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Weibull distribution: Some stochastic comparisons results

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Abstract

Let $X_1, \ldots, X_n$ be independent random variables such that X_i has Weibull distribution with shape parameter α and scale parameter λ_i, $i = 1, \ldots, n$. Let $X_1^*, \ldots, X_n^*$ be another set of independent Weibull random variables with the same common shape parameter α, but with scale parameters as $\boldsymbol{\lambda}^* = (\lambda_1^*, \ldots, \lambda_n^*)$. Suppose that $\boldsymbol{\lambda} \overset{\text{m}}{\succcurlyeq} \boldsymbol{\lambda}^*$. We prove that when $0 < \alpha < 1$, $(X_{(1)}, \ldots, X_{(n)}) \overset{\text{st}}{\succcurlyeq} (X_{(1)}^*, \ldots, X_{(n)}^*)$. For $\alpha \geqslant 1$, we prove that $X_{(1)} \leqslant_{\text{hr}} X_{(1)}^*$, whereas the inequality is reversed when $\alpha \leqslant 1$. Let $Y_1, \ldots, Y_n$ be a random sample of size n from a Weibull distribution with shape parameter α and scale parameter $\tilde{\lambda} = (\prod_{i=1}^n \lambda_i)^{1/n}$, the geometric mean of the λ_i's. It is shown that $X_{(n)} \geqslant_{\text{hr}} Y_{(n)}$ for all values of α and in case $\alpha \leqslant 1$, we also have that $X_{(n)}$ is greater than $Y_{(n)}$ according to dispersive ordering. In the process, we also prove some new results on stochastic comparisons of order statistics for the proportional hazards family.

Keywords: Proportional hazards family; Hazard rate ordering; Dispersive ordering; Schur functions; Majorization and *p*-larger ordering

1. Introduction

Weibull distribution is perhaps the most commonly used distribution in reliability and life testing. In the MathSciNet, we found 475 entries that contain the word Weibull in their

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titles. The recent book by Prabhakar Murthy et al. (2004) is solely devoted to the study of Weibull models. In its standard form it has the probability density function

$$f(x,\alpha,\lambda)=\alpha x^{\alpha-1}\lambda^{\alpha}\mathrm{e}^{-(x\lambda)^{\alpha}},\quad x>0.$$

Here, $\alpha(>0)$ is the shape parameter and $\lambda(>0)$ is the scale parameter. We shall use the notation $W(\alpha,\lambda)$ to denote such a random variable. It is a very flexible family of distributions, having decreasing, constant, and increasing failure rates when $0<\alpha<1$, $\alpha=1$, and $\alpha>1$, respectively.

An assumption often made in reliability models is that the components have lifetimes with proportional hazards. Let X_i denote the lifetime of the ith component of a reliability system with survival function $\overline{F}_i(t)$, $i=1,\ldots,n$. Then they have proportional hazard rates (PHR) if there exist constants $\lambda_1,\ldots,\lambda_n$ and a (cumulative hazard) function $R(t)\geqslant 0$ such that $\overline{F}_i(t)=\mathrm{e}^{-\lambda_i R(t)}$ for $i=1,\ldots,n$. If $X_1,\ldots,X_n$ are independent random variables such that $X_i\sim W(\alpha,\lambda_i)$ for $i=1,\ldots,n$, then they belong to the PHR family with $R(t)=t^{\alpha}$ and a *new* parameter vector $(\mu_1,\ldots,\mu_n)$, where $\mu_i=\lambda_i^{\alpha}$, $i=1,\ldots,n$, but not with the original parameters.

Let $X_1,\ldots,X_n$ be n random variables and let $X_{(i)}$ denote their ith order statistic. A k-out-of-n system of n components functions if at least k of n components function. The time of a k-out-of-n system of n components with lifetimes $X_1,\ldots,X_n$ corresponds to the $(n-k+1)$th order statistic. Thus, the study of lifetimes of k-out-of-n systems is equivalent to the study of the stochastic properties of order statistics. In particular, a 1-out-of-n system corresponds to a parallel system and an n-out-of-n system corresponds to a series system. A lot of work has been done in the literature on different aspects of order statistics when the observations are i.i.d. In many practical situations, like in reliability theory, the observations are not necessarily i.i.d. Because of the complicated nature of the problem, not much work has been done for the non-i.i.d. case. Some interesting partial ordering results on order statistics when the parent observations are independent with proportional hazard rates have been obtained by Pledger and Proschan (1971), Proschan and Sethuraman (1976), Boland et al. (1994), Dykstra et al. (1997), and Khaledi and Kochar (2000a), among others. In this paper, we obtain some new results on stochastic comparisons of order statistics and sample range when the parent observations are independent Weibull with a common shape parameter α, but with different scale parameters.

Now we introduce notations and recall some definitions. Throughout this paper *increasing* means *nondecreasing* and *decreasing* means *nonincreasing*, and we shall be assuming that all distributions under study are absolutely continuous. Let X and Y be univariate random variables with distribution functions F and G, survival functions $\overline{F}$ and $\overline{G}$, density functions f and g, and hazard rates r_F $(=f/\overline{F})$ and r_G $(=g/\overline{G})$, respectively. Let l_X (l_Y) and u_X (u_Y) be the left and the right endpoints of the support of X (Y). The random variable X is said to be *stochastically* smaller than Y (denoted by $X\leqslant_{\mathrm{st}}Y$) if $\overline{F}(x)\leqslant\overline{G}(x)$ for all x. This is equivalent to saying that $Eg(X)\leqslant Eg(Y)$ for any increasing function g for which expectations exist. X is said to be smaller than Y in *hazard rate* ordering (denoted by $X\leqslant_{\mathrm{hr}}Y$) if $\overline{G}(x)/\overline{F}(x)$ is increasing in $x\in(-\infty,\max(u_X,u_Y))$. In case the hazard rates exist, it is easy to see that $X\leqslant_{\mathrm{hr}}Y$ if and only if $r_G(x)\leqslant r_F(x)$ for every x. Note that hazard rate ordering implies stochastic ordering. The *reversed hazard rate* of a life distribution F

is defined as $\tilde{r}_F(x) = f(x)/F(x)$. Let $\tilde{r}_G(x)$ denote the reversed hazard rate of G. Then X is said to be *smaller than Y in the reversed hazard rate order* (and written as $X \leqslant_{\text{rh}} Y$) if $\tilde{r}_F(x) \leqslant \tilde{r}_G(x)$, for all x, or equivalently, if $F(x)/G(x)$ is decreasing in x. The reversed hazard rate ordering also implies stochastic ordering, but in general there are no implications between hazard rate and reversed hazard rate orderings. A random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is said to be smaller than another random vector $\mathbf{Y} = (Y_1, \ldots, Y_n)$ according to multivariate stochastic ordering (denoted by $\mathbf{X} \overset{\text{st}}{\preccurlyeq} \mathbf{Y}$) if $h(\mathbf{X}) \leqslant_{\text{st}} h(\mathbf{Y})$ for all increasing functions h. It is easy to see that multivariate stochastic ordering implies componentwise stochastic ordering. For more details on stochastic orderings, see Chapters 1 and 4 of Shaked and Shanthikumar (1994).

Let F^{-1} and G^{-1} be the right continuous inverses (quantile functions) of F and G, respectively. We say that X is less *dispersed* than Y (denoted by $X \leqslant_{\text{disp}} Y$) if $F^{-1}(\beta) - F^{-1}(\alpha) \leqslant G^{-1}(\beta) - G^{-1}(\alpha)$, for all $0 \leqslant \alpha \leqslant \beta \leqslant 1$. A consequence of $X \leqslant_{\text{disp}} Y$ is that $|X_1 - X_2| \leqslant_{\text{st}} |Y_1 - Y_2|$, which in turn implies that $\text{var}(X) \leqslant \text{var}(Y)$ as well as $E[|X_1 - X_2|] \leqslant E[|Y_1 - Y_2|]$, where X_1, X_2 (Y_1, Y_2) are two independent copies of X (Y). For details, see Section 2.B of Shaked and Shanthikumar (1994).

One of the basic tools in establishing various inequalities in statistics and probability is the notion of majorization. Let $\{x_{(1)} \leqslant \cdots \leqslant x_{(n)}\}$ denote the increasing arrangement of the components of a vector $\mathbf{x} = (x_1, \ldots, x_n)$. A vector $\mathbf{x}$ is said to majorize another vector $\mathbf{y}$ (written $\mathbf{x} \overset{\text{m}}{\succcurlyeq} \mathbf{y}$) if $\sum_{i=1}^{j} x_{(i)} \leqslant \sum_{i=1}^{j} y_{(i)}$ for $j = 1, \ldots, n-1$ and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$. Functions that preserve the majorization ordering are called Schur-convex functions. Marshall and Olkin (1979) provide extensive and comprehensive details on the theory of majorization and its applications in statistics. A vector $\mathbf{x}$ in $\mathbb{R}^{+^n}$ is said to be *p-larger* than another vector $\mathbf{y}$ also in $\mathbb{R}^{+^n}$ (written $\mathbf{x} \overset{p}{\succcurlyeq} \mathbf{y}$) if $\prod_{i=1}^{j} x_{(i)} \leqslant \prod_{i=1}^{j} y_{(i)}$, $j = 1, \ldots, n$. As shown in Khaledi and Kochar (2002) when $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{+^n}$, $\mathbf{x} \overset{\text{m}}{\succcurlyeq} \mathbf{y} \Longrightarrow \mathbf{x} \overset{p}{\succcurlyeq} \mathbf{y}$. The converse is, however, not true. The proofs of some of the results in this paper hinge on the following result.

Theorem 1.1 (*Marshall and Olkin, 1979, p. 57*). *Let $I \subset \mathbb{R}$ be an open interval and let $\phi : I^n \to \mathbb{R}$ be continuously differentiable. Necessary and sufficient conditions for ϕ to be Schur-convex on I^n are: ϕ is symmetric on I^n and for all $i \neq j$,*

$$(z_i - z_j)[\phi_{(i)}(z) - \phi_{(j)}(z)] \geqslant 0 \quad \textit{for all } z \in I^n,$$

where $\phi_{(i)}(z)$ denotes the partial derivative of ϕ with respect to its ith argument.

The organization of this paper is as follows. In Section 2, we stochastically compare the order statistics corresponding to two sets of independent Weibull random variables with a common shape parameter but when their scale parameters majorize each other. In Section 3, we prove some new results for the order statistics corresponding to two random samples when their distributions belong to the proportional hazards family. We then use these results to get bounds on the hazard rates of k-out-of-n systems made out of independent components with Weibull distributions. We also obtain a bound on the survival function of the sample range when the component lifetimes differ in their scale parameters.

2. Stochastic comparisons of order statistics from Weibull distributions

In this section, we study the stochastic properties of order statistics associated with independent random variables $X_1, \ldots, X_n$ when $X_i \sim W(\alpha, \lambda_i)$ for $i = 1, \ldots, n$. It is of interest to investigate the effect on the survival function, the hazard rate function and other characteristics of the time to failure of a system consisting of such components when we switch the vector $(\lambda_1, \ldots, \lambda_n)$ to another vector say $(\lambda_1^*, \ldots, \lambda_n^*)$.

Pledger and Proschan (1971) proved the following result for the PHR model which contains exponential distributions as a special case.

Theorem 2.1. *Let $(X_1, \ldots, X_n)$ and $(X_1^*, \ldots, X_n^*)$ be two random vectors of independent lifetimes with proportional hazards and with $(\lambda_1, \ldots, \lambda_n)$ and $(\lambda_1^*, \ldots \lambda_n^*)$ as the constants of proportionality. Then*

$$\boldsymbol{\lambda} \overset{\mathrm{m}}{\succcurlyeq} \boldsymbol{\lambda}^* \Longrightarrow X_{(i)} \geqslant_{\mathrm{st}} X_{(i)}^*, \quad i = 1, \ldots, n. \tag{2.1}$$

Proschan and Sethuraman (1976) extended this result from componentwise stochastic ordering to multivariate stochastic ordering. That is, under the assumptions of Theorem 2.1, they proved that

$$(X_{(1)}, \ldots, X_{(n)}) \overset{\mathrm{st}}{\succcurlyeq} (X_{(1)}^*, \ldots, X_{(n)}^*). \tag{2.2}$$

It follows from Theorem 2.1 that in the case of Weibull distributions with a common shape parameter α and with scale parameters as $(\lambda_1, \ldots, \lambda_n)$ and $(\lambda_1^*, \ldots, \lambda_n^*)$, (2.1) and (2.2) hold if $(\lambda_1^\alpha, \ldots, \lambda_n^\alpha) \overset{\mathrm{m}}{\succcurlyeq} (\lambda_1^{*\alpha}, \ldots, \lambda_n^{*\alpha})$. In the next theorem, we prove that a similar result holds in the Weibull case also when the two original vectors of scale parameters majorize each other and $0 < \alpha \leqslant 1$.

Theorem 2.2. *Let $X_1, \ldots, X_n$ be independent random variables with $X_i \sim W(\alpha, \lambda_i)$, $i = 1, \ldots, n$. Let $X_1^*, \ldots, X_n^*$ be another set of independent random variables with $X_i^* \sim W(\alpha, \lambda_i^*)$, $i = 1, \ldots, n$. Then for $0 < \alpha \leqslant 1$,*

$$\boldsymbol{\lambda} \overset{\mathrm{m}}{\succcurlyeq} \boldsymbol{\lambda} \Rightarrow (X_{(1)}, \ldots, X_{(n)}) \overset{\mathrm{st}}{\succcurlyeq} (X_{(1)}^*, \ldots, X_{(n)}^*).$$

Proof. First we prove the result for $n = 2$. According to Theorem 5.4.13 of Barlow and Proschan (1975), in order to prove the required result, it is sufficient to prove that for $0 < \alpha \leqslant 1$,

(a) $X_{(1)} \geqslant_{\mathrm{st}} X_{(1)}^*$,
(b) for $x \leqslant x'$, $\{X_{(2)} | X_{(1)} = x\} \leqslant_{\mathrm{st}} \{X_{(2)} | X_{(1)} = x'\}$, and
(c) $\{X_{(2)} | X_{(1)} = x\} \geqslant_{\mathrm{st}} \{X_{(2)}^* | X_{(1)}^* = x\}$.

Proving (a) is equivalent to proving that $\bar{F}_{X_{(1)}}(x)$, the survival function of $X_{(1)}$, is Schur-convex in (λ_1, λ_2). To prove it, we use Theorem 1.1. The partial derivative of $\bar{F}_{X_{(1)}}(x)$ with respect to λ_i is

$$\frac{\partial \bar{F}_{X_{(1)}}(x)}{\partial \lambda_i} = -\alpha\lambda_i^{\alpha-1}x^{\alpha}\mathrm{e}^{-x^{\alpha}(\lambda_1^{\alpha}+\lambda_2^{\alpha})}, \quad i=1,2.$$

This leads to

$$(\lambda_1-\lambda_2)\left(\frac{\partial \bar{F}_{X_{(1)}}}{\partial \lambda_1} - \frac{\partial \bar{F}_{X_{(2)}}}{\partial \lambda_2}\right) \geqslant 0,$$

thus proving (a).

The conditional survival function of $X_{(2)}|X_{(1)}=x$,

$$\begin{aligned}\bar{F}_{X_{(2)}|X_{(1)}=x}(z) &= \frac{\lambda_1^{\alpha}\mathrm{e}^{-(x\lambda_1)^{\alpha}-(z\lambda_2)^{\alpha}} + \lambda_2^{\alpha}\mathrm{e}^{-(x\lambda_2)^{\alpha}-(z\lambda_1)^{\alpha}}}{(\lambda_1^{\alpha}+\lambda_2^{\alpha})(\mathrm{e}^{-(x\lambda_1)^{\alpha}-(x\lambda_2)^{\alpha}})}\\ &= \frac{\lambda_1^{\alpha}}{\lambda_1^{\alpha}+\lambda_2^{\alpha}}\,\mathrm{e}^{-(z\lambda_2)^{\alpha}+(x\lambda_2)^{\alpha}} + \frac{\lambda_2^{\alpha}}{\lambda_1^{\alpha}+\lambda_2^{\alpha}}\,\mathrm{e}^{-(z\lambda_1)^{\alpha}+(x\lambda_1)^{\alpha}},\end{aligned}$$

is increasing in x, thus proving (b). Proving (c) is equivalent to proving that $\bar{F}_{X_{(2)}|X_{(1)}=x}(z)$ is Schur convex in (λ_1, λ_2). Its partial derivatives with respect to λ_1 and λ_2, respectively, are

$$\begin{aligned}\frac{\partial}{\partial \lambda_1}\,\bar{F}_{X_{(2)}|X_{(1)}=x}(z) &= \frac{\alpha\lambda_1^{\alpha-1}(\lambda_1^{\alpha}+\lambda_2^{\alpha}) - \alpha\lambda_1^{2\alpha-1}}{(\lambda_1^{\alpha}+\lambda_2^{\alpha})^2}\,\mathrm{e}^{-\lambda_2^{\alpha}(z^{\alpha}-x^{\alpha})} - \frac{\alpha\lambda_1^{\alpha-1}\lambda_2^{\alpha}}{(\lambda_1^{\alpha}+\lambda_2^{\alpha})^2}\\ &\quad\times \mathrm{e}^{-\lambda_1^{\alpha}(z^{\alpha}-x^{\alpha})} - (z^{\alpha}-x^{\alpha})\alpha\lambda_1^{\alpha-1}\mathrm{e}^{-\lambda_1^{\alpha}(z^{\alpha}-x^{\alpha})}\,\frac{\lambda_2^{\alpha}}{\lambda_1^{\alpha}+\lambda_2^{\alpha}}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial \lambda_2}\,\bar{F}_{X_{(2)}|X_{(1)}=x}(z) &= \frac{\alpha\lambda_2^{\alpha-1}(\lambda_1^{\alpha}+\lambda_2^{\alpha}) - \alpha\lambda_2^{2\alpha-1}}{(\lambda_1^{\alpha}+\lambda_2^{\alpha})^2}\,\mathrm{e}^{-\lambda_1^{\alpha}(z^{\alpha}-x^{\alpha})} - \frac{\alpha\lambda_2^{\alpha-1}\lambda_1^{\alpha}}{(\lambda_1^{\alpha}+\lambda_2^{\alpha})^2}\\ &\quad\times \mathrm{e}^{-\lambda_2^{\alpha}(z^{\alpha}-x^{\alpha})} - (z^{\alpha}-x^{\alpha})\alpha\lambda_2^{\alpha-1}\mathrm{e}^{-\lambda_2^{\alpha}(z^{\alpha}-x^{\alpha})}\,\frac{\lambda_1^{\alpha}}{\lambda_1^{\alpha}+\lambda_2^{\alpha}}.\end{aligned}$$

Now the difference between these two derivatives is

$$\begin{aligned}&\frac{\partial}{\partial \lambda_1}\,\bar{F}_{X_{(2)}|X_{(1)}=x}(z) - \frac{\partial}{\partial \lambda_2}\,\bar{F}_{X_{(2)}|X_{(1)}=x}(z)\\ &\quad= \frac{\alpha\lambda_1^{\alpha-1}\lambda_2^{\alpha-1}}{(\lambda_1^{\alpha}+\lambda_2^{\alpha})^2}\{\mathrm{e}^{-\lambda_2^{\alpha}(z^{\alpha}-x^{\alpha})}(\lambda_1+\lambda_2+\lambda_1(z^{\alpha}-x^{\alpha})(\lambda_1^{\alpha}+\lambda_2^{\alpha}))\\ &\quad - \mathrm{e}^{\lambda_1^{\alpha}(z^{\alpha}-x^{\alpha})}(\lambda_1+\lambda_2+\lambda_2(z^{\alpha}-x^{\alpha})(\lambda_1^{\alpha}+\lambda_2^{\alpha}))\}.\end{aligned}$$

If $\lambda_1>\lambda_2$, then $\mathrm{e}^{-\lambda_2^{\alpha}(z^{\alpha}-x^{\alpha})} \geqslant \mathrm{e}^{-\lambda_1^{\alpha}(z^{\alpha}-x^{\alpha})}$, since $z>x$. If $\lambda_1<\lambda_2$, then the above inequality is reversed. That is,

$$(\lambda_1-\lambda_2)\left(\frac{\partial}{\partial \lambda_1}\,\bar{F}_{X_{(2)}|X_{(1)}=x}(z) - \frac{\partial}{\partial \lambda_2}\,\bar{F}_{X_{(2)}|X_{(1)}=x}(z)\right) \geqslant 0.$$

The proof of part (c) again follows from Theorem 1.1. This completes the proof in the case of $n=2$. The proof for $n>2$ follows from this and using similar kinds of arguments as used in Theorem 3.4 of Proschan and Sethuraman (1976). □

For comparing two series systems with independent Weibull components, we have the following stronger result.

Theorem 2.3. *Let $X_1,\ldots,X_n$ be independent random variables with $X_i \sim W(\alpha,\lambda_i)$, $i=1,\ldots,n$. Let $X_1^*,\ldots,X_n^*$ be another set of independent random variables with $X_i^* \sim W(\alpha,\lambda_i^*)$, $i=1,\ldots,n$. Then $\boldsymbol{\lambda} \overset{\mathrm{m}}{\succcurlyeq} \boldsymbol{\lambda}^*$ implies that $X_{(1)} \geqslant_{\mathrm{hr}} X_{(1)}^*$ for $0<\alpha\leqslant 1$ and $X_{(1)} \leqslant_{\mathrm{hr}} X_{(1)}^*$ for $\alpha \geqslant 1$.*

Proof. The hazard rate of $X_{(1)}$ is

$$r_{X_{(1)}}(x;\lambda_1,\ldots\lambda_n)=\sum_{i=1}^{n}\alpha x^{\alpha-1}\lambda_i^{\alpha}.$$

The function $g(\lambda)=\alpha x^{\alpha-1}\lambda_i^{\alpha}$ is concave (convex) in λ for $0<\alpha\leqslant 1$ ($\alpha\geqslant 1$). It follows from Proposition C.1. of Marshall and Olkin (1979, p. 64) that $\sum_{i=1}^{n} g(\lambda_i)$ is Schur concave (convex). This completes the proof. □

It will be interesting to see whether the above result can be extended to other order statistics. Boland et al. (1994) have proved that in case $\alpha=1$, such a result does not hold for parallel systems with more than two components.

3. Some new results for the PHR model

Khaledi and Kochar (2000a) studied the problem of stochastically comparing the largest order statistic of a set of n independent and non-identically distributed exponential random variables with that corresponding to a set of n independent and identically distributed exponential random variables. They proved the following result.

Theorem 3.1. *Let $X_1,\ldots,X_n$ be independent exponential random variables with X_i having hazard rate λ_i, $i=1,\ldots,n$. Let $Y_1,\ldots,Y_n$ be a random sample of size n from an exponential distribution with common hazard rate $\tilde{\lambda}=(\prod_{i=1}^{n}\lambda_i)^{1/n}$. Then*

(a) $X_{(n)} \geqslant_{\mathrm{hr}} Y_{(n)}$ *and*
(b) $X_{(n)} \geqslant_{\mathrm{disp}} Y_{(n)}$.

In Theorem 3.3, we extend this result from exponential to the PHR model. To prove this we need the following result due to Rojo and He (1991).

Theorem 3.2. *Let X and Y be two random variables such that $X \leqslant_{\mathrm{st}} Y$. Then $X \leqslant_{\mathrm{disp}} Y$ implies that $\gamma(X) \leqslant_{\mathrm{disp}} \gamma(Y)$ where γ is a nondecreasing convex function.*

Theorem 3.3. *Let* $X_1, \ldots, X_n$ *be independent random variables with* X_i *having survival function* $\overline{F}^{\lambda_i}(x), i=1, \ldots, n$. *Let* $Y_1, \ldots, Y_n$ *be a random sample of size n from a distribution with survival function* $\overline{F}^{\tilde{\lambda}}(x)$, *where* $\tilde{\lambda} = (\prod_{i=1}^{n} \lambda_i)^{1/n}$. *Then*

(a) $X_{(n)} \geqslant_{\text{hr}} Y_{(n)}$ *and*
(b) *if F is* DFR, *then* $X_{(n)} \geqslant_{\text{disp}} Y_{(n)}$.

Proof. (a) Let $H(x) = -\log \overline{F}(x)$ denote the cumulative hazard of F. Let $Z_i = H(X_i)$, $i=1, \ldots, n$, and $W_i = H(Y_i)$, $i=1, \ldots, n$. Since the X_i's follow the PHR model, it is easy to show that Z_i is exponential with hazard rate λ_i, $i=1, \ldots, n$. Similarly, W_i is exponential with hazard rate $\tilde{\lambda}$, $i=1, \ldots, n$. It follows from Theorem 3.1(a) that $Z_{(n)} \geqslant_{\text{hr}} W_{(n)}$. Using this fact (since H^{-1}, the right inverse of H, is nondecreasing), it is easy to show that $H^{-1}(Z_{(n)}) \geqslant_{\text{hr}} H^{-1}(W_{(n)})$, from which part (a) follows.

(b) Theorem 3.1(a) and (b), respectively, imply that $Z_{(n)} \geqslant_{\text{st}} W_{(n)}$ and $Z_{(n)} \geqslant_{\text{disp}} W_{(n)}$. The function $H^{-1}(x)$ is convex, since F is DFR, and is nondecreasing. Using these observations, it follows from Theorem 3.2 that $H^{-1}(Z_{(n)}) \geqslant_{\text{disp}} H^{-1}(W_{(n)})$, which is equivalent to $X_{(n)} \geqslant_{\text{disp}} Y_{(n)}$. □

We show with the help of the next example that the DFR condition in the above theorem cannot be dispensed with.

Example 3.1. Let X_1 and X_2 be independent random variables with X_i having survival function $\overline{F}_i(x) = (1-x)^{\lambda_i}$, $0 \leqslant x \leqslant 1$, $i = 1, 2$. Let Y_1 and Y_2 be independent random variables with common survival function $\overline{G}(x) = (1-x)^{(\lambda_1 \lambda_2)^{1/2}}$, $0 \leqslant x \leqslant 1$. Let $\lambda_1 = 1$ and $\lambda_2 = 4$. Under this setting, it is easy to find that $\text{var}(X_{(2)}) = \frac{43}{720} < \frac{11}{225} = \text{var}(Y_{(2)})$, from which it follows that part (b) of Theorem 3.3 may not hold for the case when F, the baseline distribution, is not DFR. Note that in this example F being uniform distribution on (0, 1) is IFR.

Corollary 3.1. *Let* $X_1, \ldots, X_n$ *be independent random variables with* $X_i \sim W(\alpha, \lambda_i)$, $i = 1, \ldots, n$. *Let* $Y_1, \ldots, Y_n$ *be a random sample of size n from a* $W(\alpha, \tilde{\lambda})$ *distribution, where* $\tilde{\lambda} = (\prod_{i=1}^{n} \lambda_i)^{1/n}$. *Then*

(a) *for any* $\alpha > 0$, $X_{(n)} \geqslant_{\text{hr}} Y_{(n)}$,
(b) *for* $0 < \alpha \leqslant 1$, $X_{(n)} \geqslant_{\text{disp}} Y_{(n)}$, *and*
(c) *for* $0 < \alpha \leqslant 1$, $X_{(n)} - X_{(1)} \geqslant_{\text{st}} Y_{(n)} - Y_{(1)}$.

Proof. The proof of (a) and (b) follows from Theorem 3.3 since for any $\alpha > 0$, the geometric mean of $\lambda_1^{\alpha}, \ldots, \lambda_n^{\alpha}$ is $\tilde{\lambda}^{\alpha}$ and the fact that the Weibull distribution is DFR when $0 < \alpha \leqslant 1$. The proof of (c) follows from Theorem 3.1 of Khaledi and Kochar (2000b). □

In Theorem 3.4 below, we prove that for the largest order statistic, the conclusion of Theorem 2.1 holds under the weaker p-larger ordering. To prove this, we use the following lemma.

Lemma 3.1 (Khaledi and Kochar, 2002). *The function* $\psi : \mathbb{R}^{+^n} \to \mathbb{R}$ *satisfies*

$$\mathbf{x} \overset{p}{\succcurlyeq} \mathbf{y} \Longrightarrow \psi(\mathbf{x}) \geqslant \psi(\mathbf{y}) \tag{3.1}$$

if and only if

(i) $\psi(e^{a_1}, \ldots, e^{a_n})$ *is Schur-convex in* $(a_1, \ldots, a_n)$ *and*
(ii) $\psi(e^{a_1}, \ldots, e^{a_n})$ *is decreasing in* a_i, *for* $i = 1, \ldots, n$,

where $a_i = \log x_i$, *for* $i = 1, \ldots, n$.

Theorem 3.4. *Let* $X_1, \ldots, X_n$ *be independent random variables with* X_i *having survival function* $\overline{F}^{\lambda_i}(x)$, $i = 1, \ldots, n$. *Let* $X_1^*, \ldots, X_n^*$ *be another set of independent random variables with* X_i^* *having survival function* $\overline{F}^{\lambda_i^*}(x)$, $i = 1, \ldots, n$. *Then*

$$\boldsymbol{\lambda} \overset{p}{\succcurlyeq} \boldsymbol{\lambda}^* \Longrightarrow X_{(n)} \geqslant_{\text{st}} X_{(n)}^*.$$

Proof. The survival function of $X_{(n)}$ can be written as

$$\overline{F}_{X_{(n)}}(x) = 1 - \prod_{i=1}^{n} (1 - e^{-e^{a_i} H(x)}), \tag{3.2}$$

where $a_i = \log \lambda_i$, $i = 1, \ldots, n$ and $H(x) = -\log \overline{F}(x)$.

Using Lemma 3.1, we find that it is enough to show that the function $\overline{F}_{X_{(n)}}$ given by (3.2) is Schur-convex and decreasing in a_i's. To prove its Schur-convexity, it follows from Theorem 1.1 that we have to show that for $i \neq j$,

$$(a_i - a_j)\left(\left(\frac{\partial \overline{F}_{X_{(n)}}}{\partial a_i}\right) - \left(\frac{\partial \overline{F}_{X_{(n)}}}{\partial a_j}\right)\right) \geqslant 0.$$

That is, for $i \neq j$,

$$H(x)(a_i - a_j)\left(\prod_{i=1}^{n}(1 - e^{-e^{a_i} H(x)})\right)\left(\frac{e^{a_j} e^{-e^{a_j} H(x)}}{1 - e^{-e^{a_j} H(x)}} - \frac{e^{a_i} e^{-e^{a_i} H(x)}}{1 - e^{-e^{a_i} H(x)}}\right) \geqslant 0. \tag{3.3}$$

It is easy to see that the function $b e^{-bH(x)}/(1 - e^{-bH(x)})$ is decreasing in b, for each fixed $x > 0$. Replacing b with e^{a_i}, it follows that the function $e^{a_i} e^{-e^{a_i} H(x)}/(1 - e^{-e^{a_i} H(x)})$ is also decreasing in a_i for $i = 1, \ldots, n$. This proves that (3.3) holds. The partial derivative of $\overline{F}_{X_{(n)}}$ with respect to a_i is negative, which in turn implies that the survival function of $X_{(n)}$ is decreasing in a_i for $i = 1, \ldots, n$. This completes the proof. □

Since for any $\alpha > 0$,

$$(\lambda_1, \ldots, \lambda_n) \overset{p}{\succcurlyeq} (\lambda_1^*, \ldots, \lambda_n^*) \Leftrightarrow (\lambda_1^\alpha, \ldots, \lambda_n^\alpha) \overset{p}{\succcurlyeq} (\lambda_1^{*\alpha}, \ldots, \lambda_n^{*\alpha}),$$

the proof of the following corollary immediately follows from the above theorem.

Corollary 3.2. *Let* $X_1, \ldots, X_n$ *be independent random variables with* $X_i \sim W(\alpha, \lambda_i)$, $i = 1, \ldots, n$. *Let* $X_1^*, \ldots, X_n^*$ *be another set of independent random variables with* $X_i^* \sim W(\alpha, \lambda_i^*)$, $i = 1, \ldots, n$. *Then for any* $\alpha > 0$,

$$\boldsymbol{\lambda} \overset{p}{\succcurlyeq} \boldsymbol{\lambda}^* \Longrightarrow X_{(n)} \geqslant_{\text{st}} X_{(n)}^*.$$

Khaledi and Kochar (2000a) proved a special case of the above corollary when $\alpha = 1$. Boland et al. (1994) pointed out that for $n > 2$, (2.1) cannot be strengthened from stochastic ordering to hazard rate ordering. Since majorization implies p-larger ordering, it follows that, in general, Theorem 3.4 cannot be strengthened to hazard rate ordering.

As shown in the next example, a result similar to Theorem 3.4 may not hold for other order statistics.

Example 3.2. Let X_1, X_2, X_3 be independent exponential random variables with $\boldsymbol{\lambda}^* = (0.1, 1, 7.9)$ and X_1^*, X_2^*, X_3^* be independent exponential random variables with $\boldsymbol{\lambda} = (1, 2, 5)$. It is easy to see that $\boldsymbol{\lambda} \overset{p}{\succcurlyeq} \boldsymbol{\lambda}^*$. Then $X_{(1)}$ and $X_{(1)}^*$ have exponential distributions with respective hazard rates 9 and 8, which implies that $X_{(1)}^* \geqslant_{\text{st}} X_{(1)}$.

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