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On the right spread order of convolutions of heterogeneous exponential random variables

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ABSTRACT

A sufficient condition for comparing convolutions of heterogeneous exponential random variables in terms of right spread order is established. As a consequence, it is shown that a convolution of heterogeneous independent exponential random variables is more skewed than that of homogeneous exponential random variables in the sense of NBUE order. This gives a new insight into the distribution theory of convolutions of independent random variables. A sufficient condition is also derived for comparing such convolutions in terms of Lorenz order.

1. Introduction

The exponential distribution is one of the most popular distributions in probability and statistics. In reliability theory, it is well known for its "non-aging" property and also has many interesting applications in operation research. It has also been widely used in queuing theory, survival analysis and physics. Please refer to [1,2] for more details. The convolution of exponential random variables has attracted considerable attention in the literature due to its typical applications in many areas. For example, in reliability theory, it arises in the study of redundant standby systems with exponential components (cf. [3]); in queuing theory, it is used to model the total service time of an agent in a system; in insurance, it is used to model total claims on a number of policies in the individual risk model (cf. [4]).

Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i, $i = 1, \ldots, n$, and $Y_1, \ldots, Y_n$ be another set of independent exponential random variables with Y_i having hazard rate λ_i', $i = 1, \ldots, n$. Boland et al. [5] showed that under the condition of the majorization order,

$$(\lambda_1, \ldots, \lambda_n) \succeq_m (\lambda_1', \ldots, \lambda_n') \Longrightarrow \sum_{i=1}^n X_i \geq_{\text{lr}} \sum_{i=1}^n Y_i.$$

(Stochastic orders mentioned in this section will be reviewed in Section 2.) Under the same condition, Kochar and Ma [6] proved that

$$(\lambda_1, \ldots, \lambda_n) \succeq_{\text{m}} (\lambda_1', \ldots, \lambda_n') \Longrightarrow \sum_{i=1}^n X_i \geq_{\text{disp}} \sum_{i=1}^n Y_i. \tag{1.1}$$

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This topic has been extensively investigated by Bon and Păltănea [3]. They pointed out that, under the p order, which is a weaker order than the majorization order,

$$(\lambda_1, \ldots, \lambda_n) \overset{p}{\succeq} (\lambda'_1, \ldots, \lambda'_n) \Longrightarrow \sum_{i=1}^{n} X_i \geq_{\text{hr}} \sum_{i=1}^{n} Y_i.$$

This result has been strengthened by Khaledi and Kochar [7] as

$$(\lambda_1, \ldots, \lambda_n) \overset{p}{\succeq} (\lambda'_1, \ldots, \lambda'_n) \Longrightarrow \sum_{i=1}^{n} X_i \geq_{\text{disp}} \sum_{i=1}^{n} Y_i. \tag{1.2}$$

More recently, Zhao and Balakrishnan [8] proved that, under the condition of reciprocal order,

$$(\lambda_1, \ldots, \lambda_n) \overset{\text{rm}}{\succeq} (\lambda'_1, \ldots, \lambda'_n) \Longrightarrow \sum_{i=1}^{n} X_i \geq_{\text{mrl}} \sum_{i=1}^{n} Y_i. \tag{1.3}$$

The right spread order was introduced by Fernández-Ponce et al. [9], and independently by Shaked and Shanthikumar [10], where it was termed as *excess wealth* order. The right spread order serves as a very useful tool for comparing variability in two distributions. It has also been widely used in insurance, reliability theory and economics. One may refer to Section 3.C of [11] for a comprehensive review.

Observing (1.2), one natural question is to find sufficient condition on the parameters which will imply right spread order (which is weaker than dispersive order) between two such convolutions of independent exponential random variables. This has been an open problem for a long time because it is very complicated to check it from the definition of right spread order. In this paper, we will solve this problem by showing that,

$$(\lambda_1, \ldots, \lambda_n) \overset{\text{rm}}{\succeq} (\lambda'_1, \ldots, \lambda'_n) \Longrightarrow \sum_{i=1}^{n} X_i \geq_{\text{RS}} \sum_{i=1}^{n} Y_i. \tag{1.4}$$

This new result not only complements the existing results on variability between convolutions in the literature, it also gives a new insight into the distribution theory of convolutions. Using (1.4), it is further proved that,

$$\left(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n}\right) \succeq_m \left(\frac{1}{\lambda'_1}, \ldots, \frac{1}{\lambda'_n}\right) \Longrightarrow \sum_{i=1}^{n} X_i \geq_{\text{NBUE}} \sum_{i=1}^{n} Y_i,$$

where "$\geq_{\text{NBUE}}$" means NBUE order. NBUE order is a partial order to compare the aging of two distributions. It can be used to compare relative skewness in two distributions (cf. [12,13]). It will be shown that the convolution of heterogeneous exponential random variables is more skewed than that of homogeneous exponential random variables. Since NBUE order implies Lorenz ordering, we prove

$$\left(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n}\right) \succeq_m \left(\frac{1}{\lambda'_1}, \ldots, \frac{1}{\lambda'_n}\right) \Longrightarrow \sum_{i=1}^{n} X_i \geq_{\text{Lorenz}} \sum_{i=1}^{n} Y_i,$$

where "$\geq_{\text{Lorenz}}$" is the well-known Lorenz order, which may be of independent interest in economics.

The rest of paper is organized as follows. Section 2 introduces some notions of stochastic orders and majorization orders. Comparing convolutions in terms of right spread order is investigated in Section 3. Section 4 gives some interesting applications of the main results. In the last section, we present some discussion for further work.

Throughout this paper, the notions *increasing* and *decreasing* mean *nondecreasing* and *nonincreasing*, respectively.

2. Preliminaries

In this section, we will review some notions of stochastic orders and majorization orders, which are used in this paper.

Assume that random variables X and Y have distribution functions F and G, survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, and density functions f and g, respectively.

The following order, called dispersive order, is used to compare the variability of two distributions.

Definition 2.1. X is said to be less dispersed than Y (denoted by $X \leq_{\text{disp}} Y$) if

$$F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha)$$

for all $0 < \alpha \leq \beta < 1$, where F^{-1} and G^{-1} denote their corresponding right continuous inverses.

A weaker order, called right spread order, has also been proposed to compare the variability of two distributions.

Definition 2.2. X is said to be less right spread than Y (denoted by $X \le_{\text{RS}} Y$) if

$$\int_{F^{-1}(p)}^{\infty} \bar{F}(x)\,\mathrm{d}x \le \int_{G^{-1}(p)}^{\infty} \bar{G}(x)\mathrm{d}x, \quad \text{for all } 0 \le p \le 1.$$

Define

$$W_X(t) = \int_t^{\infty} \bar{F}_X(u)\mathrm{d}u,$$

and

$$W_Y(t) = \int_t^{\infty} \bar{F}_Y(u)\mathrm{d}u.$$

It is seen that $X \le_{\text{RS}} Y$, if and only if, (cf., [11], p. 164),

$$W_Y^{-1}(z) - W_X^{-1}(z) \quad \text{is decreasing in } z \ge 0. \tag{2.1}$$

It is known that

$$X \le_{\text{disp}} Y \Longrightarrow X \le_{\text{RS}} Y \Longrightarrow \text{Var}(X) \le \text{Var}(Y).$$

A shape order closely related to right spread order, called NBUE order, is defined as follows.

Definition 2.3. X is said to be more NBUE (new better than used in expectation) than Y or X is smaller than Y in the NBUE order (written as $X \le_{\text{NBUE}} Y$) if

$$\frac{1}{\mu_F}\int_{F^{-1}(u)}^{\infty} \bar{F}(x)\mathrm{d}x \le \frac{1}{\mu_G}\int_{G^{-1}(u)}^{\infty} \bar{G}(x)\mathrm{d}x, \quad \text{for all } u \in (0, 1],$$

where $\mu_F(\mu_G)$ denotes the expectation of $X(Y)$. Note that X is NBUE if and only if $X \le_{\text{NBUE}} Y$, where Y is an exponential random variable. The NBUE order has a good interpretation in reliability theory since it compares relative aging of two systems in the sense of more "NBUE".

When $\text{E}(X) = \text{E}(Y)$, the order $\le_{\text{RS}}$ is equivalent to the order $\le_{\text{NBUE}}$. However, they are distinct when $\text{E}(X) \ne \text{E}(Y)$. For more details, please refer to [14].

It is pointed out in [15] that the NBUE order implies the HNBUE order, which is equivalent to the Lorenz order, a well-known order in economics. It is also known that ([13], p. 69),

$$X \le_{\text{Lorenz}} Y \Longrightarrow cv(X) \le cv(Y),$$

where $cv(X)(cv(Y))$ denotes the coefficient of variation of $X(Y)$. The NBUE and HNBUE orders were introduced in [12] and studied further in [15]. For more details on shape orders measuring skewness, please refer to Section 2.C of Marshall and Olkin [13] and Shaked and Shanthikumar [11].

The following orders are usually used to compare the magnitude of random variables.

Definition 2.4 (*Shaked and Shanthikumar [11]*). X is said to be smaller than Y in the

(i) likelihood ratio order (denoted by $X \le_{\text{lr}} Y$) if $g(x)/f(x)$ is increasing in x;
(ii) hazard rate order (denoted by $X \le_{\text{hr}} Y$) if $\bar{G}(x)/\bar{F}(x)$ is increasing in x;
(iii) mean residual life order, denoted by $X \le_{\text{mrl}} Y$, if

$$\frac{\int_t^{\infty} \bar{F}(x)\mathrm{d}x}{\bar{F}(t)} \le \frac{\int_t^{\infty} \bar{G}(x)\mathrm{d}x}{\bar{G}(t)}.$$

It is known that (cf. [11]),

$$X \le_{\text{lr}} Y \Rightarrow X \le_{\text{hr}} Y \Rightarrow X \le_{\text{mrl}} Y \Rightarrow \text{E}X \le \text{E}Y.$$

We shall also be using the concept of majorization in our discussion. Let $\{x_{(1)}, x_{(2)}, \ldots, x_{(n)}\}$ denote the increasing arrangement of the components of the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

Definition 2.5. The vector $\mathbf{x}$ is said to be majorized by the vector $\mathbf{y}$ (denoted by $\mathbf{x} \preceq_m \mathbf{y}$) if

$$\sum_{i=1}^{j} x_{(i)} \ge \sum_{i=1}^{j} y_{(i)}$$

for $j = 1, \ldots, n-1$ and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$.

Two weak majorization orders follow by relaxing the equality condition.

Definition 2.6. The vector $\mathbf{x}$ is said to be

- weakly supermajorized by vector $\mathbf{y}$ (denoted by $\mathbf{x} \overset{w}{\preceq} \mathbf{y}$) if

$$\sum_{i=1}^{j} x_{(i)} \geq \sum_{i=1}^{j} y_{(i)}$$

  for $j = 1, \ldots, n$;
- weakly submajorized by vector $\mathbf{y}$ (denoted by $\mathbf{x} \preceq_w \mathbf{y}$) if

$$\sum_{i=1}^{j} x_{[i]} \leq \sum_{i=1}^{j} y_{[i]}$$

  for $j = 1, \ldots, n$, where $\{x_{[1]}, x_{[2]}, \ldots, x_{[n]}\}$ denotes the decreasing arrangement of the components of the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

For extensive and comprehensive details on the theory of majorization orders and their applications, please refer to the excellent book of Marshall and Olkin [16].

Another interesting weaker order related to the majorization order introduced by Bon and Păltănea [3] is the p order.

Definition 2.7. A vector $\mathbf{x}$ in $\mathbb{R}^{+^n}$ is said to be p-smaller than another vector $\mathbf{y}$ in $\mathbb{R}^{+^n}$ (denoted by $\mathbf{x} \overset{p}{\preceq} \mathbf{y}$) if

$$\prod_{i=1}^{j} x_{(i)} \geq \prod_{i=1}^{j} y_{(i)}, \quad j = 1, \ldots, n.$$

Zhao and Balakrishnan [8] introduced the following order of reciprocal majorization.

Definition 2.8. A vector $\mathbf{x}$ in $\mathbb{R}^{+^n}$ is said to be reciprocally majorized by another vector $\mathbf{y}$ in $\mathbb{R}^{+^n}$ (denoted by $\mathbf{x} \overset{\mathrm{rm}}{\preceq} \mathbf{y}$) if

$$\sum_{i=1}^{j} \frac{1}{x_{(i)}} \leq \sum_{i=1}^{j} \frac{1}{y_{(i)}}, \quad j = 1, \ldots, n.$$

They also wondered about the relation between p order and the reciprocal order. In the following, we will answer this question.

Note that

$$\mathbf{x} \overset{p}{\preceq} \mathbf{y} \Longleftrightarrow (\log(x_1), \ldots, \log(x_n)) \overset{w}{\preceq} (\log(y_1), \ldots, \log(y_n)). \tag{2.2}$$

It is known that by A.2.g of Marshall and Olkin ([16], p. 117),

$$\mathbf{x} \overset{w}{\preceq} \mathbf{y} \Longrightarrow (\mathrm{e}^{-x_1}, \ldots, \mathrm{e}^{-x_n}) \preceq_w (\mathrm{e}^{-y_1}, \ldots, \mathrm{e}^{-y_n}). \tag{2.3}$$

Combining (2.2) and (2.3), it follows that,

$$\mathbf{x} \overset{p}{\preceq} \mathbf{y} \Longrightarrow \left(\frac{1}{x_1}, \ldots, \frac{1}{x_n}\right) \preceq_w \left(\frac{1}{y_1}, \ldots, \frac{1}{y_n}\right). \tag{2.4}$$

That is,

$$\mathbf{x} \overset{p}{\preceq} \mathbf{y} \Longrightarrow \mathbf{x} \overset{\mathrm{rm}}{\preceq} \mathbf{y}.$$

However, the converse is not true as pointed out in [8] through a counterexample.

3. Right spread order

Saunders and Moran [17] established an equivalent characterization of dispersive order in one parameter family. The following lemma gives an equivalent characterization of right spread order in one parameter family.

Lemma 3.1. *Let $\{F_a | a \in R\}$ be a class of distribution functions, such that F_a is supported on some interval $(x_-^{(a)}, x_+^{(a)}) \subseteq (0, \infty)$, where $x_-^{(a)}$ and $x_+^{(a)}$ mean the left and right end points, respectively. Then,*

$$X_{a^*} \leq_{\text{RS}} X_a, \quad a \leq a^*,$$

if and only if

$$\frac{W_a'(x)}{\bar{F}_a(x)} \quad \text{is decreasing in } x,$$

where W_a' is the derivative of W_a with respect to a.

Proof. According to (2.1), $X_{a^*} \leq_{\text{RS}} X_a$ for $a \leq a^*$, if and only if,

$$W_a^{-1}(\beta) - W_{a^*}^{-1}(\beta) \leq W_a^{-1}(\alpha) - W_{a^*}^{-1}(\alpha), \quad \beta \geq \alpha$$

i.e., for $\alpha \leq \beta$,

$$W_a^{-1}(\alpha) - W_a^{-1}(\beta) \tag{3.1}$$

is decreasing in a. This condition is equivalent to,

$$W_a(W_a^{-1}(\alpha) + c), \quad \text{decreasing in } a \text{ for } c \geq 0. \tag{3.2}$$

To see this, let $\alpha = W_a(W_a^{-1}(\beta) + c)$ as $W_a^{-1}(\cdot)$ is a decreasing function. If (3.2) holds, for any $\lambda \geq 0$,

$$W_{a+\lambda}\left(W_{a+\lambda}^{-1}(\beta) + c\right) \leq W_a\left(W_a^{-1}(\beta) + c\right) = \alpha = W_{a+\lambda}\left(W_{a+\lambda}^{-1}(\alpha)\right),$$

i.e., for $\alpha \leq \beta$,

$$W_{a+\lambda}^{-1}(\alpha) - W_{a+\lambda}^{-1}(\beta) \leq c = W_a^{-1}(\alpha) - W_a^{-1}(\beta), \quad \lambda \geq 0.$$

So, (3.2) implies (3.1). Reversing the above argument leads (3.1) to imply (3.2).

Next, by taking the derivative with respect to a, (3.2) is equivalent to

$$W_a'(W_a^{-1}(\alpha) + c) - \bar{F}_a(W_a^{-1}(\alpha) + c)W_a'^{-1}(\alpha) \leq 0, \tag{3.3}$$

where $W_a' = \partial W_a / \partial a$ and $W'^{-1}(\alpha) = \partial W_a^{-1} / \partial a$.

Observe that,

$$W_a(W_a^{-1}(\alpha)) = \alpha.$$

Taking the derivative with respect to a on both sides, it gives,

$$W_a'(W_a^{-1}(\alpha)) - \bar{F}_a(W_a^{-1}(\alpha))W_a'^{-1}(\alpha) = 0.$$

Use this equation for (3.3), it follows that, for $c \geq 0$,

$$\frac{W_a'(W_a^{-1}(\alpha) + c)}{\bar{F}_a(W_a^{-1}(\alpha) + c)} \leq \frac{W_a'(W_a^{-1}(\alpha))}{\bar{F}_a(W_a^{-1}(\alpha))}.$$

Hence, the required result follows immediately. ∎

Now, let us discuss convolutions of independent exponential random variables when $n = 2$.

Theorem 3.2. *Let $X_{\lambda_1}, X_{\lambda_2}, X_{\lambda_1'}, X_{\lambda_2'}$ be independent exponential random variables with parameters $\lambda_1, \lambda_2, \lambda_1', \lambda_2'$, respectively. Then,*

$$(\lambda_1, \lambda_2) \overset{\text{rm}}{\succeq} (\lambda_1', \lambda_2') \Longleftrightarrow X_{\lambda_1} + X_{\lambda_2} \geq_{RS} X_{\lambda_1'} + X_{\lambda_2'}.$$

Proof. "Sufficiency $\Longrightarrow$". Without loss of generality, assume that $\lambda_1 \leq \lambda_2$ and $\lambda_1' \leq \lambda_2'$. As

$$(\lambda_1, \lambda_2) \overset{\text{rm}}{\succeq} (\lambda_1', \lambda_2'),$$

it holds that

$$\frac{1}{\lambda_1} \geq \frac{1}{\lambda_1'},$$

and

$$\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\geq\frac{1}{\lambda_1'}+\frac{1}{\lambda_2'}. \tag{3.4}$$

Now, let us discuss several cases.

(i) *Case* 1:

$$\frac{1}{\lambda_2}\geq\frac{1}{\lambda_1'}+\frac{1}{\lambda_2'}.$$

Since

$$\frac{1}{\lambda_1}\geq\frac{1}{\lambda_1'}\Longrightarrow X_{\lambda_1}\geq_{\text{RS}}X_{\lambda_1'},$$

it holds that

$$X_{\lambda_1}\geq_{\text{RS}}X_{\lambda_1'},\qquad X_{\lambda_2}\geq_{\text{RS}}X_{\lambda_2'}.$$

From Theorem 3.C.7 of Shaked and Shanthikumar [11] (see also [18]), it follows that,

$$X_{\lambda_1}+X_{\lambda_2}\geq_{\text{RS}}X_{\lambda_1'}+X_{\lambda_2'}.$$

(ii) *Case* 2:

$$\frac{1}{\lambda_2}<\frac{1}{\lambda_1'}+\frac{1}{\lambda_2'},$$

but

$$\frac{1}{\lambda_1}\geq\frac{1}{\lambda_1'}+\frac{1}{\lambda_2'}.$$

Denote

$$\frac{1}{\lambda_1^*}=\frac{1}{\lambda_1'}+\frac{1}{\lambda_2'}.$$

From Theorem 3.C.8 of Shaked and Shanthikumar [11],

$$X_{\lambda_1}+X_{\lambda_2}\geq_{\text{RS}}X_{\lambda_1},$$

observing that,

$$\frac{1}{\lambda_1}\geq\frac{1}{\lambda_1^*}\Longrightarrow X_{\lambda_1}\geq_{\text{RS}}X_{\lambda_1^*},$$

it is sufficient to show

$$X_{\lambda_1^*}\geq_{\text{RS}}X_{\lambda_1'}+X_{\lambda_2'}.$$

Now, since

$$\frac{1}{\lambda_1^*}=\frac{1}{\lambda_1'}+\frac{1}{\lambda_2'},$$

i.e.,

$$\mathrm{E}\left(X_{\lambda_1^*}\right)=\mathrm{E}\left(X_{\lambda_1'}\right)+\mathrm{E}\left(X_{\lambda_2'}\right),$$

it holds that

$$X_{\lambda_1^*}\geq_{\text{RS}}X_{\lambda_1'}+X_{\lambda_2'}\Longleftrightarrow X_{\lambda_1^*}\geq_{\text{NBUE}}X_{\lambda_1'}+X_{\lambda_2'}.$$

Hence, it is enough to show that $X_{\lambda_1'}+X_{\lambda_2'}$ is NBUE as $X_{\lambda_1^*}$ is exponential. This will follow from the fact that the density function of convolution of exponential random variables is logconcave which implies NBUE property [1].

(iii) *Case* 3:

$$\frac{1}{\lambda_2}<\frac{1}{\lambda_1'}+\frac{1}{\lambda_2'},$$

and

$$\frac{1}{\lambda_1}<\frac{1}{\lambda_1'}+\frac{1}{\lambda_2'}.$$

By (3.4), let us assume that there exists an exponential random $X_{\lambda_2^*}$ which is independent of X_{λ_1}, where

$$\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\geq\frac{1}{\lambda_1}+\frac{1}{\lambda_2^*}=\frac{1}{\lambda_1'}+\frac{1}{\lambda_2'}.$$

Using the similar argument in Case 1, it holds that

$$X_{\lambda_1}+X_{\lambda_2}\geq_{\mathrm{RS}}X_{\lambda_1}+X_{\lambda_2^*}.$$

Hence, it is enough to show that

$$X_{\lambda_1}+X_{\lambda_2^*}\geq_{\mathrm{RS}}X_{\lambda_1'}+X_{\lambda_2'}.$$

So, for convenience, we can assume that,

$$\frac{1}{\lambda_1}+\frac{1}{\lambda_2}=\frac{1}{\lambda_1'}+\frac{1}{\lambda_2'}=c,$$

and prove

$$X_{\lambda_1}+X_{\lambda_2}\geq_{\mathrm{RS}}X_{\lambda_1'}+X_{\lambda_2'}.$$

Now, let $\lambda=\lambda_1$ and $\lambda'=\lambda_1'$, then $1/\lambda_2=c-1/\lambda$ and $1/\lambda_2'=c-1/\lambda'$. It holds that,

$$c>\frac{1}{\lambda}\geq\frac{1}{\lambda'}\geq\frac{c}{2}>0.$$

In the rest of the proof, we only discuss the case that $1/\lambda'>c/2$, as the limiting argument applies when $1/\lambda'=c/2$. Let F_λ be the distribution function of $X_{\lambda_1}+X_{\lambda_2}$. Note that, for $t\geq 0$,

$$\bar{F}_\lambda(t)=\frac{1}{\frac{2}{\lambda}-c}\left[\frac{1}{\lambda}\exp\{-\lambda t\}-\left(c-\frac{1}{\lambda}\right)\exp\left\{-\frac{t}{c-\frac{1}{\lambda}}\right\}\right].$$

Hence, for $t\geq 0$,

$$\begin{aligned}W_\lambda(t)&=\frac{1}{\frac{2}{\lambda}-c}\left[\frac{1}{\lambda}\int_t^\infty\exp\{-\lambda x\}\mathrm{d}x-\left(c-\frac{1}{\lambda}\right)\int_t^\infty\exp\left\{-\frac{x}{c-\frac{1}{\lambda}}\right\}\mathrm{d}x\right]\\&=\frac{1}{2\lambda-c\lambda^2}\exp\{-\lambda t\}-\frac{(\lambda c-1)^2}{2\lambda-c\lambda^2}\exp\left\{-\frac{t}{c-\frac{1}{\lambda}}\right\}\\&=\frac{\exp\{-\lambda t\}-(\lambda c-1)^2\exp\left\{-\frac{\lambda t}{\lambda c-1}\right\}}{2\lambda-c\lambda^2}.\end{aligned}$$

So, taking the derivative with respect to λ, for $t\geq 0$,

$$\begin{aligned}W_\lambda'(t)&=-\frac{t\exp\{-\lambda t\}+2(\lambda c-1)c\exp\{-\frac{\lambda t}{\lambda c-1}\}+t\exp\{-\frac{\lambda t}{\lambda c-1}\}}{2\lambda-c\lambda^2}\\&\quad-\frac{\left[\exp\{-\lambda t\}-(\lambda c-1)^2\exp\{-\frac{\lambda t}{\lambda c-1}\}\right](2-2\lambda c)}{(2\lambda-c\lambda^2)^2}.\end{aligned}$$

Note that, for $t\geq 0$,

$$\begin{aligned}-\frac{W_\lambda'(t)}{\bar{F}_\lambda(t)}&\propto\frac{t\exp\{-\lambda t\}(2\lambda-c\lambda^2)+[2(\lambda c-1)c+t](2\lambda-c\lambda^2)\exp\{-\frac{\lambda t}{\lambda c-1}\}}{\exp\{-\lambda t\}-(\lambda c-1)\exp\left\{-\frac{\lambda t}{\lambda c-1}\right\}}\\&\quad+\frac{\left[\exp\{-\lambda t\}-(\lambda c-1)^2\exp\{-\frac{\lambda t}{\lambda c-1}\}\right](2-2\lambda c)}{\exp\{-\lambda t\}-(\lambda c-1)\exp\left\{-\frac{\lambda t}{\lambda c-1}\right\}}\\&=\frac{(2-\lambda c)\lambda t\left[1+\exp\left\{-\lambda t\left(\frac{2-\lambda c}{\lambda c-1}\right)\right\}\right]-2(\lambda c-1)\left[1-\exp\left\{-\lambda t\left(\frac{2-\lambda c}{\lambda c-1}\right)\right\}\right]}{1-(\lambda c-1)\exp\left\{-\lambda t\left(\frac{2-\lambda c}{\lambda c-1}\right)\right\}}\\&=\frac{(2-\lambda c)\lambda t\left[1+\exp\left\{-\lambda t\left(\frac{2-\lambda c}{\lambda c-1}\right)\right\}\right]-2\left[1-(\lambda c-1)\exp\left\{-\lambda t\left(\frac{2-\lambda c}{\lambda c-1}\right)\right\}\right]+2(2-\lambda c)}{1-(\lambda c-1)\exp\left\{-\lambda t\left(\frac{2-\lambda c}{\lambda c-1}\right)\right\}},\end{aligned}$$

by Lemma 3.1, it is enough to show that,

$$h(t)=\frac{\lambda t\left[1+\exp\left\{-\lambda t\left(\frac{2-\lambda c}{\lambda c-1}\right)\right\}\right]+2}{1-(\lambda c-1)\exp\left\{-\lambda t\left(\frac{2-\lambda c}{\lambda c-1}\right)\right\}},$$

is increasing in $t \geq 0$. Denote

$$\gamma = \frac{2 - \lambda c}{\lambda c - 1} > 0.$$

Thus,

$$h(t) = \frac{\lambda t \left(1 + e^{-\lambda t \gamma}\right) + 2}{1 - \frac{1}{\gamma+1} e^{-\lambda t \gamma}}.$$

Taking the derivative with respect to t, it holds that,

$$h'(t) = \frac{\lambda \left(1 + e^{-\lambda t \gamma}\right) - \lambda^2 t \gamma e^{-\lambda t \gamma}}{1 - \frac{1}{\gamma+1} e^{-\lambda t \gamma}} - \frac{\lambda t \left(1 + e^{-\lambda t \gamma}\right) + 2}{\left(1 - \frac{1}{\gamma+1} e^{-\lambda t \gamma}\right)^2} \frac{\lambda \gamma}{\gamma + 1} e^{-\lambda t \gamma}.$$

Hence, $h'(t) \geq 0$ is equivalent to, for $t \geq 0$,

$$\left(1 + e^{-\lambda t \gamma} - \lambda t \gamma e^{-\lambda t \gamma}\right) \left(1 - \frac{1}{\gamma + 1} e^{-\lambda t \gamma}\right) \geq \frac{\gamma}{\gamma + 1} e^{-\lambda t \gamma} \left[2 + \lambda t \left(1 + e^{-\lambda t \gamma}\right)\right],$$

i.e.,

$$\left(1 + e^{-\lambda t \gamma} - \lambda t \gamma e^{-\lambda t \gamma}\right) \left(\frac{\gamma + 1}{\gamma} e^{\lambda t \gamma} - \frac{1}{\gamma}\right) \geq 2 + \lambda t \left(1 + e^{-\lambda t \gamma}\right),$$

which could be further simplified as, for $t \geq 0$,

$$e^{\lambda t \gamma} + \frac{1}{\gamma} \left(e^{\lambda t \gamma} - e^{-\lambda t \gamma}\right) \geq 1 + (\gamma + 2) \lambda t. \tag{3.5}$$

Observing that, for $t \geq 0$,

$$\sinh(\lambda t \gamma) = \frac{e^{\lambda t \gamma} - e^{-\lambda t \gamma}}{2} \geq \lambda t \gamma,$$

and

$$e^{\lambda t \gamma} \geq 1 + \lambda t \gamma,$$

the inequality (3.5) is true.

"Necessity $\Longleftarrow$". Observing that the convolution of exponential distributions is IFR (increasing failure rate) which implies DMRL [1], according to Theorem 3.C.5 of Shaked and Shanthikumar [11], it holds that

$$X_{\lambda_1} + X_{\lambda_2} \geq_{RS} X_{\lambda'_1} + X_{\lambda'_2} \Rightarrow X_{\lambda_1} + X_{\lambda_2} \geq_{\text{mrl}} X_{\lambda'_1} + X_{\lambda'_2}.$$

The required result follows from Theorem 3.1 of Zhao and Balakrishnan [8]. ■

Using the similar argument in Theorem 1 of Bon and Păltănea [3] or Theorem 4.1 in Zhao and Balakrishnan [8], one could easily extend Theorem 3.2 to the case of $n \geq 3$. The proof is omitted for briefness.

Theorem 3.3. *Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i and let $Y_1, \ldots, Y_n$ be independent exponential random variables with Y_i having hazard rate λ'_i, for $i = 1, \ldots, n$. Then,*

$$(\lambda_1, \ldots, \lambda_n) \overset{\text{rm}}{\succeq} (\lambda'_1, \ldots, \lambda'_n) \Longrightarrow \sum_{i=1}^{n} X_i \geq_{\text{RS}} \sum_{i=1}^{n} Y_i. \quad ■$$

As mentioned is Section 2, the NBUE order is equivalent to the right spread order with the same mean. Observing that

$$(1/\lambda_1, \ldots, 1/\lambda_n) \succeq_m (1/\lambda'_1, \ldots, 1/\lambda'_n) \Longrightarrow (\lambda_1, \ldots, \lambda_n) \overset{\text{rm}}{\succeq} (\lambda'_1, \ldots, \lambda'_n),$$

the following result is a direct consequence of Theorem 3.3.

Corollary 3.4. *Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i and let $Y_1, \ldots, Y_n$ be independent exponential random variables with Y_i having hazard rate λ'_i, for $i = 1, \ldots, n$. Then,*

$$(1/\lambda_1, \ldots, 1/\lambda_n) \succeq_m (1/\lambda'_1, \ldots, 1/\lambda'_n) \Longrightarrow \sum_{i=1}^{n} X_i \geq_{\text{NBUE}} \sum_{i=1}^{n} Y_i.$$

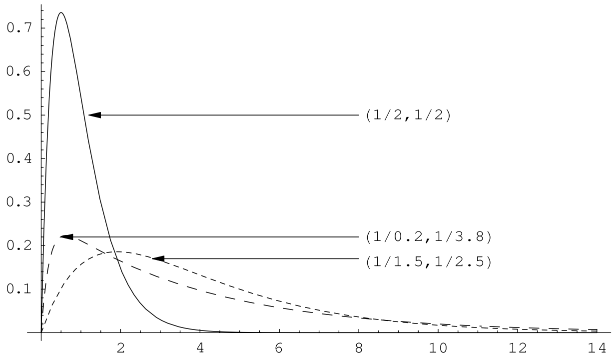

Fig. 1. The highest peak curve is the density function with $(\lambda_1, \lambda_2) = (1/2, 1/2)$; the middle peak curve is the density function with $(\lambda_1, \lambda_2) = (1/0.2, 1/3.8)$; the lowest peak curve is the density function with $(\lambda_1, \lambda_2) = (1/1.5, 1/2.5)$.

The following result is a direct consequence of Corollary 3.4 as the NBUE order implies the Lorenz order, which is of great interest in economics.

Corollary 3.5. *Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i and let $Y_1, \ldots, Y_n$ be independent exponential random variables with Y_i having hazard rate λ'_i for $i = 1, \ldots, n$. Then,*

$$(1/\lambda_1, \ldots, 1/\lambda_n) \succeq_m (1/\lambda'_1, \ldots, 1/\lambda'_n) \Longrightarrow \sum_{i=1}^n X_i \geq_{\text{Lorenz}} \sum_{i=1}^n Y_i.$$

As stated in Section 1, the NBUE order can be used to compare the shapes of distributions. The following theorem reveals that the density function of convolution of heterogeneous exponential random variables is more skewed than that of homogeneous exponential random variables.

Corollary 3.6. *Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i for $i = 1, \ldots, n$ and $Z_1, \ldots, Z_n$ be independent and identically distributed exponential random variables. Then,*

$$\sum_{i=1}^n X_i \geq_{\text{NBUE}} \sum_{i=1}^n Z_i.$$

Proof. Let $Y_1, \ldots, Y_n$ be independent and identical exponential random variables with the same hazard rate λ', where

$$(1/\lambda_1, \ldots, 1/\lambda_n) \succeq_m (1/\lambda', \ldots, 1/\lambda').$$

According to Corollary 3.4,

$$\sum_{i=1}^n X_i \geq_{\text{NBUE}} \sum_{i=1}^n Y_i.$$

Note that $\sum_{i=1}^n Y_i$ is a gamma random variable with shape parameter n and scale parameter $1/\lambda'$. Sine the NBUE order is scale invariant, it follows that

$$\sum_{i=1}^n X_i \geq_{\text{NBUE}} \sum_{i=1}^n Y_i =_{\text{NBUE}} \sum_{i=1}^n Z_i.$$

Hence, the required result follows immediately. ■

Fig. 1 gives an illustration of the above result. It is seen that

$$(0.2, 3.8) \succeq_m (1.5, 2.5) \succeq_m (2, 2).$$

The following result gives an equivalent characterization of right spread order between convolutions of two exponential samples when one sample has heterogeneous hazard rates but the other has homogeneous hazard rates.

Corollary 3.7. *Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i for $i = 1, \ldots, n$ and let $Y_1, \ldots, Y_n$ be independent and identical exponential random variables with the same hazard rate λ. Then,*

$$\sum_{i=1}^n X_i \geq_{RS} \sum_{i=1}^n Y_i \Longleftrightarrow \mathrm{E}\left(\sum_{i=1}^n X_i\right) \geq \mathrm{E}\left(\sum_{i=1}^n Y_i\right).$$

Proof. Theorem 4.3 in [9] shows that $X \geq_{\text{NBUE}} Y$ and $\text{E}X \geq \text{E}Y$ imply $X \geq_{\text{RS}} Y$. Since $X \geq_{\text{RS}} Y$ implies $\text{E}X \geq \text{E}Y$ when X and Y have the same left support, the result follows from Corollary 3.6. ■

As the convolution of exponential distributions is DMRL [1], using Theorem 3.C.5 of Shaked and Shanthikumar [11], one can easily derive the following result, which has been reported by Zhao and Balakrishnan [8].

Corollary 3.8. *Let $X_1, \ldots, X_n$ be independent exponential random variables such that X_i has hazard rate λ_i, $i = 1, \ldots, n$ and let $Y_1, \ldots, Y_n$ be independent exponential random variables with Y_i having hazard rate λ'_i, for $i = 1, \ldots, n$. Then,*

$$(\lambda_1, \ldots, \lambda_n) \overset{\text{rm}}{\succeq} (\lambda'_1, \ldots, \lambda'_n) \Longrightarrow \sum_{i=1}^{n} X_i \geq_{\text{mrl}} \sum_{i=1}^{n} Y_i.$$

Theorem 3.3 can be easily extended to convolutions of Erlang distributions (Gamma distributions with integer valued shape parameters) as follows.

Corollary 3.9. *Let $X_1, \ldots, X_n$ be independent Gamma random variables such that X_i has integer shape parameter k_i and scale parameter λ_i, i.e., $X_i \sim \Gamma(k_i, \lambda_i)$, and let $Y_1, \ldots, Y_n$ be independent Gamma random variables such that Y_i has integer shape parameter k_i and scale parameter λ'_i, i.e., $Y_i \sim \Gamma(k_i, \lambda'_i)$ for $i = 1, \ldots, n$. Then, for $i = 1, \ldots, n$,*

$$(\lambda_1, \ldots, \lambda_n) \overset{\text{rm}}{\succeq} (\lambda'_1, \ldots, \lambda'_n) \Longrightarrow \sum_{i=1}^{n} X_i \geq_{\text{RS}} \sum_{i=1}^{n} Y_i.$$

Proof. Note that X_i can be expressed as a sum of k_i independent exponential random variables with the same hazard rate λ_i. Since

$$(\lambda_1, \ldots, \lambda_n) \overset{\text{rm}}{\succeq} (\lambda'_1, \ldots, \lambda'_n)$$

implies

$$(\underbrace{\lambda_1, \ldots, \lambda_1}_{k_1}, \ldots, \underbrace{\lambda_n, \ldots, \lambda_n}_{k_n}) \overset{\text{rm}}{\succeq} (\underbrace{\lambda'_1, \ldots, \lambda'_1}_{k_1}, \ldots, \underbrace{\lambda'_n, \ldots, \lambda'_n}_{k_n}),$$

the required result follows from Theorem 3.3. ■

4. Applications

In this section, we will give some applications of our main results.

4.1. Reliability theory

Suppose a redundant standby system is composed of different exponential components (which is often a common assumption in a large system). When a component fails, one standby component is immediately put into operation. So the lifetime of the system is the convolution of the component lifetimes. Theorem 3.3 states that greater the degree of heterogeneity (as reflected by the reciprocal majorization order) among means of different components, greater is the degree of variability in the system. In practice, the engineer may only know the average lifetime of the system. Corollary 3.7 provides a simple lower bound on the variance of such a system.

Corollary 4.1. *Let $X_1, \ldots, X_n$ be independent exponential components with respective hazard rates $\lambda_i, \ldots, \lambda_n$, respectively. Then,*

$$\text{Var}\left(\sum_{i=1}^{n} X_i\right) \geq \frac{n}{\hat{\lambda}^2},$$

where $\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} \frac{1}{\lambda_i}}$ is the mean lifetime of the redundant standby system.

This bound is sharper than the one given in [6] which is in terms of the arithmetic mean of the λ_i's.

Corollary 3.6 has an interesting interpretation in reliability theory: a redundant standby system composed of nonidentical exponential components exhibits more "NBUE" aging property than such a system composed of homogeneous exponential components. As a further consequence, this result provides a lower bound on the coefficient of variation of the redundant standby system as stated in Section 2.

Corollary 4.2. *Let $X_1, \ldots, X_n$ be independent exponential components with respective hazard rates $\lambda_1, \ldots, \lambda_n$. Then,*

$$cv\left(\sum_{i=1}^{n} X_i\right) \geq \frac{1}{\sqrt{n}}.$$

4.2. Economics

Lorenz curve is a graph (x, y) that shows, for the bottom $100x\%$ of the households, the percentage $100y\%$ of the total income which they posses. If every one in the population has the same income, the bottom $p\%$ of the population would always have $p\%$ of the total income and the Lorenz curve would be the diagonal line $y = x$. The more the Lorenz curve is below the diagonal line, the more is the disparity between the incomes. To compare the extent of inequality that exists between two incomes, the Lorenz order is used. If X and Y denote two incomes, $X \leq_{\text{Lorenz}} Y$ means that X shows less inequality than Y. Now, suppose that each individual in the population has income coming from different sources (e.g. salary, stocks, bonus, etc.), which could be represented as the sum of different exponential or Erlang variables. Corollary 3.5 reveals that more diverse the different component distributions are, the more is the extent of inequality between the population incomes.

4.3. Actuarial science

In actuarial science, people are always interested in the following question: how much can we expect to lose with a given probability? This introduces the concept of value-at-risk (VaR), which has become the benchmark risk measure. For more details about VaR, please refer to [19]. The VaR is defined as

$$\text{VaR}[X;\ p] = F^{-1}(p).$$

As the VaR at a fixed level only gives local information about the underlying distribution, actuaries proposed the so-called *expected shortfall* to overcome this shortcoming. Expected shortfall at probability level p is the stop-loss premium with retention $\text{VaR}[X;\ p]$, that is,

$$\begin{aligned}\text{ES}[X; p] &= \text{E}\,(X - \text{VaR}[X;\ p])_+ \\ &= \int_{F^{-1}(p)}^{\infty} \bar{F}(x)\text{d}x\end{aligned}$$

where $(X)_+ = \max\{X, 0\}$. Now, suppose that a total claim is composed of several subclaims which come from different exponential or Erlang distributions. The actuary wants to know the properties of expected shortfall in order to make a good policy for the insurance company. Theorem 3.3 (or Corollary 3.9) states that greater the degree of heterogeneity among subclaims, the larger the expected shortfall is. If the actuary is able to estimate the mean of heterogeneous subclaims, Corollary 3.7 provides a sharp lower bound for the expected shortfall of subclaims at each probability level p.

For example, suppose that the total claim is composed of three subclaims coming from exponential distributions with parameters λ_1, λ_2 and λ_3. Then the distribution function of $X_1 + X_2 + X_3$ is

$$F(x) = 1 - \frac{\lambda_2\lambda_3}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)}\text{e}^{-\lambda_1 x} - \frac{\lambda_1\lambda_3}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)}\text{e}^{-\lambda_2 x} - \frac{\lambda_1\lambda_2}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)}\text{e}^{-\lambda_3 x}.$$

Let us assume that

$$(\lambda_1, \lambda_2, \lambda_3) = (1, 2, 3).$$

Then, the arithmetic mean and the harmonic mean are 2 and 18/11, respectively. In Fig. 2, we used Mathematica to plot the expected shortfalls of the total claim when the parameters are (1, 2, 3), and their arithmetic mean and harmonic mean. It is seen that the harmonic mean provides a sharper bound for the expected shortfall as stated in Corollary 3.7.

5. Discussion

Let $X_1, \ldots, X_n$ $(Y_1, \ldots, Y_n)$ be independent exponential random variables with hazard rates $\lambda_1, \ldots, \lambda_n$ $(\lambda'_1, \ldots, \lambda'_n)$, respectively. In Corollary 3.4, we used the condition,

$$(1/\lambda_1, \ldots, 1/\lambda_n) \succeq_{\text{m}} (1/\lambda'_1, \ldots, 1/\lambda'_n). \tag{5.1}$$

One may wonder whether this condition implies the p order? The answer is negative, as shown by the following example.

Let $(\lambda_1, \lambda_2, \lambda_3) = (1/2, 2, 3)$ and $(\lambda'_1, \lambda'_2, \lambda'_3) = (6/7, 1, 3/2)$. It can be checked that

$$(1/\lambda_1, 1/\lambda_2, 1/\lambda_3) \succeq_m (1/\lambda'_1, 1/\lambda'_2, 1/\lambda'_3).$$

However, the p order is not satisfied. It is also seen that the majorization order does not imply (5.1) either. For example, let $(\lambda_1, \lambda_2, \lambda_3) = (0.2, 1, 9)$ and $(\lambda'_1, \lambda'_2, \lambda'_3) = (0.2, 4, 6)$. It is easily seen that

$$(\lambda_1, \lambda_2, \lambda_3) \succeq_{\text{m}} (\lambda'_1, \lambda'_2, \lambda'_3).$$

However, (5.1) is not satisfied.

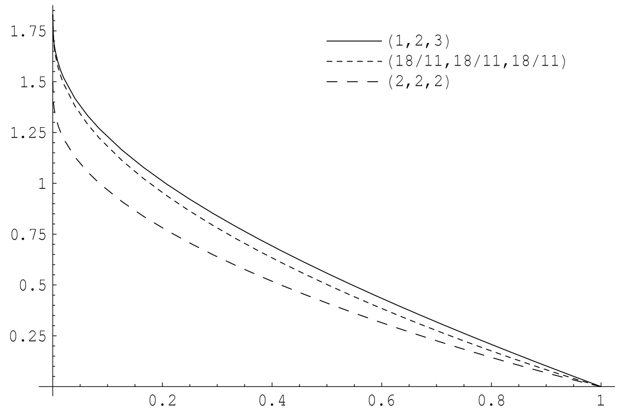

Fig. 2. Plot of the expected shortfall of the total claim of three subclaims with exponential parameters (1, 2, 3), the harmonic mean parameters (18/11, 18/11, 18/11) and the arithmetic mean parameters (2, 2, 2).

Our main result Theorem 3.3 complements the existing results on variability orderings among convolutions of independent random variables. In the literature not much attention has been paid to skewness orders between convolutions. Corollary 3.4 gives an insight into the distribution theory of convolutions through NBUE order. In statistics, the other two popular shape orders for comparing skewness are convex transform order and star transform order. One may refer to [13] for discussion about those orders on measuring skewness. This topic is also of interest in insurance as it provides information on tail risks. The authors could extend Corollary 3.4 to the star transform order for $n = 2$ under the condition of majorization order. However, it is still an open problem for arbitrary n.

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References

[1] R.E. Barlow, F. Proschan, Statistical Theory of Reliability and Life Testing, To Begin with, Silver Spring, MD, 1981.
[2] N. Balakrishnan, A.P. Basu (Eds.), The Exponential Distribution: Theory, Methods and Applications, Gordon and Breach, Newark, New Jersey, 1995.
[3] J.L. Bon, E. Păltănea, Ordering properties of convolutions of exponential random variables, Lifetime Data Analysis 5 (1999) 185–192.
[4] R. Kaas, M. Goovaerts, J. Dhaene, M. Denuit, Modern Actuarial Risk Theory, Kluwer Academic Publishers, 2001.
[5] P.J. Boland, E. El-Neweihi, F. Proschan, Schur properties of convolutions of exponential and geometric random variables, Journal of Multivariate Analysis 48 (1994) 157–167.
[6] S.C. Kochar, C. Ma, Dispersive ordering of convolutions of exponential random variables, Statistics & Probability Letters 43 (1999) 321–324.
[7] B.-E. Khaledi, S.C. Kochar, Ordering convolutions of gamma random variables, Sankhyā 66 (2004) 466–473.
[8] P. Zhao, N. Balakrishnan, Mean residual life order of convolutions of heterogeneous exponential random variables, Journal of Multivariate Analysis. doi:10.1016/j.jmva.2009.02.009.
[9] J.M. Fernández-Ponce, S.C. Kochar, J. Muñoz-Perez, Partial orderings of distributions based on right-spread functions, Journal of Applied Probability 35 (1998) 221–228.
[10] M. Shaked, J.G. Shanthikumar, Two variability orders, Probability in the Engineering and Informational Sciences 12 (1998) 1–23.
[11] M. Shaked, J.G. Shanthikumar, Stochastic Orders and their Applications, Springer, New York, 2007.
[12] S.C. Kochar, D.P. Wiens, Partial orderings of life distributions with respect to their aging properties, Naval Research Logistics 34 (1987) 823–829.
[13] A.W. Marshall, I. Olkin, Life Distributions, Springer, New York, 2007.
[14] S.C. Kochar, X. Li, M. Shaked, The total time on test transform and the excess wealth stochastic order of distributions, Advances in Applied Probability 34 (2002) 826–845.
[15] S.C. Kochar, On extensions of DMRL and related partial orderings of life distributions, Communications in Statistics—Stochastic Models 5 (1989) 235–245.
[16] A.W. Marshall, I. Olkin, Inequalities: Theory of Majorization and its Applications, Academic Press, New York, 1979.
[17] I.W. Saunders, P.A. Moran, On the quantiles of the gamma and F distributions, Journal of Applied Probability 15 (1978) 426–432.
[18] T. Hu, J. Chen, J. Yao, Preservation of the location independent risk order under convolution, Insurance: Mathematics & Economics 38 (2006) 406–412.
[19] M. Denuit, J. Dhaene, M. Goovaerts, R. Kaas, Actuarial Theory for Dependent Risks: Measures, Orders and Models, John Wiley & Sons, 2005.