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Dispersive ordering among linear combinations of uniform random variables

Baha-Eldin Khaledi[a], Subhash Kochar[b,*]

[a] Department of Mathematics, University of Arizona, Tucson, AZ 85721, USA

[b] Indian Statistical Institute, 7, SJS Sansanwal Marg, New Delhi-110016, India

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Abstract

Let $a_{(i)}$ and $b_{(i)}$ be the ith smallest components of $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ respectively, where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^{+n}$. The vector $\mathbf{a}$ is said to be p-larger than $\mathbf{b}$ (denoted by $\mathbf{a} \overset{\text{p}}{\succcurlyeq} \mathbf{b}$) if $\prod_{i=1}^{k} a_{(i)} \leqslant \prod_{i=1}^{k} b_{(i)}$, for $k = 1, \ldots, n$. Let $U_1, \ldots, U_n$ be independent $U(0,1)$ random variables. It is shown that if $\boldsymbol{\lambda}$, $\boldsymbol{\lambda}^*$ belonging to $\mathbb{R}^{+n}$ are such that $\boldsymbol{\lambda} \overset{\text{p}}{\succcurlyeq} \boldsymbol{\lambda}^*$, then $\sum_{i=1}^{n} U_i/\lambda_i$ is greater than $\sum_{i=1}^{n} U_i/\lambda_i^*$ according to dispersive as well as hazard rate orderings. These results give simple bounds on various quantities of interest associated with these statistics.

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1. Introduction

Stochastic models are usually sufficiently complex in various fields of statistics. Obtaining bounds and approximations for some of their characteristics of interest is of practical importance. That is, the approximation of a stochastic model either by a simpler model or by a model with simple constituent components might lead to convenient bounds and approximations for some particular and desired characteristics of the model. Lot of work has been done in the literature on this problem.

Statistics which are linear combinations of random variables, arise frequently in statistics and their distribution theory can be quite complicated in many cases. From time to time attempts have been made in the literature to obtain bounds and approximations for their distributions. Some relevant references are Proschan (1965), Bock

* Corresponding author. Tel.: +91-11-656-4741; fax: +91-11-685-6779.

E-mail addresses: bkhaledi@math.arizona.edu (B.-E. Khaledi), kochar@isid.ac.in (S. Kochar).

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et al. (1987), Tong (1988), Boland et al. (1994), Kochar and Ma (1999) and Ma (2000) among others.

In this paper, we will concentrate only on linear combinations of independent and identically distributed uniform random variables with positive coefficients (or equivalently convolutions of independent uniform random variables differing in their scale parameters) and obtain new dispersive ordering results for them when the vectors of their coefficients satisfy certain order restrictions. These results lead to simple bounds on various quantities of interest associated with these statistics. Throughout this paper, 'increasing' means nondecreasing and 'decreasing' means nonincreasing. First we review the necessary definitions and concepts.

Let X and Y be two random variables with distribution functions F and G, respectively. Let F^{-1} and G^{-1} be their right continuous inverses. X is said to be more dispersed than Y (denoted by $X \geqslant_{\text{disp}} Y$) if

$$F^{-1}(v) - F^{-1}(u) \geqslant G^{-1}(v) - G^{-1}(u) \quad \text{for } 0 \leqslant u \leqslant v \leqslant 1. \tag{1.1}$$

This means that the difference between any two quantiles of F is at least as much as the difference between the corresponding quantiles of G. From this one can easily see that

$$X \geqslant_{\text{disp}} Y \quad \Leftrightarrow \quad F^{-1}(x) - G^{-1}(x) \quad \text{is increasing in } x \in (0,1). \tag{1.2}$$

A consequence of $X \geqslant_{\text{disp}} Y$ is that $|X_1 - X_2| \geqslant_{\text{st}} |Y_1 - Y_2|$ and which in turn implies $\text{var}(X) \geqslant \text{var}(Y)$ as well as $E[|X_1 - X_2|] \geqslant E[|Y_1 - Y_2|]$, where $X_1, X_2\,(Y_1, Y_2)$ are two independent copies of $X(Y)$, and 'st' represents the usual stochastic order.

By taking $u = 0$ in (1.1), it follows that for nonnegative random variables, $X \geqslant_{\text{disp}} Y \Rightarrow X \geqslant_{\text{st}} Y$. Recall that a random variable X with survival function $\bar{F}$ is said to be larger than another random variable Y with survival function $\bar{G}$ in *hazard rate* ordering (denoted by $X \geqslant_{\text{hr}} Y$) if $\bar{F}(x)/\bar{G}(x)$ is increasing in x. It is easy to see that for nonnegative random variables, $X \geqslant_{\text{hr}} Y \Rightarrow X \geqslant_{\text{st}} Y$. Bagai and Kochar (1986) noted the following connection between hazard rate ordering and dispersive ordering.

Lemma 1.1. *Let X and Y be two nonnegative random variables. If $X \geqslant_{\text{disp}} Y$ and X or Y is* IFR, *then $X \geqslant_{\text{hr}} Y$.*

For details, see Chapters 1 and 2 of Shaked and Shanthikumar (1994).

One of the tools which is useful for deriving inequalities in statistics and probability is the notion of majorization. Let $\{x_{(1)} \leqslant x_{(2)} \leqslant \cdots \leqslant x_{(n)}\}$ denote the increasing arrangements of the components of the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. The vector $\mathbf{x}$ is said to majorize the vector $\mathbf{y}$ (written $\mathbf{x} \overset{\text{m}}{\succcurlyeq} \mathbf{y}$) if $\sum_{i=1}^{j} x_{(i)} \leqslant \sum_{i=1}^{j} y_{(i)}$ for $j = 1, \ldots, n-1$ and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$. Functions that preserve the majorization ordering are called Schur-convex functions. The vector $\mathbf{x}$ is said to majorize the vector $\mathbf{y}$ *weakly* (written if $\sum_{i=1}^{j} x_{(i)} \leqslant \sum_{i=1}^{j} y_{(i)}$ for $j = 1, \ldots, n$. See Marshall and Olkin (1979, Chapter 3) for properties and more details on these partial orderings. Recently Bon and Paltanea (1999) have considered a pre-order on $\mathbb{R}^{+n}$, which they call as a p-*larger order*. A vector $\mathbf{x}$ in $\mathbb{R}^{+n}$ is said to be p-larger than another vector $\mathbf{y}$ also in $\mathbb{R}^{+n}$ (written $\mathbf{x}\overset{\text{p}}{\succcurlyeq}\mathbf{y}$) if $\prod_{i=1}^{j} x_{(i)} \leqslant \prod_{i=1}^{j} y_{(i)}$, $j = 1,\ldots,n$. Let $\log(\mathbf{x})$ denote the vector of the logarithms of the coordinates of $\mathbf{x}$. It is easy to verify that

$$\mathbf{x}\overset{\text{p}}{\succcurlyeq}\mathbf{y} \;\Leftrightarrow\; \log(\mathbf{x})\overset{\text{w}}{\succcurlyeq}\log(\mathbf{y}). \tag{1.3}$$

It is known that $\mathbf{x}\overset{\text{m}}{\succcurlyeq}\mathbf{y} \Rightarrow (g(x_1),\ldots,g(x_n))\overset{\text{w}}{\succcurlyeq}(g(y_1),\ldots,g(y_n))$ for all concave functions g (cf. Marshall and Olkin, 1979, p. 115). From this and (1.3), it follows that for $\mathbf{x},\mathbf{y} \in \mathbb{R}^{+n}$

$$\mathbf{x}\overset{\text{m}}{\succcurlyeq}\mathbf{y} \;\Rightarrow\; \mathbf{x}\overset{\text{p}}{\succcurlyeq}\mathbf{y}.$$

The converse is, however, not true. For example, the vectors $(0.2,1,5)\overset{\text{p}}{\succcurlyeq}(1,2,3)$ but majorization does not hold between these two vectors. Obviously, for any vector $\boldsymbol{\lambda} \in \mathbb{R}^{+n}$, $(\lambda_1,\ldots,\lambda_n)\overset{\text{p}}{\succcurlyeq}(\tilde{\lambda},\ldots,\tilde{\lambda})$, where $\tilde{\lambda}$ denotes the geometric mean of the components of the vector $\boldsymbol{\lambda}$.

Let $U_1,\ldots,U_n$ be independent $U(0,1)$ random variables and let $S(\lambda_1,\ldots,\lambda_n) = \sum_{i=1}^{n} U_i/\lambda_i$, where $\lambda_i > 0$ for $i = 1,\ldots,n$. Note that $S(\lambda_1,\ldots,\lambda_n) \equiv_{\text{dist}} \sum_{i=1}^{n} X_{\lambda_i}$, where $X_{\lambda_i},\ldots,X_{\lambda_n}$ are independent random variables such that for $i=1,\ldots,n$, $X_{\lambda_i}=_{\text{dist}}U(0,1/\lambda_i)$. In Section 2, we prove that

$$\boldsymbol{\lambda}\overset{\text{p}}{\succcurlyeq}\boldsymbol{\lambda}^* \Rightarrow S(\lambda_1,\ldots,\lambda_n) \geqslant_{\text{disp}} S(\lambda_1^*,\ldots,\lambda_n^*). \tag{1.4}$$

We also show that

$$\boldsymbol{\lambda}\overset{\text{p}}{\succcurlyeq}\boldsymbol{\lambda}^* \Rightarrow S(\lambda_1,\ldots,\lambda_n) \geqslant_{\text{hr}} S(\lambda_1^*,\ldots,\lambda_n^*),$$

where $\geqslant_{\text{hr}}$ denotes hazard rate ordering.

2. Main results

To prove the desired results we need the following theorems.

Theorem 2.1 (Lewis and Thompson, 1981). *Let Z be a random variable independent of random variables X and Y. If $X \geqslant_{\text{disp}} Y$ and Z has a log-concave density, then*

$$X + Z \geqslant_{\text{disp}} Y + Z.$$

Theorem 2.2 (Lewis and Thompson, 1981). *Let for $n \geqslant 1$, X_n, Y_n, X and Y be random variables such that $X_n \to X$ and $Y_n \to Y$, weakly. Then $X_n \geqslant_{\text{disp}} Y_n$, $n \geqslant 1$ implies $X \geqslant_{\text{disp}} Y$.*

Theorem 2.3 (Marshall and Olkin, 1979, p. 59). *A real valued function ϕ on the set $A \subset R^n$ satisfies*

$$\mathbf{x}\overset{\text{w}}{\succcurlyeq}\mathbf{y} \text{ on } A \Rightarrow \phi(\mathbf{x}) \geqslant \phi(\mathbf{y}),$$

if and only if, ϕ is decreasing and Schur-convex on A.

Lemma 2.1. *The function* $\psi : \mathbb{R}^{+n} \to \mathbb{R}$ *satisfies*

$$\mathbf{x} \overset{\text{p}}{\succcurlyeq} \mathbf{y} \Rightarrow \psi(\mathbf{x}) \geqslant \psi(\mathbf{y}) \tag{2.1}$$

if and only if,

(i) $\psi(e^{a_1}, \ldots, e^{a_n})$ *is Schur-convex in* $(a_1, \ldots, a_n)$
(ii) $\psi(e^{a_1}, \ldots, e^{a_n})$ *is decreasing in* a_i, *for* $i = 1, \ldots, n$,

where $a_i = \log x_i$, *for* $i = 1, \ldots, n$.

Proof. Using relation (1.3), we see that (2.1) is equivalent to

$$\mathbf{a} \overset{\text{w}}{\succcurlyeq} \mathbf{b} \Rightarrow \psi(e^{a_1}, \ldots, e^{a_n}) \geqslant \psi(e^{b_1}, \ldots, e^{b_n}), \tag{2.2}$$

where $a_i = \log x_i$ and $b_i = \log y_i$, for $i = 1, \ldots, n$. Taking $\phi(a_1, \ldots, a_n) = \psi(e^{a_1}, \ldots, e^{a_n})$ in Theorem 2.3, we get the required result. □

Let X_{λ_1} and X_{λ_2} be independent $U(0, 1/\lambda_1)$ and $U(0, 1/\lambda_2)$ random variables, respectively. Without loss of generality assume that $\lambda_1 \geqslant \lambda_2$. The density function and the distribution function of $S(\lambda_1, \lambda_2) = X_{\lambda_1} + X_{\lambda_2}$ are, respectively,

$$g(\lambda_1, \lambda_2; x) = \begin{cases} \lambda_1 \lambda_2 x, & 0 \leqslant x < 1/\lambda_1, \\ \lambda_2, & 1/\lambda_1 \leqslant x \leqslant 1/\lambda_2, \\ \lambda_1 + \lambda_2 - \lambda_1 \lambda_2 x, & 1/\lambda_2 \leqslant x \leqslant 1/\lambda_1 + 1/\lambda_2, \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

and

$$G(\lambda_1, \lambda_2; x) = \begin{cases} 0, & x \leqslant 0, \\ \lambda_1 \lambda_2 x^2/2, & 0 \leqslant x \leqslant 1/\lambda_1, \\ \lambda_2 x - \lambda_2/2\lambda_1, & 1/\lambda_1 \leqslant x \leqslant 1/\lambda_2, \\ 1 - (\lambda_1 + \lambda_2 - \lambda_1 \lambda_2 x)^2/2\lambda_1 \lambda_2, & 1/\lambda_2 \leqslant x \leqslant 1/\lambda_1 + 1/\lambda_2, \\ 1, & \text{otherwise.} \end{cases} \tag{2.4}$$

The right inverse function of G is

$$G^{-1}(\lambda_1, \lambda_2; x) = \begin{cases} (2x/\lambda_1 \lambda_2)^{1/2}, & 0 \leqslant x \leqslant \lambda_2/2\lambda_1, \\ (x + \lambda_2/2\lambda_1)/\lambda_2, & \lambda_2/2\lambda_1 \leqslant x \leqslant 1 - \lambda_2/2\lambda_1, \\ 1/\lambda_1 + 1/\lambda_2 - (2(1-x)/\lambda_1 \lambda_2)^{1/2}, & 1 - \lambda_2/2\lambda_1 \leqslant x < 1. \end{cases} \tag{2.5}$$

Now we prove the main result of this section.

Theorem 2.4. *Let* $X_{\lambda_1},\ldots,X_{\lambda_n}$ *be independent random variables such that* X_{λ_i} *has* $U(0,1/\lambda_i)$ *distribution, for* $i=1,\ldots,n$. *Then,* $\boldsymbol{\lambda} \overset{\mathrm{p}}{\succcurlyeq} \boldsymbol{\lambda}^*$ *implies* $S(\lambda_1,\ldots,\lambda_n) \geqslant_{\mathrm{disp}} S(\lambda_1^*,\ldots,\lambda_n^*)$.

Proof. We first prove the result for $n=2$ and then extend it to arbitrary positive integer $n>2$. Without loss of generality, let us assume that $\lambda_1 \geqslant \lambda_2$ and $\lambda_1^* \geqslant \lambda_2^*$.

Case (a): $\lambda_1 > \lambda_2$ and $\lambda_1^* > \lambda_2^*$.

We first consider the case when $\lambda_1 \neq \lambda_1^*$ and $\lambda_2 \neq \lambda_2^*$ and shall discuss the other possibilities later. It follows from Lemma 2.1 that it is sufficient to show that

(i) the function

$$G^{-1}(\mathrm{e}^{a_1},\mathrm{e}^{a_2};\beta) - G^{-1}(\mathrm{e}^{a_1},\mathrm{e}^{a_2};\alpha) \tag{2.6}$$

is Schur-convex in (a_1,a_1) for all $0<\alpha\leqslant\beta<1$, where $a_i=\log\lambda_i$, $i=1,2$,

(ii) the function defined in (2.6) is decreasing in a_i, $i=1,2$.

Without loss of generality, assume that $a_1+a_2=c$. From the assumptions we have $a_1 \geqslant a_2$ and $a_1^* \geqslant a_2^*$ and $a_1 \neq a_2$ and $a_1^* \neq a_2^*$. Using relation (1.2), it is easy to see that (i) is equivalent to saying that the function

$$h(x) = G^{-1}(\mathrm{e}^{a_1},\mathrm{e}^{a_2};x) - G^{-1}(\mathrm{e}^{a_1^*},\mathrm{e}^{a_2^*};x) \quad \text{is increasing in } x\in(0,1). \tag{2.7}$$

The function h in (2.7) can be written as

$$h(x) = \begin{cases} 0, & 0 \leqslant x \leqslant \frac{\mathrm{e}^{a_2-a_1}}{2}, \\ \mathrm{e}^{-a_2}\left(x+\frac{\mathrm{e}^{a_2-a_1}}{2}\right) - (2x\mathrm{e}^{-c})^{1/2}, & \frac{\mathrm{e}^{a_2-a_1}}{2} < x \leqslant \frac{\mathrm{e}^{a_2^*-a_1^*}}{2}, \\ \mathrm{e}^{-a_2}\left(x+\frac{\mathrm{e}^{a_2-a_1}}{2}\right) - \mathrm{e}^{-a_2^*}\left(x+\frac{\mathrm{e}^{a_2^*-a_1^*}}{2}\right), & \frac{\mathrm{e}^{a_2^*-a_1^*}}{2} \leqslant x \leqslant 1-\frac{\mathrm{e}^{a_2^*-a_1^*}}{2}, \\ \mathrm{e}^{-a_2}\left(x+\frac{\mathrm{e}^{a_2-a_1}}{2}\right) - \mathrm{e}^{-c}(\mathrm{e}^{a_1^*}+\mathrm{e}^{a_2^*}) & \\ \quad +(2(1-x)\mathrm{e}^{-c})^{1/2}, & 1-\frac{\mathrm{e}^{a_2^*-a_1^*}}{2} < x \leqslant 1-\frac{\mathrm{e}^{a_2-a_1}}{2}, \\ \mathrm{e}^{-c}(\mathrm{e}^{a_1}+\mathrm{e}^{a_2}) - (2(1-x)\mathrm{e}^{-c})^{1/2}, & \\ -\mathrm{e}^{-c}(\mathrm{e}^{a_1^*}+\mathrm{e}^{a_2^*}) + (2(1-x)\mathrm{e}^{-c})^{1/2}, & 1-\frac{\mathrm{e}^{a_2-a_1}}{2} < x < 1. \end{cases} \tag{2.8}$$

Under the constraints of majorization between (a_1,a_2) and (a_1^*,a_2^*), it is easy to see that the function h is increasing in x. This proves (i). It is worth noting that (ii) is equivalent to saying that $S(\mathrm{e}^{a_1},\mathrm{e}^{a_2})$ is decreasing in a_1 and a_2 according to dispersive ordering. Now let $a_1 > a_1'$. It is easy to see that $X_{\mathrm{e}^{a_1'}} \geqslant_{\mathrm{disp}} X_{\mathrm{e}^{a_1}}$. The random variables X's are independent and $X_{\mathrm{e}^{a_2}}$ has a log-concave density. Combining these facts, it follows from Theorem 2.1 that $S(\mathrm{e}^{a_1'},\mathrm{e}^{a_2}) \geqslant_{\mathrm{disp}} S(\mathrm{e}^{a_1},\mathrm{e}^{a_2})$. Similarly one can prove that $S(\mathrm{e}^{a_1},\mathrm{e}^{a_2})$ is decreasing in a_2. The required result follows from these. Now if $\lambda_1=\lambda_1^*$, then $(\lambda_1,\lambda_2) \overset{\mathrm{p}}{\succcurlyeq} (\lambda_1^*,\lambda_2^*)$ implies that $\lambda_2 < \lambda_2^*$ and which in turn implies that

$X_{\lambda_2} \geqslant_{\rm disp} X_{\lambda_2^*}$. The required result in this case follows from Theorem 2.1, since the random variables X's have log-concave densities and they are independent. The last possibility is $\lambda_1 = \lambda_2^*$. In this case $\lambda_2 < \lambda_1 = \lambda_2^* < \lambda_1^*$. Again the required result follows from Theorem 2.1. This completes the proof of case (a).

Case (b): $\lambda_1 = \lambda_2$ and $\lambda_1^* > \lambda_2^*$.

Noting that $\lambda_1 = \lambda_2^* < \lambda_1^*$ or $\lambda_1 < \lambda_2^* < \lambda_1^*$, the result follows from Theorem 2.1.

Case (c): $\lambda_1 > \lambda_2$ and $\lambda_1^* = \lambda_2^*$.

Again the required result for the case when $\lambda_1 = \lambda_1^*$ immediately follows from Theorem 2.1. Now let $\lambda_1 \neq \lambda_1^*$. In this case $(\lambda_1, \lambda_2) \overset{\rm p}{\succcurlyeq} (\lambda_1^*, \lambda_2^*)$ implies that $\lambda_1^* \geqslant \tilde{\lambda}$, where $\tilde{\lambda} = (\lambda_1 \lambda_1)^{1/2}$, the geometric mean of λ_1, λ_2. First we prove the result for the case when $\lambda_1^* = \tilde{\lambda}$. It is easy to see that, for $n \geqslant 1$, $(\lambda_1, \lambda_2) \overset{\rm p}{\succcurlyeq} (\tilde{\lambda}, \tilde{\lambda} + 1/n)$. Using this observation, it follows from case (a) that, for $n \geqslant 1$,

$$X_{\lambda_1} + X_{\lambda_2} \geqslant_{\rm disp} X_{\tilde{\lambda}} + X_{\tilde{\lambda}+1/n}.$$

Using the fact that $X_{\tilde{\lambda}+1/n} \to X_{\tilde{\lambda}}$, weakly, it follows that $X_{\tilde{\lambda}} + X_{\tilde{\lambda}+1/n} \to S(\tilde{\lambda}, \tilde{\lambda})$ weakly. Combining these observations, the required result in this case follows from Theorem 2.2. The result for the case when $\lambda_1^* > \tilde{\lambda}$ follows from the above case, the fact that $X_{\tilde{\lambda}} \geqslant_{\rm disp} X_{\lambda_1^*}$ and again using Theorem 2.1. This completes the proof of this case.

Case (d): $\lambda_1 = \lambda_2$ and $\lambda_1^* = \lambda_2^*$.

Using (2.5) in this case, it is easy to see that (1.2) holds.

This completes the proof for $n = 2$.

Now we prove the result for $n > 2$. As in the proof for $n = 2$, we show that

(i)

$$\mathbf{a} \overset{\rm m}{\succcurlyeq} \mathbf{a}^* \Rightarrow S(\mathrm{e}^{a_1}, \ldots, \mathrm{e}^{a_n}) \geqslant_{\rm disp} S(\mathrm{e}^{a_1^*}, \ldots, \mathrm{e}^{a_n^*}).$$

where $a_i = \log \lambda_i$ and $a_i^* = \log \lambda_i^*$, $i = 1, , \ldots, n$,

(ii) $S(\mathrm{e}^{a_1}, \ldots, \mathrm{e}^{a_n})$ is decreasing in a_i, for $i = 1, \ldots, n$ according to dispersive ordering.

To prove (i), it is sufficient to consider the case when $(a_1, a_2) \overset{\rm m}{\succcurlyeq} (a_1^*, a_2^*)$, and $a_i = a_i^*$, $i = 3, \ldots, n$. Then it follows from the case $n = 2$ that $S(\mathrm{e}^{a_1}, \mathrm{e}^{a_2}) \geqslant_{\rm disp} S(\mathrm{e}^{a_1^*}, \mathrm{e}^{a_2^*})$. The random variable $S(\mathrm{e}^{a_3}, \ldots, \mathrm{e}^{a_n})$ has a log-concave density, since the class of distributions with log-concave densities is closed under convolutions (cf. Dharmadhikari and Joag-dev, 1988, p. 17). Adding $S(\mathrm{e}^{a_3}, \ldots, \mathrm{e}^{a_n})$ to both sides of the above inequality, we find that the required result follows from Theorem 2.1. The proof of (ii) here is similar to that of (ii) in case $n = 2$. Using (i) and (ii), again the main result follows from Theorem 2.3. □

The following result immediately follows from the above results.

Corollary 2.1. *Let $X_{\lambda_1}, \ldots, X_{\lambda_n}$ be independent random variables such that $X_{\lambda_i} =_{\rm dist} U(0, 1/\lambda_i)$ for $i = 1, \ldots, n$. Then, $\boldsymbol{\lambda} \overset{\rm p}{\succcurlyeq} \boldsymbol{\lambda}^*$ implies $S(\lambda_1, \ldots, \lambda_n) \geqslant_{\rm hr} S(\lambda_1^*, \ldots, \lambda_n^*)$.*

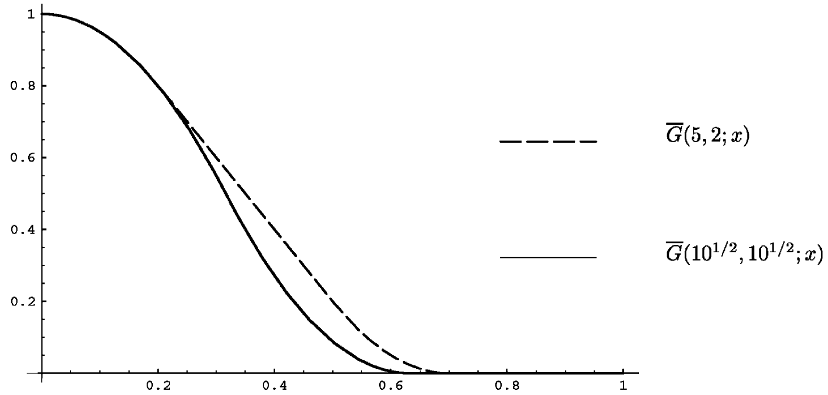

Fig. 1. Graphs of survival functions of $S(\lambda_1, \lambda_2)$.

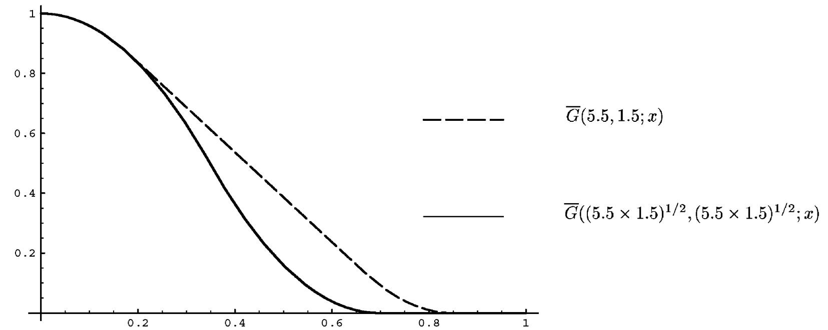

Fig. 2. Graphs of survival functions of $S(\lambda_1, \lambda_2)$.

Proof. Since $S(\lambda_1,\ldots,\lambda_n)$ has a log-concave density, it is IFR (cf. Barlow and Proschan, 1981). From Theorem 2.4, $S(\lambda_1,\ldots,\lambda_n) \geqslant_{\text{disp}} S(\lambda_1^*,\ldots,\lambda_n^*)$. The required result then follows from Lemma 1.1. $\Box$

Let $\tilde{\lambda}$ denote the geometric mean of $\lambda_1,\ldots,\lambda_n$. Then since $(\lambda_1,\ldots,\lambda_n)\overset{\text{p}}{\succcurlyeq}(\tilde{\lambda},\ldots,\tilde{\lambda})$, we get the following lower bounds on various quanties of interest associated with convolutions of uniform random variables.

Corollary 2.2. *Let $X_{\lambda_1},\ldots,X_{\lambda_n}$ be independent random variables such that $X_{\lambda_i} =_{\text{dist}} U(0,1/\lambda_i)$ for $i=1,\ldots,n$. Then,*

(a) $S(\lambda_1,\ldots,\lambda_n) \geqslant_{\text{disp}} S(\tilde{\lambda},\ldots,\tilde{\lambda})$,
(b) $S(\lambda_1,\ldots,\lambda_n) \geqslant_{\text{hr}} S(\tilde{\lambda},\ldots,\tilde{\lambda})$,
(c) $S(\lambda_1,\ldots,\lambda_n) \geqslant_{\text{st}} S(\tilde{\lambda},\ldots,\tilde{\lambda})$.

In Figs. 1 and 2, we plot the survival functions (denoted by $\bar{G}(\lambda_1,\lambda_2;x)$) of convolutions of two independent uniform random variables along with the bounds given

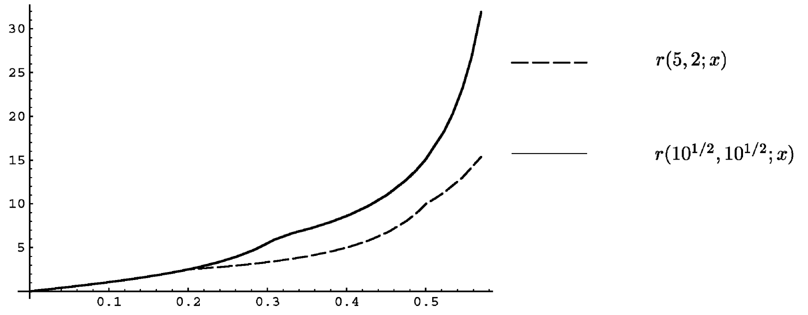

Fig. 3. Graphs of hazard rate functions of $S(\lambda_1,\lambda_2)$.

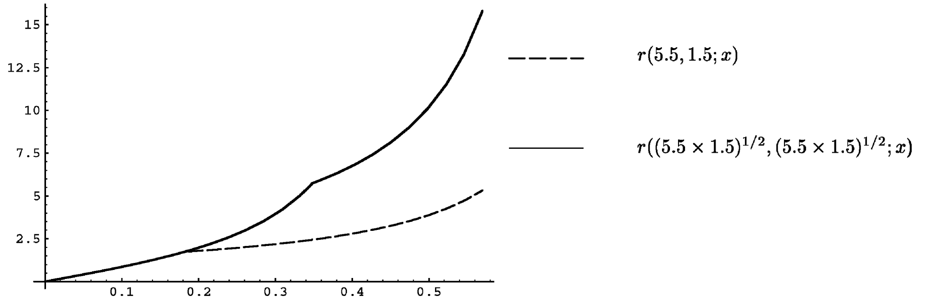

Fig. 4. Graphs of hazard rate functions of $S(\lambda_1,\lambda_2)$.

by Corollary 2.2(c). In Figs. 3 and 4, we plot the hazard rate functions (denoted by $r(\lambda_1,\lambda_2;x)$) of convolutions of two independent uniform random variables along with the bounds given by Corollary 2.2(b).

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