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On the range of heterogeneous samples

Christian Genest *, Subhash C. Kochar, Maochao Xu

Université Laval, Canada
Portland State University, United States

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ABSTRACT

Let R_n be the range of a random sample $X_1, \ldots, X_n$ of exponential random variables with hazard rate λ. Let S_n be the range of another collection $Y_1, \ldots, Y_n$ of mutually independent exponential random variables with hazard rates $\lambda_1, \ldots, \lambda_n$ whose average is λ. Finally, let r and s denote the reversed hazard rates of R_n and S_n, respectively. It is shown here that the mapping $t \mapsto s(t)/r(t)$ is increasing on $(0, \infty)$ and that as a result, $R_n = X_{(n)} - X_{(1)}$ is smaller than $S_n = Y_{(n)} - Y_{(1)}$ in the likelihood ratio ordering as well as in the dispersive ordering. As a further consequence of this fact, $X_{(n)}$ is seen to be more stochastically increasing in $X_{(1)}$ than $Y_{(n)}$ is in $Y_{(1)}$. In other words, the pair $(X_{(1)}, X_{(n)})$ is more dependent than the pair $(Y_{(1)}, Y_{(n)})$ in the monotone regression dependence ordering. The latter finding extends readily to the more general context where $X_1, \ldots, X_n$ form a random sample from a continuous distribution while $Y_1, \ldots, Y_n$ are mutually independent lifetimes with proportional hazard rates.

1. Introduction

Let $X_1, \ldots, X_n$ be a random sample of exponential random variables with hazard rate λ. Let $Y_1, \ldots, Y_n$ be mutually independent exponential random variables with hazard rates $\lambda_1, \ldots, \lambda_n$ such that

$$(\lambda_1 + \cdots + \lambda_n)/n = \lambda. \tag{1}$$

It seems plausible that on average, the homogeneous sample would be less variable than the heterogeneous sample. This intuition was confirmed by Kochar and Rojo [1], who exhibited a stochastic order relation between the ranges

$$R_n = X_{(n)} - X_{(1)}, \qquad S_n = Y_{(n)} - Y_{(1)} \tag{2}$$

derived from the sets of order statistics $X_{(1)} < \cdots < X_{(n)}$ and $Y_{(1)} < \cdots < Y_{(n)}$. Specifically, they established that for all $t \in \mathbb{R}$,

$$F(t) \equiv \Pr(R_n \leqslant t) \geqslant \Pr(S_n \leqslant t) \equiv G(t). \tag{3}$$

This result was recently extended by Kochar and Xu [2], who proved that the mapping $t \mapsto G(t)/F(t)$ is increasing on $\mathbb{R}_+ = (0, \infty)$. Thus R_n is smaller than S_n in the reversed hazard rate ordering.

The main purpose of this note is to strengthen relation (3) in two directions, one of which involves the densities f and g of R_n and S_n, respectively. More precisely, it is shown here that R_n is smaller than S_n in the likelihood ratio ordering and in

* Corresponding address: Université Laval, Département de mathématiques et de statistique, 1045, avenue de la Médecine, Québec, Canada G1V 0A6.
E-mail address: Christian.Genest@mat.ulaval.ca (C. Genest).

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the dispersive ordering, viz.

(i) $R_n \prec_{\text{LR}} S_n$, i.e., the mapping $t \mapsto g(t)/f(t)$ is increasing on $\mathbb{R}_+$;
(ii) $R_n \prec_{\text{DISP}} S_n$, i.e., $F^{-1}(\beta) - F^{-1}(\alpha) \leqslant G^{-1}(\beta) - G^{-1}(\alpha)$ for all $0 < \alpha < \beta < 1$.

See Shaked and Shanthikumar [3] for a review of these orderings.

Define the reversed hazard rates of R_n and S_n at $t \in \mathbb{R}_+$ by

$$r(t) = f(t)/F(t), \qquad s(t) = g(t)/G(t),$$

respectively. It is proved in Section 2 that the mapping $t \mapsto s(t)/r(t)$ is increasing on $\mathbb{R}_+$. It is then shown in Section 3 that statements (i) and (ii) are immediate consequences of this fact.

A further implication of this result is presented in Section 4, where $X_{(n)}$ is seen to be more stochastically increasing in $X_{(1)}$ than $Y_{(n)}$ is in $Y_{(1)}$ with respect to the monotone regression dependence ordering as defined, e.g., by Capéraà and Genest [4] or Avérous et al. [5]. In view of the work of Dolati et al. [6], the conclusion extends readily to the more general context where $X_1, \ldots, X_n$ form a random sample from a continuous distribution while $Y_1, \ldots, Y_n$ are mutually independent lifetimes with proportional hazard rates.

2. The ratio s/r is increasing

Let $X_1, \ldots, X_n$ be a random sample of exponential random variables with hazard rate λ, and let R_n be its range, as defined by (2). The reversed hazard rate of R_n at arbitrary $t \in \mathbb{R}_+$ is then given by

$$r(t) = \frac{(n-1)\lambda e^{-\lambda t}}{1 - e^{-\lambda t}} \equiv (n-1) b_\lambda(t). \tag{4}$$

More generally, let $Y_1, \ldots, Y_n$ be mutually independent exponential random variables with hazard rates $\lambda_1, \ldots, \lambda_n$ satisfying condition (1), and let S_n be the corresponding range defined in (2). Kochar and Xu [2] show that the reversed hazard rate of S_n is then of the form

$$s(t) = \sum_{i \neq j} a_{\lambda_i}(t) b_{\lambda_j}(t) \Big/ \sum_k a_{\lambda_k}(t), \tag{5}$$

for all $t \in \mathbb{R}_+$, where here and in what follows, the sums run over all possible indices in $\{1, \ldots, n\}$, and

$$a_{\lambda_i}(t) = \frac{\lambda_i}{1 - e^{-\lambda_i t}}, \qquad b_{\lambda_j}(t) = \frac{\lambda_j e^{-\lambda_j t}}{1 - e^{-\lambda_j t}}.$$

This section contains a proof of the following result.

Proposition 1. *Let $r(t)$ and $s(t)$ be defined by (4) and (5) for all $t \in \mathbb{R}_+$. The mapping $t \mapsto s(t)/r(t)$ is increasing on $\mathbb{R}_+$.*

Remark 1. Note that this result does not extend to the case where for each $i \in \{1, \ldots, n\}$, X_i is exponential with hazard rate λ_i^*, and $\Lambda^* = (\lambda_1^*, \ldots, \lambda_n^*) \prec (\lambda_1, \ldots, \lambda_n) = \Lambda$ in the majorization ordering of Marshall and Olkin [7]. In other words, if s_{Λ^*} and s_Λ are the reversed hazard rates of $R_n = X_{(n)} - X_{(1)}$ and $S_n = Y_{(n)} - Y_{(1)}$, respectively, the mapping $t \mapsto s_\Lambda(t)/s_{\Lambda^*}(t)$ may not be monotone; take, e.g., $n = 3$ and $\Lambda^* = (0.1, 4, 6) \prec \Lambda = (0.1, 1, 9)$.

Proof of Proposition 1. Let $\Lambda = (\lambda_1, \ldots, \lambda_n)$ and introduce

$$u_\Lambda(t) = \sum_{i \neq j} a_{\lambda_i}(t) b_{\lambda_j}(t), \qquad v_\Lambda(t) = b_\lambda(t) \sum_k a_{\lambda_k}(t),$$

so that $(n-1)s(t)/r(t) = u_\Lambda(t)/v_\Lambda(t)$. The mapping $t \mapsto s(t)/r(t)$ is increasing if for all $t \in \mathbb{R}_+$,

$$u_\Lambda(t)\, v'_\Lambda(t) \leqslant v_\Lambda(t)\, u'_\Lambda(t). \tag{6}$$

Upon differentiation with respect to t, one gets

$$a'_{\lambda_i}(t) = -a_{\lambda_i}(t) b_{\lambda_i}(t), \qquad b'_{\lambda_j}(t) = -a_{\lambda_j}(t) b_{\lambda_j}(t),$$

and $b'_\lambda(t) = -a_\lambda(t) b_\lambda(t)$, where $a_\lambda(t) \equiv \lambda/(1 - e^{-\lambda t})$ for all $t \in \mathbb{R}_+$. Consequently,

$$u'_\Lambda(t) = -\sum_{i \neq j} \{a_{\lambda_i}(t) a_{\lambda_j}(t) b_{\lambda_j}(t) + a_{\lambda_i}(t) b_{\lambda_i}(t) b_{\lambda_j}(t)\}$$

and

$$v'_\Lambda(t) = -b_\lambda(t) \sum_k \{a_{\lambda_k}(t) a_\lambda(t) + a_{\lambda_k}(t) b_{\lambda_k}(t)\}.$$

Now observe that if $\Lambda^* = (\lambda_1^*, \ldots, \lambda_n^*) = (t\lambda_1, \ldots, t\lambda_n)$, one has

$$t^2 u_\Lambda(t) = u_{\Lambda^*}(1), \qquad t^2 v_\Lambda(t) = v_{\Lambda^*}(1)$$

and

$$t^3 u'_\Lambda(t) = u'_{\Lambda^*}(1), \qquad t^3 v'_\Lambda(t) = v'_{\Lambda^*}(1).$$

Thus upon multiplying by t^5 on both sides of (6) and expanding, one can see that this inequality holds if and only if

$$u_{\Lambda^*}(1)\, v'_{\Lambda^*}(1) \leqslant v_{\Lambda^*}(1)\, u'_{\Lambda^*}(1). \tag{7}$$

Hence it suffices to prove relation (7) for all choices of $\lambda_1^*, \ldots, \lambda_n^*$ or equivalently, to show inequality (6) for arbitrary $\lambda_1, \ldots, \lambda_n$ and $t = 1$.

Let $a = a_\lambda(1)$, $b = b_\lambda(1)$, $a_i = a_{\lambda_i}(1)$ and $b_j = b_{\lambda_j}(1)$ for all $i, j \in \{1, \ldots, n\}$. It will be seen in the following subsections that

$$-b\left(\sum_k a_k\right)\left(a\sum_{i\neq j} a_i b_j\right) \leqslant -b\left(\sum_k a_k\right)\left(\sum_{i\neq j} a_i a_j b_j\right) \tag{8}$$

and

$$-b\left(\sum_{i\neq j} a_i b_j\right)\left(\sum_k a_k b_k\right) \leqslant -b\left(\sum_{i\neq j} a_i b_i b_j\right)\left(\sum_k a_k\right). \tag{9}$$

This will be enough to conclude, because the terms on the left-hand side of relations (8) and (9) add up to $u_\Lambda(1)\, v'_\Lambda(1)$ while the terms on the right-hand side sum up to $v_\Lambda(1)\, u'_\Lambda(1)$. □

Proof of inequality (8). This inequality is equivalent to

$$\sum_{i\neq j} a_i a_j b_j \leqslant a \sum_{i\neq j} a_i b_j.$$

The latter is an immediate consequence of the following chain:

$$\frac{\sum_{i\neq j} a_i a_j b_j}{\sum_{i\neq j} a_i b_j} \leqslant \frac{\sum_k a_k b_k}{\sum_k b_k} \leqslant \frac{n}{\sum_k 1/a_k} \leqslant a. \tag{10}$$

The right-most inequality in (10) states that

$$\frac{1}{a} \leqslant \frac{1}{n}\sum_k \frac{1}{a_k}$$

or

$$\frac{1-\mathrm{e}^{-\lambda}}{\lambda} \leqslant \frac{1}{n}\sum_k \frac{1-\mathrm{e}^{-\lambda_k}}{\lambda_k}.$$

As the mapping $t \mapsto (1-\mathrm{e}^{-t})/t$ is convex, this is an immediate consequence of Jensen's inequality.

The middle inequality in (10) may be expressed alternatively as

$$\left(\frac{1}{n}\sum_k \frac{1}{a_k}\right)\left(\frac{1}{n}\sum_k a_k b_k\right) \leqslant \frac{1}{n}\sum_k b_k$$

or

$$\left(\frac{1}{n}\sum_k p(\lambda_k)\right)\left(\frac{1}{n}\sum_k q(\lambda_k)\right) \leqslant \frac{1}{n}\sum_k p(\lambda_k) q(\lambda_k), \tag{11}$$

where

$$p(\lambda_k) = \frac{1}{a_k} = \frac{1-\mathrm{e}^{-\lambda_k}}{\lambda_k} \quad \text{and} \quad q(\lambda_k) = a_k b_k = \frac{\lambda_k^2 \mathrm{e}^{-\lambda_k}}{(1-\mathrm{e}^{-\lambda_k})^2}.$$

As shown, e.g., by Khaledi and Kochar [8], the mappings

$$t \mapsto p(t) = \frac{1 - e^{-t}}{t} \quad \text{and} \quad t \mapsto q(t) = \frac{t^2 e^{-t}}{(1 - e^{-t})^2}$$

are both decreasing on $\mathbb{R}_+$. Inequality (11) thus follows from an application of Čebyšev's sum inequality; see p. 36 of Mitrinović [9].

Finally, the left-most inequality in (10) amounts to

$$\left(\sum_{i\neq j} a_i a_j b_j\right)\left(\sum_k b_k\right) \leqslant \left(\sum_{i\neq j} a_i b_j\right)\left(\sum_k a_k b_k\right).$$

Rewrite the left-hand side of this inequality as

$$\left\{\left(\sum_i a_i\right)\left(\sum_j a_j b_j\right) - \left(\sum_k a_k^2 b_k\right)\right\}\left(\sum_k b_k\right)$$

and the right-hand side as

$$\left\{\left(\sum_i a_i\right)\left(\sum_j b_j\right) - \left(\sum_k a_k b_k\right)\right\}\left(\sum_k a_k b_k\right).$$

Upon canceling the first summand, which is common to both sides, one sees that inequality (10) holds provided that

$$\left(\sum_k a_k b_k\right)^2 \leqslant \left(\sum_k a_k^2 b_k\right)\left(\sum_k b_k\right),$$

but the latter follows from the classical Cauchy–Schwarz inequality. This completes the proof of inequality (8). □

Proof of inequality (9). This inequality is equivalent to

$$\left(\sum_{i\neq j} a_i b_i b_j\right)\left(\sum_k a_k\right) \leqslant \left(\sum_{i\neq j} a_i b_j\right)\left(\sum_k a_k b_k\right).$$

In order to establish this fact, first observe that

$$\frac{n-1}{n}\left(\sum_k a_k\right)\left(\sum_k b_k\right) \leqslant \sum_{i\neq j} a_i b_j. \tag{12}$$

Indeed, the right-hand side can be written alternatively as

$$\left(\sum_i a_i\right)\left(\sum_j b_j\right) - \sum_k a_k b_k,$$

and hence inequality (12) is equivalent to

$$\sum_k a_k b_k \leqslant \frac{1}{n}\left(\sum_k a_k\right)\left(\sum_k b_k\right).$$

The latter may be re-expressed as

$$\frac{1}{n}\sum_k p(\lambda_k) q(\lambda_k) \leqslant \left(\frac{1}{n}\sum_k p(\lambda_k)\right)\left(\frac{1}{n}\sum_k q(\lambda_k)\right),$$

where

$$p(\lambda_k) = a_k = \frac{\lambda_k}{1 - e^{-\lambda_k}} \quad \text{and} \quad q(\lambda_k) = b_k = \frac{\lambda_k e^{-\lambda_k}}{1 - e^{-\lambda_k}}.$$

Inequality (12) then follows from Čebyšev's sum inequality, because the mapping $t \mapsto p(t)$ is increasing while the mapping $t \mapsto q(t)$ is decreasing on $\mathbb{R}_+$. In the light of (12), inequality (9) is valid if

$$\left(\sum_{i\neq j} a_i b_i b_j\right)\left(\sum_k a_k\right) \leqslant \frac{n-1}{n}\left(\sum_k a_k\right)\left(\sum_k b_k\right)\left(\sum_k a_k b_k\right)$$

or

$$\left(\sum_{i\neq j} a_i b_i b_j\right) \leqslant \frac{n-1}{n}\left(\sum_k b_k\right)\left(\sum_k a_k b_k\right). \tag{13}$$

Upon writing the left-hand side of the latter inequality in the form

$$\left(\sum_i a_i b_i\right)\left(\sum_j b_j\right) - \sum_k a_k b_k^2,$$

one can see that inequality (13) reduces to

$$\frac{1}{n}\left(\sum_k b_k\right)\left(\sum_k a_k b_k\right) \leqslant \sum_k a_k b_k^2$$

or

$$\left(\frac{1}{n}\sum_k p(\lambda_k)\right)\left(\frac{1}{n}\sum_k q(\lambda_k)\right) \leqslant \frac{1}{n}\sum_k p(\lambda_k)q(\lambda_k),$$

where

$$p(\lambda_k) = b_k = \frac{\lambda_k e^{-\lambda_k}}{1-e^{-\lambda_k}} \quad \text{and} \quad q(\lambda_k) = a_k b_k = \frac{\lambda_k^2 e^{-\lambda_k}}{(1-e^{-\lambda_k})^2}.$$

Now the mappings $t \mapsto p(t)$ and $t \mapsto q(t)$ are both decreasing on $\mathbb{R}_+$. Thus, inequality (13) is yet another consequence of Čebyšev's sum inequality. This establishes inequality (9) and completes the proof that the mapping $t \mapsto s(t)/r(t)$ is increasing on $\mathbb{R}_+$. □

3. Consequences on variability

Formally stated and proved here are consequences (i) and (ii) of Proposition 1 announced in the Introduction. Both of them describe the effect of heterogeneity on the degree of dispersion of the range in a sample of exponential random variables.

Proposition 2. *Let R_n be the range of a random sample of exponential random variables with hazard rate λ. Let S_n be the range of another set of mutually independent exponential random variables with hazard rates $\lambda_1, \ldots, \lambda_n$ meeting condition (1). Then $R_n \prec_{\mathrm{LR}} S_n$.*

Proof. Let F and G be defined as per (3). If r and s denote the corresponding reversed hazard rates, it must be seen that the mapping

$$t \mapsto \frac{g(t)}{f(t)} = \frac{G(t)}{F(t)} \times \frac{s(t)}{r(t)}$$

is increasing on $\mathbb{R}_+$. This comes from Proposition 1 and the fact that the mapping $t \mapsto G(t)/F(t)$ is increasing on $\mathbb{R}_+$. □

Remark 2. The increasingness of the mapping $t \mapsto G(t)/F(t)$ on $\mathbb{R}_+$ means that R_n is smaller than S_n in the reversed hazard rate ordering. Thus Proposition 2 may be viewed as deriving from Proposition 1 together with Theorem 1.C.4 in Shaked and Shanthikumar [3]. Also since $X_{(1)}$ and $Y_{(1)}$ have the same log-concave distribution, viz.

$$\Pr(X_{(1)} > t) = \Pr(Y_{(1)} > t) = e^{-(\lambda_1+\cdots+\lambda_n)t} = e^{-n\lambda t}$$

for all $t \in \mathbb{R}_+$, results from Shanthikumar and Yao [10] imply that $X_{(n)} \prec_{\mathrm{LR}} Y_{(n)}$, as already pointed out by Kochar and Xu [2].

Proposition 3. *Under the same conditions as above, $R_n \prec_{\mathrm{DISP}} S_n$, and in particular* $\mathrm{var}(R_n) \leqslant \mathrm{var}(S_n)$.

Proof. Put $\ell = 1/(n-1) > 0$ and let X_ℓ and Y_ℓ be distributed as F^ℓ and G^ℓ, respectively. The ratio of their densities may then be expressed as a product of positive increasing functions, viz.

$$\frac{\ell\, G^{\ell-1}(t) g(t)}{\ell\, F^{\ell-1}(t) f(t)} = \left(\frac{G(t)}{F(t)}\right)^{\ell} \times \frac{s(t)}{r(t)}.$$

Accordingly, one has $X_\ell \prec_{\mathrm{LR}} Y_\ell$ and hence X_ℓ is smaller than Y_ℓ in the hazard rate ordering. Thus the mapping

$$t \mapsto H_\ell(t) = \frac{1-G^\ell(t)}{1-F^\ell(t)} = \frac{1-G^\ell(t)}{e^{-\lambda t}}$$

is increasing on $\mathbb{R}_+$. Clearly, $\mathrm{d}H_\ell(t)/\mathrm{d}t \geqslant 0$ if and only if

$$g(t) \leqslant \lambda(n-1)G(t)\{G^{-\ell}(t)-1\}$$

for all $t \in \mathbb{R}_+$. As the right-hand side equals $f \circ F^{-1} \circ G(t)$, it follows that $g \circ G^{-1}(u) \leqslant f \circ F^{-1}(u)$ for all $u \in (0,1)$, whence $R_n \prec_{\mathrm{DISP}} S_n$ by Equation (3.B.11) of Shaked and Shanthikumar [3]. □

Remark 3. In the case $n = 2$, Proposition 3 also follows from Theorem 3.7 in Kochar and Korwar [11], which states that under the conditions of Proposition 2, the normalized spacings of the homogeneous sample are less dispersed than those of the heterogeneous sample.

4. Consequences on dependence

As shown by Dolati et al. [6], Proposition 3 is equivalent to the following result, whose scope extends well beyond the exponential case.

Proposition 4. *Let $X_1, \ldots, X_n$ be a random sample from some continuous distribution while $Y_1, \ldots, Y_n$ are mutually independent lifetimes with proportional hazards, i.e., there exist a baseline survival function $\bar{H}$ and positive scalars $\lambda_1, \ldots, \lambda_n$ for which $\Pr(Y_i > t) = \{\bar{H}(t)\}^{\lambda_i}$ holds for all $t \in \mathbb{R}$. Then $Y_{(n)}|Y_{(1)} \prec_{\mathrm{MRD}} X_{(n)}|X_{(1)}$.*

This observation strengthens the conclusions of Dolati et al. [6], who established the same relation for the weaker right-tail increasing ordering. Under the conditions of Proposition 4, therefore, one has $\kappa(Y_{(1)}, Y_{(n)}) \leqslant \kappa(X_{(1)}, X_{(n)})$ for any margin-free measure of concordance κ satisfying the axioms of Scarsini [12], such as Spearman's rho or Kendall's tau.

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