9261	1.0831	1.2288
F_2	1.606	1.6913	1.8112

5 Extension to Dependent Samples

In this section, we prove that $U_{\alpha,m}(\underline{X})$ (and thus $U_{\alpha,n}(\underline{Y})$) remains asymptotically normal when $X_1, \ldots, X_m$ are no longer independent, instead $X_1, \ldots, X_n$ is assumed to have been taken from a double-infinite sequence of random variables satisfying the following definition:

Definition 5.1: For any $-\infty \leqslant a < b \leqslant \infty$, let σ_α^b denote the sigma field generated by $X_a, \ldots, X_b$. The sequence $\{X_j\}_{-\infty}^{\infty}$ is said to be strong mixing if for any events $A \in \sigma_{-\infty}^a$ and $B \in \sigma_{a+n}^{\infty}$ we have

$$|P(AB) - P(A)P(B)| < \alpha(n), \tag{5.1}$$

where $\alpha(\cdot)$ is a function defined on the integers and is decreasing to 0.

Definition 5.2: The sequence $\{X_j\}_{-\infty}^{\infty}$ is said to be strictly stationary if the distribution of $\{X_{a+1}, \ldots, X_{a+n}\}$ is independent of a for any n.

The following result gives the conditions under which the asymptotic normality of $\sqrt{m}\,(U_{\alpha,m}(\underline{X}) - \delta_{\alpha,F})$ is obtained and gives the formula of the variance.

Theorem 5.1: Assume that $\{X_j\}_{-\infty}^{\infty}$ is a strictly stationary strongly mixing sequence of random variables such that $\sum_{n=1}^{\infty} n\alpha(n) < \infty$. Then $\sqrt{m}\,(U_{\alpha,m}(\underline{X}) - \delta_{\alpha,F})$ is asymptotically normal with mean 0 and variance $\sigma_\alpha^2 = 9\,\gamma_{\alpha,p}^2$, where

$$\gamma_{\alpha,p}^2 = \mathrm{Var}\,\Psi_1^2(X_1) + 2 \sum_{k=1}^{\infty} \mathrm{Cov}\,(\Psi_1^2(X_1), \Psi_1^2(X_{k+1})),$$

with $\psi_1(x_1) = E_F\{\phi(x_1, X_2, X_3)\}$.

Before proving this theorem, let us briefly discuss its implications and relation with results in the literature. First observe that for both the one-sample and the two-sample problems discussed in Sections 2 and 3 above, the test statistics remain asymptotically normal even when we sample from a strongly mixing process. Also, it shall be clear from the proof that similar argument can be said about other U-statistics based test procedures such as Hollander and Proschan (1972) test for NBU, Ahmad (1975) test for IFR and Hollander, Park, and Proschan (1982) test for F is more NBU than G.

Note also that if in (5.1) we require the stronger bound $P(A)\alpha(n)$, then the sequence $\{X_j\}$ is called uniform mixing. In this case and assuming that $\alpha(n) = 0(n^{-(2+\gamma)/\gamma})$ for some $0 < \gamma < 1$, Theorem 5.1's conclusion reduces to Theorem 1 of Yoshihara (1976). Thus our theorem is an extension of his result to strong mixing with weaker condition $\Sigma\, n\alpha(n) < \infty$. Our method of proof is also different.

We shall only sketch the proof of Theorem 5.1 and refer the reader to Ahmad and Kochar (1988) for further details.

Sketch of Proof of Theorem 5.1

Note that $U_{\alpha,m}(\underline{X})$ is an estimate of $\delta_{\alpha,F} = P(X_1 > \alpha X_2 + (1-\alpha)X_3)$ and that $U_{\alpha,m}(\underline{X})$ and $V_{\alpha,m}(\underline{X})$ have same limiting distribution, where $V_{\alpha,m}(\underline{X}) = \int_0^\infty \int_0^\infty \bar{F}_m(\alpha x_1 + (1-x)x_2)\, dF_m(x_1) dF_m(x_2)$. Let $W_{\alpha,m} = \int_0^\infty \int_0^\infty \bar{F}(\alpha(x_1 + (1-\alpha)x_2) dF(x_1) dF_m(x_2)$. The method of proof is to note that $\sqrt{m}\,(W_{\alpha,m} - \delta_{\alpha,F})$ is asymptotically normal with mean 0 and variance σ_α^2 and that $mE(W_{\alpha,m} - U_{\alpha,m})^2 \to 0$ as $m \to \infty$. Recall that $\phi_\alpha^*(x_1, x_2, x_3) = 1$ if $x_1 > \alpha x_2 + (1-\alpha)x_2$ and writing that $\tau(x_1, x_2, x_3) = \phi_\alpha^*(x_1, x_2, x_3) - E\phi_\alpha^*(x_1, x_2, x_3)$ we get

$$m\{E(V_{\alpha,m} - W_{\alpha,m})^2\} = m^{-5} \sum_{i_1} \sum_{i_2} \sum_{i_3} \sum_{j_1} \sum_{j_2} \sum_{j_3} E\tau(X_{i_1}, X_{i_2}, X_{i_3})\tau(X_{j_1}, X_{j_2}, X_{j_3}). \tag{5.2}$$

But $|E\tau(X_{i_1}, X_{i_2}, X_{i_3})\tau(X_{j_1}, X_{j_2}, X_{j_3})| \leqslant I + II + III + IV + V$, where

$$I = |E\phi^*(X_{i_1}, X_{i_2}, X_{i_3})\phi^*(X_{j_1}, X_{j_2}, X_{j_3}) - E\phi^*(X_{i_1}, X_{i_2}, X_{i_3})E\phi^*(X_{j_1}, X_{j_2}, X_{j_3})|,$$

$$II = |E\psi_1(X_i)\phi^*(X_{j_1}, X_{j_2}, X_{j_3}) - E\psi_1(X_i)E\phi^*(X_{j_1}, X_{j_2}, X_{j_3})|,$$

$$III = |E\phi^*(X_{i_1}, X_{i_2}, X_{i_3})\psi_1(X_j) - E\phi^*(X_{i_1}, X_{i_2}, X_{i_3})E\psi_1(X_j)|,$$

$$IV = |E\psi_1(X_i)\psi_1(X_i) - \delta_{\alpha,F}^2|,$$

and

$$V = |E\phi^*(X_{i_1}, X_{i_2}, X_{i_3} - \delta_{\alpha,F}|\ |E\phi^*(X_{j_1}, X_{j_2}, X_{j_3}) - \delta_{\alpha,F}|.$$

Now, be Lemma 2 of Billingsley (1968) p. 171, it follows that (see Ahmad and Kochar 1988), $I < 2\alpha(\max[|i_1 - j_1|, |i_2 - j_2|, |i_3 - j_3|])$. Note also that II, III, IV all have similar bounds as I. Thus

$$\sum_{i_1}\sum_{i_2}\sum_{i_3}\sum_{j_2}\sum_{j_3}(I + \ldots + IV) \leqslant K_m, \quad \text{say,} \tag{5.3}$$

where

$$K_m = 16m^{-2}\sum_{i_1}\sum_{j_1}\alpha(|i_1 - j_1|) + 8m^{-3}\sum_{i_1}\sum_{i_2}\sum_{j_1 \neq i_1}\sum_{j_2 \neq i_2}\alpha(\max[|i_1 - j_1|, |i_2 - j_2|]).$$

But $\sum_{i_1}\sum_{j_1}\alpha(|i_1 - j_1|) = 0(m)$, and $\sum_{i_1}\sum_{i_2}\sum_{j_1 \neq i_1}\sum_{j_2 \neq i_2}\alpha(\max[|i_1 - j_1|, |i_2 - j_2|]) = 0(m^2)$ (see Ahmad and Kochar 1988). Thus $k_m = 0(m^{-1})$. Finally we have

$$|E\phi_\alpha^*(X_{i_1}, X_{i_2}, X_{i_3}) - \delta_{\alpha,F}| \leqslant \alpha(|i_1 - i_2|) + \alpha(i_1 - i_3|). \tag{5.4}$$

Thus $m^{-5/2}\sum_{i_1}\sum_{i_2}\sum_{i_3}|E\phi_\alpha^*(X_{i_1}, X_{i_2}, X_{i_3}) - \delta_{\alpha,F}| \leqslant 2m^{-1/2}\sum\alpha(m) + 2m^{-3/2}\sum m\alpha(m)$ $\to 0$ as $m \to \infty$. The proof is complete. □

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References

Ahmad IA (1975) A nonparametric test for the monotonicity of the failure rate function. Commun Statist 4:967–974

Ahmad IA, Kochar SC (1988) Testing whether F is more IFR than G. Statistics Technical Report, NIU, 88-10

Barlow RE, Doksum K (1972) Isotonic tests for convex orderings. Proc Sixth Berk Symp Math Statist Prob 1:293–323

Barlow RE, Proschan F (1981) Statistical theory of reliability and life testing. To Begin with, Silver Spring: Maryland

Bickel PJ, Doksum K (1969) Tests for monotone failure rate based on normalized spacings. Ann Math Statist 40:1216–1235

Billingsley P (1968) Convergence of probability measures. Wiley & Sons, New York

Chandra M, Singpurwala ND (1981) Relationships between some notions which are common to Reliability and Economics. Math Oper Res 6:116–121

Deshpande SC, Kochar SC (1983) A test for exponentiality against IFR alternatives. IAPQR Transactions 8:1–8

Gerlach B (1986) A new test for whether F is "more NBU" than G. Statistics 17:79–86

Hoeffding W (1948) A class of statistics with asymptotically normal distribution. Ann Math Statist 19:293–325

Hollander M, Proschan F (1972) Testing whether new is better than used. Ann Math Statist 43: 1136–1146

Hollander M, Park D, Proschan F (1982) Testing whether F is "more NBU" than is G. Florida State University Report M 626

Ibragimov IA, Linnik Yn (1971) Independent and stationary random variables. North Holland, New York

Proschan F, Pyke R (1967) Tests for monotone failure rate. Proc Fifth Berk Symp Math Statist Prob 3:293–312

Puri ML, Sen PK (1971) Nonparametric methods in multivariate analysis. Wiley, New York

Van Zwet WR (1964) Convex transformations of Random variables. Mathematisch Centrum Amsterdam

Yoshihara KJ (1976) Limiting behavior of U-statistics for stationary absolutely regular processes. Z Wahrscheinlichkeitstheorie Verw Gebiete 35:237–252

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