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Stochastic comparisons of parallel systems of heterogeneous exponential components

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Abstract

Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i, $i = 1, \ldots, n$. Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$. Let $Y_1, \ldots, Y_n$ be a random sample of size n from an exponential distribution with common hazard rate $\bar{\lambda} = \sum_{i=1}^{n} \lambda_i/n$. The purpose of this paper is to study stochastic comparisons between the largest order statistics $X_{n:n}$ and $Y_{n:n}$ from these two samples. It is proved that the hazard rate of $X_{n:n}$ is smaller than that of $Y_{n:n}$. This gives a convenient upper bound on the hazard rate of $X_{n:n}$ in terms of that of $Y_{n:n}$. It is also proved that $Y_{n:n}$ is smaller than $X_{n:n}$ according to dispersive ordering. While it is known that the survival function of $X_{n:n}$ is Schur convex in $\boldsymbol{\lambda}$, Boland, El-Neweihi and Proschan [*J. Appl. Probab.* **31** (1994) 180–192] have shown that for $n > 2$, the hazard rate of $X_{n:n}$ is not Schur concave. It is shown here that, however, the reversed hazard rate of $X_{n:n}$ is Schur convex in $\boldsymbol{\lambda}$. © 1997 Elsevier Science B.V.

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1. Introduction

Order statistics play an important role in reliability theory in particular and in statistics in general. A k-out-of-n system of n components functions if and only if at

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least k of the components function. The time to failure of a k-out-of-n system of n components with lifetimes $X_1, \dots, X_n$ corresponds to the $(n - k + 1)$th order statistic, $X_{n-k+1:n}$. Thus the study of the lifetimes of k-out-of-n systems is equivalent to the study of the probability distributions of order statistics. A series system is an n-out-of-n system and a parallel system is a 1-out-of-n system. Thus the time to failure of a series system corresponds to the first order statistic while that of a parallel system corresponds to the largest order statistic. Series and parallel systems are the simplest examples of coherent systems and they have been extensively studied in the literature. Much is known about their stochastic properties when the components are independent and identically distributed. But it is not uncommon to encounter systems with components having nonidentical lifetimes. Some general results on order statistics and spacings from nonidentical distributions have been obtained by Pledger and Proschan (1971), Proschan and Sethuraman (1976), Bapat and Kochar (1994), Boland et al. (1994), Kochar and Kirmani (1995), Boland et al. (1996), Kochar and Korwar (1996) and Kochar and Rojo (1996), among others.

The exponential distribution plays a very important role in statistics. Because of its non-aging property, it has many nice properties and it often gives very convenient bounds on survival probabilities and other characteristics of interest for systems with non-exponential components. Pledger and Proschan (1971) considered the problem of stochastically comparing the order statistics and the spacings of nonidentical independent exponential random variables with those corresponding to independent and identically distributed exponential random variables. This topic is pursued further in this paper which concentrates on stochastic comparisons of the largest order statistics from heterogenous and homogeneous exponential distributions.

There are many ways in which a random variable X can be said to be smaller than another random variable Y. In the *usual* stochastic ordering case, a random variable X with distribution function F is stochastically smaller than a random variable Y with distribution function G (and written as $X \overset{\text{st}}{\preceq} Y$) if $F(t) \geqslant G(t)$ for all t. In some cases, a pair of distributions may satisfy a stronger condition called *likelihood ratio ordering*. If distributions F and G possess densities (or probability mass functions) f and g, respectively, and $f(x)/g(x)$ is nonincreasing in x, then we say that X is smaller than Y according to likelihood ratio ordering. This is denoted by $X \overset{\text{lr}}{\preceq} Y$. It is known that $X \overset{\text{lr}}{\preceq} Y$ implies that $\bar{F}(x)/\bar{G}(x)$ is nonincreasing in x, where $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$ denote the survival functions of X and Y, respectively. This latter condition defines *hazard rate ordering*. In the case of absolutely continuous distributions, this is equivalent to the hazard rate of F, $r_F(x) = f(x)/\bar{F}(x)$, being everywhere as large as $r_G(x) = g(x)/\bar{G}(x)$, the hazard rate of G. If this happens, X is said to be smaller than Y according to *hazard rate ordering* and this is denoted by $X \overset{\text{hr}}{\preceq} Y$. Note that hazard rate ordering implies stochastic ordering. The *reversed hazard rate* of a life distribution F is defined as $\tilde{r}_F(x) = f(x)/F(x)$. Let $\tilde{r}_G(x)$ denote the reversed hazard rate of G. Then X is said to be *smaller than Y in the reversed hazard rate order* (and written as $X \overset{\text{rh}}{\preceq} Y$) if $\tilde{r}_F(x) \leqslant \tilde{r}_G(x)$, for all x, or equivalently, if $F(x)/G(x)$ is nonincreasing in x. The reversed hazard rate ordering also implies stochastic ordering, but in general, there are no implications between hazard rate and reversed hazard rate orderings. It is known that $X \overset{\mathrm{lr}}{\preceq} Y$ also implies $X \overset{\mathrm{rh}}{\preceq} Y$. Lehmann and Rojo (1992) provide simple characterizations of these orderings.

The above notions of stochastic dominance among univariate random variables can be extended to the multivariate case. A random vector $\boldsymbol{X} = (X_1, \ldots, X_n)$ is *smaller than another random vector* $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ *in the multivariate stochastic order* (and written as $\boldsymbol{X} \overset{\mathrm{st}}{\preceq} \boldsymbol{Y}$) if $E[\phi(\boldsymbol{X})] \leqslant E[\phi(\boldsymbol{Y})]$ for all increasing functions ϕ whenever the expectations exist. Karlin and Rinott (1980) introduced and studied the concept of multivariate likelihood ratio ordering. Let f and g denote the density functions of $\boldsymbol{X}$ and $\boldsymbol{Y}$, respectively. Then $\boldsymbol{X}$ *is smaller than* $\boldsymbol{Y}$ *in the multivariate likelihood ratio order* (written as $\boldsymbol{X} \overset{\mathrm{lr}}{\preceq} \boldsymbol{Y}$) if

$$f(\boldsymbol{x})g(\boldsymbol{y}) \leqslant f(\boldsymbol{x} \wedge \boldsymbol{y})g(\boldsymbol{x} \vee \boldsymbol{y}) \quad \text{for every } \boldsymbol{x} \text{ and } \boldsymbol{y} \text{ in } \mathbb{R}^n, \tag{1.1}$$

where $\boldsymbol{x} \wedge \boldsymbol{y} = (\min(x_1, y_1), \ldots, \min(x_n, y_n))$, $\boldsymbol{x} \vee \boldsymbol{y} = (\max(x_1, y_1), \ldots, \max(x_n, y_n))$.

It is known that multivariate likelihood ratio ordering implies multivariate stochastic ordering, but the converse is not true. Also if two random vectors are ordered according to multivariate likelihood ratio ordering, then their corresponding subsets of components are also ordered accordingly. See Chapters 1 and 4 of Shaked and Shanthikumar (1994) for more details on various kinds of stochastic orders, their inter-relationships and their properties.

We shall also be comparing the various statistics according to the criteria of dispersiveness. Let X and Y be two random variables with distribution functions F and G, respectively. Let F^{-1} and G^{-1} be their right continuous inverses. The distribution of the random variable X is less dispersed than that of Y ($X \overset{\mathrm{disp}}{\preceq} Y$) if

$$F^{-1}(v) - F^{-1}(u) \leqslant G^{-1}(v) - G^{-1}(u), \quad \text{for } 0 \leqslant u \leqslant v \leqslant 1. \tag{1.2}$$

This means that the difference between any two quantiles of F is smaller than the difference between the corresponding quantiles of G. When X and Y have densities f and g, respectively, then $X \overset{\mathrm{disp}}{\preceq} Y$ if and only if,

$$g(x) \leqslant f(F^{-1}G(x)) \quad \text{for all } x \in (0, \infty). \tag{1.3}$$

One of the consequences of $X \overset{\mathrm{disp}}{\preceq} Y$ is that $\mathrm{var}(X) \leqslant \mathrm{var}(Y)$. For other properties of dispersive ordering, see Chapter 2 of Shaked and Shanthikumar (1994).

The concepts of majorization of vectors and Schur convexity of functions will be needed throughout this work. Let $\{x_{(1)} \leqslant x_{(2)} \leqslant \cdots \leqslant x_{(n)}\}$ denote the increasing arrangement of the components of the vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$. The vector $\boldsymbol{y}$ is said to *majorize* the vector $\boldsymbol{x}$ (written as $\boldsymbol{x} \overset{\mathrm{m}}{\preceq} \boldsymbol{y}$) if $\sum_{i=1}^{j} y_{(i)} \leqslant \sum_{i=1}^{j} x_{(i)}$, $j = 1, \ldots, n-1$ and $\sum_{i=1}^{n} y_{(i)} = \sum_{i=1}^{n} x_{(i)}$. A real-valued function ϕ defined on a set $\mathscr{A} \subset \mathbb{R}^n$ is said to be *Schur convex* (*Schur concave*) on $\mathscr{A}$ if $\boldsymbol{x} \overset{\mathrm{m}}{\preceq} \boldsymbol{y} \Rightarrow \phi(\boldsymbol{x}) \leqslant (\geqslant) \phi(\boldsymbol{y})$.

Pledger and Proschan (1971) proved the following result on order statistics from exponential distributions.

Theorem 1.1. *Let* $X_1, \ldots, X_n$ *be independent exponential random variables with* X_i *having hazard rate* λ_i; $i = 1, \ldots, n$. *Let* $Y_1, \ldots, Y_n$ *be another set of independent* *exponential random variables with λ_i^* as the hazard rate of Y_i, $1, \ldots, n$. Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and $\boldsymbol{\lambda}^* = (\lambda_1^*, \ldots, \lambda_n^*)$. Then $\boldsymbol{\lambda} \overset{\mathrm{m}}{\succeq} \boldsymbol{\lambda}^*$ implies*

$$X_{1:n} \overset{\mathrm{st}}{\equiv} Y_{1:n} \quad \text{and} \quad X_{k:n} \overset{\mathrm{st}}{\geq} Y_{k:n}, \quad \text{for } k = 2, \ldots, n.$$

Proschan and Sethuraman (1976) strengthened this result from componentwise stochastic ordering to multivariate stochastic ordering. They proved that under the conditions of the above theorem $(X_{1:n}, \ldots, X_{n:n}) \overset{\mathrm{st}}{\geq} (Y_{1:n}, \ldots, Y_{n:n})$.

The purpose of the present investigation is to see to what extent these results can be strengthened. This paper focuses on the probabilistic behavior of the largest order statistic. For the special case $n = 2$ and $k = 2$, Boland et al. (1994) partially strengthened the above result from stochastic ordering to hazard rate ordering. They proved that the hazard rate of a parallel system of two independent exponential components is Schur-concave in (λ_1, λ_2), the component hazard rates. They also concluded that the above result cannot be generalized for arbitrary n. Then the next natural problem is to compare the hazard rate of $X_{n:n}$ with that of $Y_{n:n}$, where $Y_1, \ldots, Y_n$ is a random sample of size n from an exponential distribution with hazard rate $\bar{\lambda} = \sum_{i=1}^n \lambda_i/n$. We prove in Section 2 that $Y_{n:n} \overset{\mathrm{hr}}{\leq} X_{n:n}$. This gives a convenient upper bound on the hazard rate of $X_{n:n}$ in terms of that of $Y_{n:n}$. We also prove that $Y_{n:n} \overset{\mathrm{disp}}{\leq} X_{n:n}$. In the last section it is shown that the *reversed hazard rate* of $X_{n:n}$ is Schur-convex in $\boldsymbol{\lambda}$.

2. Comparisons with i.i.d. exponentials

Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i, $i = 1, \ldots, n$. Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$. Let $Y_1, \ldots, Y_n$ be a random sample of size n from an exponential distribution with common hazard rate $\bar{\lambda} = \sum_{i=1}^n \lambda_i/n$. In this section we shall stochastically compare $X_{n:n}$ with $Y_{n:n}$. It is proved that $Y_{n:n}$ is less dispersed and has greater hazard rate than $X_{n:n}$. The proof of these results hinges on the following technical lemma from p. 73 of Mitrinovic, Pecaric and Fink (1993).

Lemma 2.1. *Let h be a nondecreasing function such that $h(0) > 0$, and let $a_i \geq 0$, $i = 1, \ldots, n$. Then*

$$\prod_{i=1}^n a_i \leq \frac{\sum_{i=1}^n a_i^n h(a_i)}{\sum_{i=1}^n h(a_i)}. \tag{2.1}$$

Now we prove the main result of this section.

Theorem 2.1. *Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i, $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be a random sample of size n from an* *exponential distribution with common hazard rate* $\bar{\lambda} = \sum_{i=1}^{n} \lambda_i / n$. *Then*

(a) $Y_{n:n} \overset{\text{disp}}{\leq} X_{n:n}$; (2.2)

(b) $Y_{n:n} \overset{\text{hr}}{\leq} X_{n:n}$. (2.3)

Proof. (a) Let F and G denote the distribution functions of $Y_{n:n}$ and $X_{n:n}$ with corresponding densities f and g, respectively.

It is easy to verify that for $x > 0$,

$$F^{-1}G(x) = -\frac{1}{\bar{\lambda}} \ln\left[1 - \prod_{i=1}^{n} (1 - e^{-\lambda_i x})^{1/n}\right],$$

and

$$f[F^{-1}G(x)] = n\bar{\lambda}\left[1 - \prod_{i=1}^{n} (1 - e^{-\lambda_i x})^{1/n}\right]\left[\prod_{i=1}^{n} (1 - e^{-\lambda_i x})^{1/n}\right]^{n-1}. \tag{2.4}$$

To prove that $Y_{n:n} \overset{\text{disp}}{\leq} X_{n:n}$, it follows from the relation (1.3) that it is sufficient to show that

$$g(x) \leqslant f[F^{-1}G(x)], \quad \text{for } x > 0.$$

That is, for $x > 0$,

$$\left[\sum_{i=1}^{n} \left(\frac{\lambda_i e^{-\lambda_i x}}{1 - e^{-\lambda_i x}}\right)\right]\left[\prod_{i=1}^{n} (1 - e^{-\lambda_i x})\right] \leqslant n\bar{\lambda}\left[1 - \prod_{i=1}^{n} (1 - e^{-\lambda_i x})^{1/n}\right]\left[\prod_{i=1}^{n} (1 - e^{-\lambda_i x})\right]^{(n-1)/n}.$$

This is equivalent to,

$$\sum_{i=1}^{n} \lambda_i \left\{\frac{1}{1 - e^{-\lambda_i x}} - 1\right\} \leqslant \left[\sum_{i=1}^{n} \lambda_i\right]\left[\prod_{i=1}^{n} \frac{1}{(1 - e^{-\lambda_i x})^{1/n}} - 1\right]; \tag{2.5}$$

i.e., if for $x > 0$,

$$\sum_{i=1}^{n} \frac{\lambda_i}{1 - e^{-\lambda_i x}} \leqslant \left[\sum_{i=1}^{n} \lambda_i\right]\left[\prod_{i=1}^{n} \frac{1}{(1 - e^{-\lambda_i x})}\right]^{1/n}. \tag{2.6}$$

Multiplying both sides by x (> 0), we see that to prove the desired result, it is sufficient to prove the following inequality for $y > 0$,

$$\sum_{i=1}^{n} \frac{y_i}{1 - \exp(-y_i)} \leqslant \left[\sum_{i=1}^{n} y_i\right]\left[\prod_{i=1}^{n} \frac{1}{1 - \exp(-y_i)}\right]^{1/n}. \tag{2.7}$$

In Lemma 2.1, let $a_i = (1 - e^{-y_i})^{1/n}$, $h(t) = -\ln(1 - t^n)/t^n$ so that $h(a_i) = y_i/(1 - \exp(-y_i))$. It is easy to verify that h is nondecreasing and $\lim_{t \to 0} h(t) = 1$. Thus all the conditions of Lemma 2.1 are satisfied. Applying the inequality (2.1) yields the required result (2.7).

(b) Bagai an Kochar (1986) (also see Theorem 2.B.13 of Shaked and Shanthikumar, 1994) proved that if $X \overset{\text{disp}}{\leq} Y$ and either X or Y has IFR (increasing failure rate) distribution, then $X \overset{\text{hr}}{\leq} Y$. It follows from Theorem 5.8 of Barlow and Proschan (1981) that $Y_{n:n}$ has IFR distribution. Using this and part (a) of this theorem we get the required result that $Y_{n:n} \overset{\text{hr}}{\leq} X_{n:n}$. □

This theorem gives a simple upper bound on the hazard rate of a parallel system of heterogeneous exponential components in terms of that of a parallel system of identically distributed exponential random variables. It also gives a lower bound on the variance of $X_{n:n}$ in terms of that of $Y_{n:n}$. These results are summarized in the following corollary.

Corollary 2.1. *Under the conditions of Theorem* 2.1,
(a) *the hazard rate* $r_{X_{n:n}}$ *of* $X_{n:n}$ *satisfies*

$$r_{X_{n:n}}(x) \leqslant \frac{n\bar{\lambda}[1-\exp(-\bar{\lambda}x)]^{n-1}\exp(-\bar{\lambda}x)}{1-[1-\exp(-\bar{\lambda}x)]^{n}}, \tag{2.8}$$

(b)

$$\operatorname{var}(X_{n:n}) \geqslant \frac{1}{\bar{\lambda}^2}\sum_{i=1}^{n}\frac{1}{(n-i+1)^2}. \tag{2.9}$$

3. Some Schur type results

In this section some new Schur type results for the hazard rate and the reversed hazard rate of the maximum of independent exponential random variables are obtained. The next theorem strengthens Theorem 2.1 of Boland et al. (1994) from hazard rate ordering to likelihood ratio ordering.

Theorem 3.1. *Let* X_1 *and* X_2 *be two independent exponential random variables with hazard rates* λ_1 *and* λ_2, *respectively. Let* Y_1 *and* Y_2 *be another set of independent exponential random variables with respective hazard rates* λ_1^* *and* λ_2^*. *Then*

$$(\lambda_1, \lambda_2) \overset{\text{m}}{\succeq} (\lambda_1^*, \lambda_2^*) \Rightarrow X_{2:2} \overset{\text{lr}}{\succeq} Y_{2:2}. \tag{3.1}$$

Proof. Kochar and Rojo (1996) have proved that $(\lambda_1, \lambda_2) \overset{\text{m}}{\succeq} (\lambda_1^*, \lambda_2^*)$ implies

$$X_{2:2} - X_{1:2} \overset{\text{lr}}{\succeq} Y_{2:2} - Y_{1:2}. \tag{3.2}$$

It follows from Theorem 2.1 of Kochar and Korwar (1996) that under the conditions of the theorem, $X_{1:2}$ is independent of $X_{2:2} - X_{1:2}$, $Y_{1:2}$ is independent of $Y_{2:2} - Y_{1:2}$ and $X_{1:2} \overset{\text{dist}}{=} Y_{1:2}$ has log-concave density.

Using Lemma 14.B.9 of Shaked and Shanthikumar (1994), it follows from (3.2) that

$$X_{2:2} = (X_{2:2} - X_{1:2}) + X_{1:2} \overset{\text{lr}}{\succeq} (Y_{2:2} - Y_{1:2}) + Y_{1:2} = Y_{2:2}.$$

That is,

$$X_{2:2} \overset{\text{lr}}{\succeq} Y_{2:2}. \quad \square$$

In the light of the counter-example of Boland et al. (1994) as discussed in the first section, the result of Theorem 3.1 on the life of a parallel system of two components, cannot be extended beyond the case $n = 2$.

Since under the assumptions of Theorem 3.1, $X_{2:2} \overset{\mathrm{lr}}{\succeq} Y_{2:2}$, and $X_{1:2} \overset{\mathrm{st}}{=} Y_{1:2}$, one might wonder whether the vector $(X_{1:2}, X_{2:2})$ is greater than the vector $(Y_{1:2}, Y_{2:2})$ in the multivariate likelihood ratio ordering sense. The answer is in the negative even in the case when the Y's are identically distributed as shown by the next counter example.

Example 3.1. Let $\lambda_1 = 3$, $\lambda_2 = 5$, $\lambda_1^* = \lambda_2^* = 4$. The joint density of $(X_{1:2}, X_{2:2})$ is

$$f(x_1, x_2) = 15[\mathrm{e}^{-(5x_1+3x_2)} + \mathrm{e}^{-(3x_1+5x_2)}], \quad \text{for } x_1 < x_2,$$

and that of $(Y_{1:2}, Y_{2:2})$ *is*

$$g(y_1, y_2) = 16(2!)\mathrm{e}^{-4(y_1+y_2)}, \quad \text{for } y_1 < y_2.$$

In order to prove that $(X_{1:2}, X_{2:2}) \overset{\mathrm{lr}}{\succeq} (Y_{1:2}, Y_{2:2})$, we have to show that

$$f(x_1, x_2)g(y_1, y_2) \leqslant f(x_1 \vee y_1, x_2 \vee y_2)g(x_1 \wedge y_1, x_2 \wedge y_2) \tag{3.3}$$

holds.

Now for $x_1 < x_2$ and $y_1 < y_2$,

$$f(x_1, x_2)g(y_1, y_2) = 15 \times 16(2!)[\mathrm{e}^{-(5x_1+3x_2+8\bar{y})} + \mathrm{e}^{-(3x_1+5x_2+8\bar{y})}], \tag{3.4}$$

and

$$\begin{aligned} &f(x_1 \vee y_1, x_2 \vee y_2)g(x_1 \wedge y_1, x_2 \wedge y_2) \\ &= 15 \times 16(2!)[\mathrm{e}^{-[5(x_1 \vee y_1)+3(x_2 \vee y_2)+8(\overline{x \wedge y})]} + \mathrm{e}^{-[3(x_1 \vee y_1)+3(x_2 \vee y_2)+8(\overline{x \wedge y})]}]. \end{aligned} \tag{3.5}$$

At $x_1 = 4$, $x_2 = 46$, $y_1 = 41$, $y_2 = 43$, we find that the value of (3.4) is $240(2!)$ $[\mathrm{e}^{-494} + \mathrm{e}^{-578}]$ and that of (3.5) is $240(2!)$ $[\mathrm{e}^{-531} + \mathrm{e}^{-541}]$ showing thereby that (3.3) does not hold in this case.

A natural question to ask is whether for $n > 2$, $(\lambda_1, \dots, \lambda_n) \overset{\mathrm{lr}}{\succeq} (\lambda_1^*, \dots, \lambda_n^*)$ implies $X_{2:n} \overset{\mathrm{lr}}{\succeq} Y_{2:n}$? The next example shows that for $n = 3$, even hazard rate ordering does not hold between $X_{2:n}$ and $Y_{2:n}$, proving thereby that for arbitrary n, Theorem 1.1 cannot be strengthened from stochastic ordering to hazard rate ordering.

Example 3.2. The survival function of $X_{2:3}$ is

$$\begin{aligned} \bar{F}_{2:3}(x) &= \bar{F}_1(x)\bar{F}_2(x)\bar{F}_3(x) + F_1(x)\bar{F}_2(x)\bar{F}_3(x) + F_2(x)\bar{F}_1(x)\bar{F}_3(x) \\ &\quad + F_3(x)\bar{F}_1(x)\bar{F}_2(x) \\ &= \bar{F}_2(x)\bar{F}_3(x) + F_2(x)\bar{F}_1(x)\bar{F}_3(x) + F_3(x)\bar{F}_1(x)\bar{F}_2(x) \\ &= \mathrm{e}^{-sx}[\mathrm{e}^{\lambda_1 x} + \mathrm{e}^{\lambda_2 x} + \mathrm{e}^{\lambda_3 x} - 2], \end{aligned} \tag{3.6}$$

in the case of exponential random variables. Here $s = \lambda_1 + \lambda_2 + \lambda_3$.

Let $\boldsymbol{\lambda} = (40, 10, 1)$ and $\boldsymbol{\lambda}^* = (40, 5.5, 5.5)$. Then obviously, $\boldsymbol{\lambda} \overset{\mathrm{m}}{\succeq} \boldsymbol{\lambda}^*$. But the ratio of the survival functions of $X_{2:3}$ to that of $Y_{2:3}$

$$\frac{e^{40x} + e^{10x} + e^{x} - 2}{e^{40x} + e^{5.5x} + e^{5.5x} - 2}$$

is not monotone in x. This can be seen by verifying that the values of this ratio at 0.01, 0.07 and 0.25 are 1.0013, 1.0084 and 1.00025, respectively. This ratio should have been increasing in x in order for the hazard rate ordering, $X_{2:3} \overset{\mathrm{hr}}{\succeq} Y_{2:3}$ to hold.

As discussed earlier and as pointed out by Boland et al. (1994), the hazard rate of $X_{n:n}$, the lifetime of a parallel system of n components is not Schur concave in $\boldsymbol{\lambda}$ for $n > 2$. However, we prove in the next theorem that the reversed hazard rate of $X_{n:n}$ is Schur convex in $\boldsymbol{\lambda}$ for any $n > 1$.

Theorem 3.2. *Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i for $i = 1, \ldots, n$. Then the reversed hazard of $X_{n:n}$ is Schur convex in $\boldsymbol{\lambda}$. That is, if $Y_1, \ldots, Y_n$ is another set of independent exponential random variables with parameters $(\lambda_i^*, \ldots, \lambda_n^*)$, then*

$$\boldsymbol{\lambda} \overset{\mathrm{m}}{\succeq} \boldsymbol{\lambda}^* \Rightarrow X_{n:n} \overset{\mathrm{rh}}{\succeq} Y_{n:n}.$$

Proof. For $x > 0$, the distribution function of $X_{n:n}$ is

$$F_{n:n}(x) = \prod_{i=1}^{n} [1 - e^{-\lambda_i x}]$$

with the reversed hazard rate

$$\tilde{r}_{n:n}(\boldsymbol{\lambda}, x) = \frac{d}{dx} \ln F_{n:n}(x) = \frac{1}{x} \sum_{i=1}^{n} \frac{(\lambda_i x) e^{-\lambda_i x}}{1 - e^{-\lambda_i x}}. \tag{3.7}$$

Since for $x > 0$, the ith term in the summation (3.7) is convex in $\lambda_i x$, $i = 1, \ldots, n$, it follows that $\tilde{r}_{n:n}(\boldsymbol{\lambda}, x)$ is Schur convex in $(\lambda_1, \ldots, \lambda_n)$ for each x (cf. Marshall and Olkin, 1979, p. 64). □

The above result partially strengthens Theorem 2.1 of Pledger and Proschan (1971) from stochastic ordering to reversed hazard rate ordering for the largest order statistics.

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