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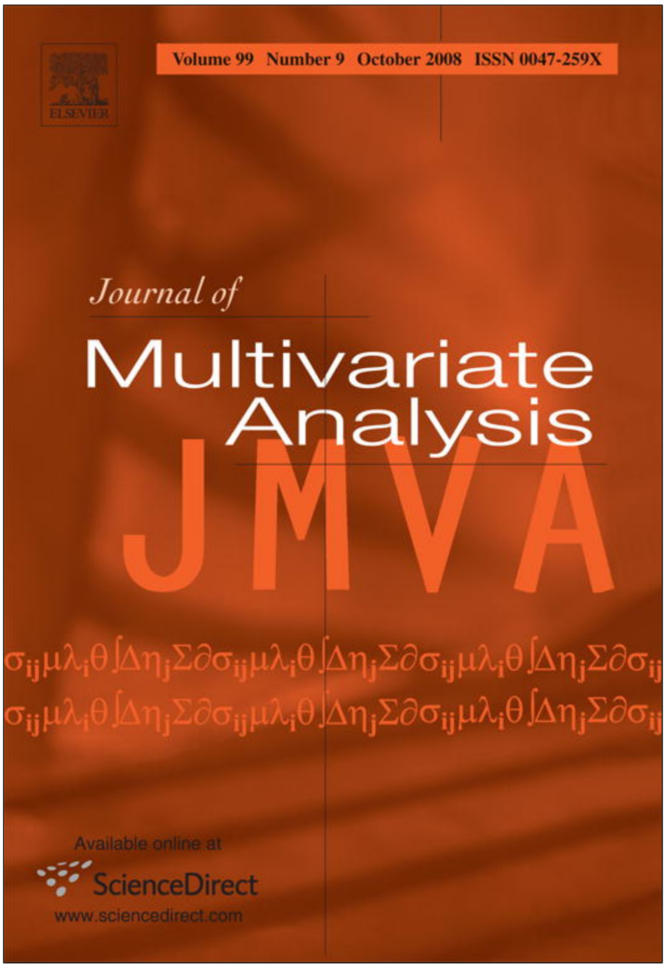

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A new dependence ordering with applications

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Abstract

In this paper, we introduce a new copula-based dependence order to compare the relative degree of dependence between two pairs of random variables. Relationship of the new order to the existing dependence orders is investigated. In particular, the new ordering is stronger than the partial ordering, *more monotone regression dependence* as developed by Avérous et al. [J. Avérous, C. Genest, S.C. Kochar, On dependence structure of order statistics, Journal of Multivariate Analysis 94 (2005) 159–171]. Applications of this partial order to order statistics, k-record values and frailty models are given.
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1. Introduction

The concept of dependence between random variables is of utmost importance and it has been well studied in the literature. Realizing the shortcomings of summary measures of dependence like Pearson coefficient of correlation, researchers starting with Lehmann [21] and Yanagimoto and Okamoto [28] started developing nonparametric functional measures of monotone dependence involving the entire joint distributions and the conditional distributions of the random variables. See Chapter 5 of Barlow and Porschan [3] for details where several functional measures of positive dependence with varying degrees of strength have been discussed. A nice thing about these monotone functional orderings is that if two random variables

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are dependent according to these functional measures, then the usual measure of dependence like Kendall's coefficient of concordance, Spearman's coefficient and many other well-defined coefficients of positive association will be nonnegative. Several partial orderings to compare the relative degree of dependence between two pairs of random variables, say (X_1, Y_1) and (X_2, Y_2) have also been studied. In most of the relevant literature, it is tactically assumed that the pairs (X_1, Y_1) and (X_2, Y_2) have the same margins, that is, $X_1 \overset{dist}{=} X_2$ and $Y_1 \overset{dist}{=} Y_2$. Refer to Kimeldorf and Sampson [18], Mosler and Scarsini [22] and Joe [13] for a unified treatment of families, orderings and measures of monotone dependence.

In many practical situations, however, we need to compare the degree of dependence between two pairs of random variables with different margins. Capéraà and Genest [5] presented several dependence orders for this problem. For example, one order called, *more stochastic increasing* (SI) is defined as follows. The dependence of Y_1 on X_1 is said to be less in the sense of SI than that of Y_2 on X_2, if and only if, for $0 < u \le 1$ and $x \le x'$,

$$H_{2[x']} \circ H_{2[x]}^{-1}(u) \le H_{1[x']} \circ H_{1[x]}^{-1}(u),$$

where $H_{i[x]}$ denotes the conditional distribution of Y_i given $X_i = x$, and the $H_{i[x]}^{-1}$ stands for the right continuous inverse of $H_{i[x]}$ for $i = 1, 2$. However, as summarized in the books by Joe [13], Nelsen [23] or Drouet-Mari and Kotz [8], many years of research into concepts and measures of association show that the proper way of comparing the relative degree of dependence between (X_1, Y_1) and (X_2, Y_2) is in terms of their associated copulas, implicitly defined in a unique fashion by the relation

$$C_i(u, v) = H_i\left\{F_i^{-1}(u), G_i^{-1}(v)\right\}, \quad u,\ v \in [0, 1],$$

where H_i is the joint cumulative distribution function of (X_i, Y_i), and F_i^{-1} and G_i^{-1} are the corresponding right continuous inverses of the marginal distributions. It is easy to see that the notion of more SI as defined by Capéraà and Genest [5] is not *copula-based*. Hence, Avérous et al. [2] suggested a modification of their definition by conditioning on the quantiles of the marginal distributions. That is, the dependence of Y_1 on X_1 is said to be less in the sense of SI than that of Y_2 on X_2, denoted by $(Y_1|X_1) \prec_{SI} (Y_2|X_2)$, if and only if, for $0 < u \le 1$ and $0 < p \le q \le 1$,

$$H_{2[\xi_{2q}]} \circ H_{2[\xi_{2p}]}^{-1}(u) \le H_{1[\xi_{1q}]} \circ H_{1[\xi_{1p}]}^{-1}(u),$$

where $\xi_{ip} = F_i^{-1}(p)$ stands for the pth quantile of the marginal distribution of X_i for $i = 1, 2$. They used this concept to compare the relative degree of dependence between two pairs of order statistics of a random sample from a continuous distribution. It is proved there for $i < j$, the dependence of the jth order statistic on the ith order statistic decreases as i and j draw apart. Subsequently, Khaledi and Kochar [17] further extended this result to the *generalized order statistics* which include order statistics, k-record values and several other cases of ordered random variables as special cases. Recently Dolati et al. [7] have used two modified weaker copula dependence orders called more right-tail increasing (RTI) and more left-tail decreasing (LTD) (see Avérous and Dortet-Bernadet [1]), to investigate the relative dependence between the extreme order statistics when the sample consists of independent but non-identically distributed observations with proportional hazard rates.

Avérous et al. [2] remark whether we could find suitable conditions under which more stochastic increasingness could be replaced by stronger dependence orderings. In this paper,

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we introduce a dependence ordering called more *reversed hazard rate* (RHR) ordering which is stronger than the more SI ordering to investigate the relative degree of dependence between two pairs of order statistics and k-record values. It is shown that for $i < j$, the dependence of the jth order statistic (k-record values) on the ith order statistic (k-record values) decreases, in the sense of RHR dependence ordering, as i and j draw apart. For example, the dependence of $X_{n:n}$ on $X_{i+1:n}$ is more than that of $X_{n:n}$ on $X_{i:n}$ for $1 \le i \le n-1$.

Definition 1. The dependence of Y_1 on X_1 is said to be less than that of Y_2 on X_2 according to RHR dependence order, denoted by $(Y_1|X_1) \prec_{RHR} (Y_2|X_2)$, if and only if, for $0 < u \le v \le 1$ and $0 < p \le q \le 1$,

$$H_{2[\xi_{2q}]} \circ H^{-1}_{2[\xi_{2p}]}(u) \times H_{1[\xi_{1q}]} \circ H^{-1}_{1[\xi_{1p}]}(v) \\ \le H_{2[\xi_{2q}]} \circ H^{-1}_{2[\xi_{2p}]}(v) \times H_{1[\xi_{1q}]} \circ H^{-1}_{1[\xi_{1p}]}(u), \tag{1}$$

where $H_{i[x]}$ denotes the conditional distribution function of Y_i given $X_i = x$, and $\xi_{ip} = F_i^{-1}(p)$ stands for the pth quantile of the marginal distribution of X_i for $i = 1, 2$.

This ordering may be attributed to Capéraà and Genest [5], although their original formulation was not *copula-based*. If Y_1 and X_1 are independent, then $(Y_1|X_1) \prec_{RHR} (Y_2|X_2)$, if and only if for $0 < p \le q \le 1$,

$$\frac{H_{2[\xi_{2q}]} \circ H^{-1}_{2[\xi_{2p}]}(u)}{u}$$

is increasing in $u \in (0, 1]$, that is, for $0 < p \le q \le 1$,

$$\frac{H_{2[\xi_{2q}]}(u)}{H_{2[\xi_{2p}]}(u)}$$

is increasing in $u \in (0, 1]$. This condition is equivalent to (X_2, Y_2) being DTP$_-$(0,1), dependent by total positivity with degree$_-$ (0,1) as introduced by Hu and Yang [11] (see also Shaked [25]). Obviously, if (1) holds and if (X_1, Y_1) is $DTP_-(0, 1)$, then so is (X_2, Y_2). Hence, this order may also be called *more* DTP$_-$(0,1) order. Setting $v = 1$ in (1), it holds that, for $u \in (0, 1]$,

$$H_{2[\xi_{2q}]} \circ H^{-1}_{2[\xi_{2p}]}(u) \le H_{1[\xi_{1q}]} \circ H^{-1}_{1[\xi_{1p}]}(u),$$

proving that $(Y_1|X_1) \prec_{RHR} (Y_2|X_2)$ implies $(Y_1|X_1) \prec_{SI} (Y_2|X_2)$ and which in turn implies the more well-known weaker dependence orderings like more PQD (or more concordance) ordering. As mentioned in Avérous et al. [2], more PQD ordering implies

$$\kappa(X_1, Y_1) \le \kappa(X_2, Y_2) \tag{2}$$

where $\kappa(X, Y)$ represents Spearman's rho, Kendall's tau, Gini's coefficient, or indeed any other copula-based measure of concordance satisfying the axioms of Scarsini [24]. In the special case where $F_1 = F_2$ and $G_1 = G_2$, it also follows more PQD ordering which implies that the pairs (X_1, Y_1) and (X_2, Y_2) are ordered by Pearson's correlation coefficient, namely

$$corr(X_1, Y_1) \le corr(X_2, Y_2).$$

Analogously one may define the more *hazard rate* dependence order (see also Capéraà and Genest [5]). The dependence of Y_1 on X_1 is said to be less than that of Y_2 on X_2 according

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to hazard rate (HR) dependence order, denoted by $(Y_1|X_1) \prec_{HR} (Y_2|X_2)$, if and only if, for $0 < v \le u \le 1$ and $0 < p \le q \le 1$,

$$\bar{H}_{2[\xi_{2q}]} \circ \bar{H}^{-1}_{2[\xi_{2p}]}(u) \times \bar{H}_{1[\xi_{1q}]} \circ \bar{H}^{-1}_{1[\xi_{1p}]}(v) \le \bar{H}_{2[\xi_{2q}]} \circ \bar{H}^{-1}_{2[\xi_{2p}]}(v) \times \bar{H}_{1[\xi_{1q}]} \circ \bar{H}^{-1}_{1[\xi_{1p}]}(u),$$

where $\bar{H}_{i[x]} = 1 - H_{i[x]}$ denotes the conditional survival function of Y_i given $X_i = x$, and $\xi_{ip} = F_i^{-1}(p)$ stands for the pth quantile of the marginal distribution of X_i for $i = 1, 2$. If Y_1 and X_1 are independent, then $(Y_1|X_1) \prec_{HR} (Y_2|X_2)$, if and only if for $0 < p \le q \le 1$,

$$\frac{\bar{H}_{2[\xi_{2q}]} \circ \bar{H}^{-1}_{2[\xi_{2p}]}(u)}{u}$$

is decreasing in $u \in (0, 1]$, that is, for $0 < p \le q \le 1$,

$$\frac{\bar{H}_{2[\xi_{2q}]}(u)}{\bar{H}_{2[\xi_{2p}]}(u)}$$

is increasing in $u \in (0, 1]$.

This condition is equivalent to (X_2, Y_2) being DTP(0, 1), dependent by total positivity with degree (0, 1) as introduced by Shaked [25]. This order may also be called *more* DTP (0, 1) order. It is easy to see that, by setting $u = 1$, the more HR dependence order also implies the more SI dependence order. However, the more HR dependence order does not imply the more RHR order, and vice verse (see Capéraà and Genest [5]).

Before stating our main results, let us review the following concepts which will be used in what follows.

Definition 2 (*Shaked and Shanthikumar [26]*). Let X and Y be two nonnegative random variables with distribution functions F and G, respectively. Then

- X is said to be less dispersed than Y (denoted by $X \le_{\text{disp}} Y$) if

  $$F^{-1}(\beta) - F^{-1}(\alpha) \le G^{-1}(\beta) - G^{-1}(\alpha)$$

  for all $0 < \alpha \le \beta < 1$, where F^{-1} and G^{-1} denote their corresponding right continuous inverses. Equivalently, one has $X \le_{\text{disp}} Y$ if and only if

  $$F\{F^{-1}(u) - c\} \le G\{G^{-1}(u) - c\}$$

  for every $c \ge 0$ and $0 < u < 1$.
- X is said to be smaller than Y in the convex transform order, denoted by $X \le_c Y$, if $G^{-1}F(x)$ is convex in $x \ge 0$.
- X is said to be smaller than Y in the star order, denoted by $X \le_* Y$, if $G^{-1}F(x)/x$ is increasing in $x \ge 0$.

Note that the convex transform order is also called *more IFR* order (cf. Kochar and Weins [19]) and X is IFR (increasing failure rate) if and only if it is convex ordered with respect to the exponential distributions. Similarly, the star order compares the relative IFRA (increasing failure rate average) property of two probability distributions.

Definition 3. A random variable X with distribution function F is said to be decreasing reversed hazard rate (DRHR) distributed if, for $u \ge 0$,

$$\frac{F(x)}{F(x+u)}$$

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is increasing in $x \geq 0$, or equivalently, if its density exists, the reversed hazard rate

$$\tilde{r}_X(x) = \frac{f(x)}{F(x)}$$

is decreasing.

For a more comprehensive discussion and other details about DRHR, one may refer to Block et al. [4].

The organization of the paper is as follows. In Section 2, we obtain some new results on dependence among order statistics, which partly strengthen the results of Avérous et al. [2]. Section 3 is devoted to studying the dependence properties of record values, which strengthens the result for k-record values in Khaledi and Kochar [17]. In Section 4, we obtain some new results on dependence for the frailty model. We conclude our discussion with some remarks in Section 5. The proofs of all the results are given in the Appendix.

2. Order statistics

In this section, we compare the relative degree of dependence between two pairs of order statistics from two populations in terms of the more RHR dependence order. To prove our main result, we will use the following lemmas.

Lemma 1. *Let X and Y be two nonnegative continuous random variables with distribution functions F and G, density functions f and g, respectively. If $X \leq_c Y$, $X \geq_{\text{disp}} Y$, and Y is DRHR, then*

$$\frac{G(G^{-1}(x) - c)}{F(F^{-1}(x) - c)}$$

is increasing in $x \in A = \left\{s \in (0, 1] : G^{-1}(s) > c \geq 0\right\}$.

Lemma 2. *Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the order statistics of a random sample of size n from a standard exponential distribution. Then, for $1 \leq i \leq n$ and arbitrary positive integer m,*

(a) $X_{i:n} \leq_c X_{1:m}$.
(b) $X_{n+1:n+1} \leq_c X_{n:n}$.

Theorem 1. *Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the order statistics of a random sample of size n from a continuous distribution with c.d.f. F and let $Y_{1:n'} \leq Y_{2:n'} \leq \cdots \leq Y_{n':n'}$ be the order statistics associated with a random sample of size n' from a continuous distribution with c.d.f. G. Then, for $1 \leq i < j \leq n$ and $1 \leq i' < j' \leq n'$,*

$$(Y_{j':n'} | Y_{i':n'}) \prec_{RHR} (X_{j:n} | X_{i:n})$$

holds for

$$i' \leq i, \quad n' - i' \geq n - i, \quad j' - i' \geq j - i = 1, \quad n - j \geq n' - j',$$

or

$$i' \leq i, \quad n' - i' \geq n - i, \quad j' = n', \quad j = n.$$

The following immediate consequences of Theorem 1 are of special interest.

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Corollary 1. *Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the order statistics corresponding to a random sample $X_1, \cdots, X_n$ from some continuous distribution. Then,*

- $(X_{n+1:n+1}|X_{i:n+1}) \prec_{RHR} (X_{n:n}|X_{i:n})$ *for* $1 \leq i \leq n$*;*
- $(X_{n:n}|X_{i:n}) \prec_{RHR} (X_{n:n}|X_{i+1:n})$ *for* $1 \leq i \leq n-1$*;*
- $(X_{j:n}|X_{i:n}) \prec_{RHR} (X_{i+1:n}|X_{i:n})$ *for* $1 \leq i \leq n-1$ *and* $j > i$*.*

3. Record values

Let $\{X_i, i \geq 1\}$ be a sequence of i.i.d random variables from a continuous population X, and let k be a positive integer. According to Dziubdziela and Kopociński [9], the nth k-record value from population X is defined as

$$R(n:k) = X_{L_n^k : L_n^k + k - 1},$$

where L_n^k are the nth occurrence times of k-record values of X,

$$L_0^k = 1,$$
$$L_{n+1}^k = \min\left\{ j > L_n^k : X_{L_n^k : L_n^k + k - 1} < X_{j:j+k-1} \right\}.$$

For $k = 1$, they are called the usual record values. It is shown in Khaledi and Kochar [17] that, for $i \geq i'$, and $j - i \leq j' - i'$,

$$\left(R(j':k)|R(i':k)\right) \prec_{SI} (R(j:k)|R(i:k)).$$

In this section, we will strengthen this result to the more RHR dependence ordering. First, let us introduce the following two Lemmas.

Lemma 3. *Let $R(n:k)$ be the nth k-record value from population X with standard exponential distribution, then, for $n' \geq n \geq 1$, and $k', k \geq 1$,*

$$R(n':k') \leq_c R(n:k).$$

Lemma 4. *Let $R(n:k)$ be the nth k-record value from population X with standard exponential distribution, then, for $n' \geq n \geq 1$, and $k' \geq k \geq 1$,*

$$R(n:k) \leq_{\text{disp}} R(n':k').$$

Now, we are ready to present the following theorem.

Theorem 2. *Let $R(n:k)$ and $R'(n':k')$ be the nth k-record value and n'th k'-record value from two continuous distributions, then,*

$$\left(R'(j':k')|R'(i':k')\right) \prec_{RHR} (R(j:k)|R(i:k)).$$

for $i \geq i' \geq 1$, $k \geq k' \geq 1$ and $j' - i' \geq j - i$.

The following results are the immediate consequences of Theorem 2.

Corollary 2. *Let $R(n:k)$ be the nth k-record value from continuous distribution, then*

- $(R(j':k)|R(i':k)) \prec_{RHR} (R(j:k)|R(i:k))$ *for* $j' - i' \geq j - i$ *and* $i \geq i'$*;*
- $(R(j:k)|R(i:k)) \prec_{RHR} (R(j:k+1)|R(i:k+1))$ *for* $j > i$*.*

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4. Frailty model

The well-known frailty model, which was first introduced by Vaupel et al. [30], is particularly useful to handle heterogeneity left unexplained by observed covariates. Given that the frailty random variable $V = v$, the conditional hazard rate function of the overall population random variable X is given by

$$\lambda(t|v) = v\lambda_0(t), \quad \text{for all } t \geq 0, \tag{3}$$

where $\lambda_0(t)$ is the hazard rate function of the baseline random variable Y. The frailty is considered basically unobservable and hence the individual level model in Eq. (3) is unobservable. Gupta and Kirmani [10] considered the population model below where the hazard function is considered to be a randomly drawn individual,

$$\overline{F}(t) = \mathrm{E}\left[\overline{G}^V(t)\right], \quad \text{for all } t \geq 0,$$

where $\overline{F}$ and $\overline{G}$ are survival functions of X and Y. Proposition 2.1 in Gupta and Kirmani [10] shows that

$$\mathrm{E}(V|X > t) \quad \text{is decreasing in } t \geq 0,$$

which implies that there is some kind of negative dependence between X and V. In fact, X and V are negatively likelihood ratio order dependence as shown by Xu and Li [27]. Now, if frailty random variables V_1 and V_2 are from two populations, we may be interested in knowing which one has more effect on its corresponding overall population random variable. The following theorem answers this question.

Theorem 3. *Let (X_i, V_i) be frailty model random vector for $i = 1, 2$. Then, $V_2 \leq_* V_1$ if and only if one of the following statements holds,*

(i) $(X_1|V_1) \prec_{RHR} (X_2|V_2)$;
(ii) $(X_1|V_1) \prec_{SI} (X_2|V_2)$.

5. Concluding remarks

This paper extends the work of Avérous et al. [2] and Khaledi and Kochar [17] for comparing the degree of dependence between two pairs of order statistics and k-record values from the same continuous distribution through the stronger new more RHR dependence ordering. We partly strengthen the main result of Avérous et al. [2] on dependence among order statistics. We also strengthen the more SI dependence ordering result for k-record values in Khaledi and Kochar [17] to the more RHR dependence ordering. We conjecture that the main result in Avérous et al. [2] can be generalized to the more RHR ordering. Actually, if one can prove that for order statistics from exponential distribution, $X_{j:m} \leq_c X_{i:n}$ holds for $i \leq j$ and $m - j \leq n - i$, the result will follow. But we have been unable to establish this result so far.

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Appendix. Proofs

Proof of Lemma 1. Note that, $X \le_c Y$, implies

$$\frac{f(F^{-1}(x))}{g(G^{-1}(x))} \quad \text{is increasing in } x \ge 0.$$

That is,

$$\frac{\tilde{r}_X(F^{-1}(x))}{\tilde{r}_Y(G^{-1}(x))} \ge \frac{\tilde{r}_X(F^{-1}(x^*))}{\tilde{r}_Y(G^{-1}(x^*))}$$

for $0 \le x^* \le x \le 1$, where $\tilde{r}_X$ and $\tilde{r}_Y$ denote the reversed hazard rates of X and Y, respectively.

Setting,

$$x^* = F(F^{-1}(x) - c), \quad c \ge 0,$$

it holds that,

$$\frac{\tilde{r}_X(F^{-1}(x))}{\tilde{r}_Y(G^{-1}(x))} \ge \frac{\tilde{r}_X(F^{-1}(x) - c)}{\tilde{r}_Y(G^{-1}(x^*))}.$$

Note that, $X \ge_{\text{disp}} Y$ implies

$$F(F^{-1}(x) - c) \ge G(G^{-1}(x) - c),$$

i.e.,

$$G^{-1}(x^*) \ge G^{-1}(x) - c.$$

According to the DRHR property of Y, it follows that,

$$\tilde{r}_Y(G^{-1}(x^*)) \le \tilde{r}_Y(G^{-1}(x) - c),$$

for $G^{-1}(x) > c \ge 0$. Thus, for $G^{-1}(x) > c \ge 0$,

$$\frac{\tilde{r}_X(F^{-1}(x))}{\tilde{r}_Y(G^{-1}(x))} \ge \frac{\tilde{r}_X(F^{-1}(x) - c)}{\tilde{r}_Y(G^{-1}(x) - c)},$$

which implies

$$\frac{G(G^{-1}(x) - c)}{F(F^{-1}(x) - c)}$$

is increasing in $x \in A = \left\{s \in (0, 1] : G^{-1}(s) > c \ge 0\right\}$. ■

Proof of Lemma 2. (a) According to Theorem 5.8 of Barlow and Proschan ([3], p. 108), $X_{i:n}$ has increasing hazard rate for $1 \le i \le n$. Hence, for arbitrary positive integer m,

$$X_{i:n} \le_c X_{1:m},$$

since $X_{1:m}$ is exponential.

(b) The distribution functions of $X_{n:n}$ and $X_{n+1:n+1}$ are

$$G(x) = P(X_{n:n} \le x) = (1 - \mathrm{e}^{-x})^n,$$

and

$$F(x) = P(X_{n+1:n+1} \le x) = (1 - \mathrm{e}^{-x})^{n+1},$$

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respectively. Thus,

$$G^{-1}(F(x)) = -\log\left[1-(1-\mathrm{e}^{-x})^{1+1/n}\right].$$

Since $(1-\mathrm{e}^{-x})^{1+1/n}$ can be regarded as the distribution function of a random variable Z, it is enough to prove that Z has increasing hazard rate. It is easy to see that the density function of Z is log-concave, which implies that Z has increasing hazard rate (see Lemma 5.8 of Barlow and Proschan [3], p. 77). Hence the result follows. ■

Proof of Theorem 1. As pointed out in Avérous et al. [2], the copula of a pair of order statistics is independent of the parent distribution in case the random samples are from continuous distributions. Since the RHR dependence order is copula-based, we can assume without loss of generality that $F = G$ is a standard exponential distribution.

According to the Markov property of order statistics (see David and Nagaraja [6]), and the non-aging property of exponential distribution, it follows that,

$$H_{2[s]}(t) = P(X_{j:n} \le t \mid X_{i:n} = s) = F_{j-i:n-i}(t-s),$$

where $F_{j-i:n-i}$ denotes the distribution function of $X_{j-i:n-i}$. Thus, it holds that, for $0 < p \le 1$,

$$H^{-1}_{2[\xi_{2p}]}(u) = F^{-1}_{j-i:n-i}(u) + \xi_{2p}.$$

Further, for $0 < p \le q \le 1$,

$$H_{2[\xi_{2q}]} \circ H^{-1}_{2[\xi_{2p}]}(u) = F_{j-i:n-i}\left(F^{-1}_{j-i:n-i}(u) - (\xi_{2q} - \xi_{2p})\right).$$

Similarly, for $0 < p \le q \le 1$,

$$H_{1[\xi_{1q}]} \circ H^{-1}_{1[\xi_{1p}]}(u) = F_{j'-i':n'-i'}\left(F^{-1}_{j'-i':n'-i'}(u) - (\xi_{1q} - \xi_{1p})\right).$$

By the definition of more RHR order, we need to prove, for $0 < u \le v \le 1$ and $0 < p \le q \le 1$,

$$\begin{aligned} &F_{j-i:n-i}\left(F^{-1}_{j-i:n-i}(u) - (\xi_{2q} - \xi_{2p})\right) \times F_{j'-i':n'-i'}\left(F^{-1}_{j'-i':n'-i'}(v) - (\xi_{1q} - \xi_{1p})\right) \\ &\quad \le F_{j-i:n-i}\left(F^{-1}_{j-i:n-i}(v) - (\xi_{2q} - \xi_{2p})\right) \\ &\qquad \times F_{j'-i':n'-i'}\left(F^{-1}_{j'-i':n'-i'}(u) - (\xi_{1q} - \xi_{1p})\right). \end{aligned} \tag{4}$$

According to Lemma 2.1 of Khaledi and Kochar [16],

$$X_{i':n'} \le_{\mathrm{disp}} X_{i:n}$$

for $i' \le i$ and $n' - i' \ge n - i$, thus,

$$\xi_{2q} - \xi_{2p} \ge \xi_{1q} - \xi_{1p}. \tag{5}$$

Also, for $j' - i' \ge j - i$ and $n' - j' \le n - j$,

$$F_{j-i:n-i}\left(F^{-1}_{j-i:n-i}(u) - c\right) \le F_{j'-i':n'-i'}\left(F^{-1}_{j'-i':n'-i'}(u) - c\right), \quad c \ge 0. \tag{6}$$

Combining Eqs. (5) and (6), it holds that,

$$\begin{aligned} F_{j-i:n-i}\left(F^{-1}_{j-i:n-i}(u) - (\xi_{2q} - \xi_{2p})\right) &\le F_{j-i:n-i}\left(F^{-1}_{j-i:n-i}(u) - (\xi_{1q} - \xi_{1p})\right) \\ &\le F_{j'-i':n'-i'}\left(F^{-1}_{j'-i':n'-i'}(u) - (\xi_{1q} - \xi_{1p})\right). \end{aligned}$$

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Hence, for $u \in \{x : F^{-1}_{j'-i':n'-i'}(x) \le \xi_{1q} - \xi_{1p}\} \bigcup \{x : F^{-1}_{j-i:n-i}(x) \le \xi_{2q} - \xi_{2p}\}$, Eq. (4) holds. Thus, it is enough to prove that,

$$\begin{aligned}\frac{H_{2[\xi_{2q}]} \circ H^{-1}_{2[\xi_{2p}]}(u)}{H_{1[\xi_{1q}]} \circ H^{-1}_{1[\xi_{1p}]}(u)} &= \frac{F_{j-i:n-i}\left(F^{-1}_{j-i:n-i}(u) - (\xi_{2q} - \xi_{2p})\right)}{F_{j'-i':n'-i'}\left(F^{-1}_{j'-i':n'-i'}(u) - (\xi_{1q} - \xi_{1p})\right)} \\ &= \frac{F_{j-i:n-i}\left(F^{-1}_{j-i:n-i}(u) - (\xi_{2q} - \xi_{2p})\right)}{F_{j'-i':n'-i'}\left(F^{-1}_{j'-i':n'-i'}(u) - (\xi_{2q} - \xi_{2p})\right)} \\ &\quad \times \frac{F_{j'-i':n'-i'}\left(F^{-1}_{j'-i':n'-i'}(u) - (\xi_{2q} - \xi_{2p})\right)}{F_{j'-i':n'-i'}\left(F^{-1}_{j'-i':n'-i'}(u) - (\xi_{1q} - \xi_{1p})\right)}\end{aligned}$$

is increasing in $u \in \{x : F^{-1}_{j'-i':n'-i'}(x) > \xi_{1q} - \xi_{1p}\} \bigcap \{x : F^{-1}_{j-i:n-i}(x) > \xi_{2q} - \xi_{2p}\}$. Since the density function of $X_{j'-i':n'-i'}$ is log-concave, it follows that

$$\frac{F_{j'-i':n'-i'}\left(F^{-1}_{j'-i':n'-i'}(u) - (\xi_{2q} - \xi_{2p})\right)}{F_{j'-i':n'-i'}\left(F^{-1}_{j'-i':n'-i'}(u) - (\xi_{1q} - \xi_{1p})\right)}$$

is increasing in $u \in \{x : F^{-1}_{j'-i':n'-i'}(x) > \xi_{1q} - \xi_{1p}\}$. Observing that, for $j' - i' \ge j - i$ and $n' - j' \le n - j$,

$$F^{-1}_{j'-i':n'-i'}(u) \ge F^{-1}_{j-i:n-i}(u), \quad u \in (0, 1],$$

it is sufficient to prove that

$$\frac{F_{j-i:n-i}\left(F^{-1}_{j-i:n-i}(u) - (\xi_{2q} - \xi_{2p})\right)}{F_{j'-i':n'-i'}\left(F^{-1}_{j'-i':n'-i'}(u) - (\xi_{2q} - \xi_{2p})\right)}$$

is increasing in $u \in \{x : F^{-1}_{j-i:n-i}(x) > \xi_{2q} - \xi_{2p}\}$. The required result follows from Lemmas 1 and 2. ■

Proof of Lemma 3. According to Kamps [15], the distribution function of the nth k-record value is,

$$P(R(n:k) \le x) = \frac{k^n}{(n-1)!}\int_0^x \mathrm{e}^{-ku} u^{n-1} \mathrm{d}u = \frac{1}{(n-1)!}\int_0^{kx} \mathrm{e}^{-u} u^{n-1} \mathrm{d}u = F_n(kx),$$

where $F_n(x)$ is the lower incomplete gamma distribution with parameter n. According to Van Zwet ([29], p. 60),

$$F_{n'}(x) \le_c F_n(x)$$

for $n' \ge n \ge 1$. Hence,

$$\frac{1}{k} F_n^{-1}(F_{n'}(k'x))$$

is convex in $x \ge 0$ for $k', k \ge 1$, since the convex order is independent of the scale parameter. That is, for $n' \ge n \ge 1$, and $k', k \ge 1$,

$$R(n' : k') \le_c R(n : k). \quad ■$$

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Proof of Lemma 4. It has been proved in Kochar [20] that, for $n' \geq n \geq 1$,

$$F_n(x) \leq_{\text{disp}} F_{n'}(x),$$

that is, for $0 < p \leq q \leq 1$,

$$F_n^{-1}(q) - F_n^{-1}(p) \leq F_{n'}^{-1}(q) - F_{n'}^{-1}(p).$$

Hence, it holds that, for $k \geq k' \geq 1$,

$$\frac{1}{k} F_n^{-1}(q) - \frac{1}{k} F_n^{-1}(p) \leq \frac{1}{k'} F_{n'}^{-1}(q) - \frac{1}{k'} F_{n'}^{-1}(p),$$

i.e., for $n' \geq n \geq 1$ and $k \geq k' \geq 1$,

$$F_n(kx) \leq_{\text{disp}} F_{n'}(kx),$$

that is,

$$R(n : k) \leq_{\text{disp}} R(n' : k'). \quad \blacksquare$$

Proof of Theorem 2. As in the proof of Theorem 1, we can assume that the parent distribution of $R(n : k)$ and $R(n' : k')$ is standard exponential. According to Kamps [15], the k-record value also has the Markov property (see also Proposition 2.1 of Hu and Zhuang [12]). It holds that for $j > i$, and $t \geq s \geq 0$,

$$H_{2[s]}(t) = P(R(j : k) \leq t | R(i : k) = s) = P(R(j - i : k) \leq t - s),$$

The rest of the proof is similar to that of Theorem 1 and it follows using Lemmas 3 and 4. It is omitted for the sake of brevity. $\blacksquare$

Proof of Theorem 3. Note that, for $s > 0$,

$$\begin{aligned} H_{2[s]}(t) &= P(X_2 \leq t | V_2 = s) \\ &= 1 - P(X_2 > t | V_2 = s) \\ &= 1 - \bar{G}_2^s(t). \end{aligned}$$

Hence, for $0 < p \leq 1$,

$$H_{2[s]}^{-1}(p) = \bar{G}_2^{-1}[(1 - p)^{1/s}],$$

and, for $0 < p \leq q \leq 1$,

$$H_{2[\xi_{2q}]} \circ H_{2[\xi_{2p}]}^{-1}(u) = 1 - (1 - u)^{\xi_{2q}/\xi_{2p}},$$

where $\xi_{2p} = K_2^{-1}(p)$, the pth quantile of the distribution of V_2.

Similarly, for $0 < p \leq q \leq 1$

$$H_{1[\xi_{1q}]} \circ H_{1[\xi_{1p}]}^{-1}(u) = 1 - (1 - u)^{\xi_{1q}/\xi_{1p}},$$

where $\xi_{1p} = K_1^{-1}(p)$, the pth quantile of the distribution of V_1.

$V_2 \leq_* V_1$ implies, for $0 < p \leq q \leq 1$,

$$\frac{\xi_{1q}}{\xi_{1p}} \geq \frac{\xi_{2q}}{\xi_{2p}},$$

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therefore,

$$(1-y)^{\xi_{iq}/\xi_{ip}-1}$$

is TP_2 in $(y, i) \in (0, 1] \times \{1, 2\}$. For the definition of TP_2, one may refer to Karlin [14]. Also observing that $1_{\{y \le u\}}$ is TP_2 in $(u, y) \in (0, 1] \times (0, 1]$. By applying basic composition formula (see Karlin[14], p. 17),

$$h(u,i) = 1-(1-u)^{\xi_{iq}/\xi_{ip}} = \frac{\xi_{iq}}{\xi_{ip}} \int_0^1 1_{\{y \le u\}}(1-y)^{\xi_{iq}/\xi_{ip}-1}\mathrm{d}y,$$

is TP_2 in $(u, i) \in (0, 1] \times \{1, 2\}$. Hence,

$$(X_1|V_1) \prec_{RHR} (X_2|V_2). \tag{7}$$

Conversely, if Eq. (7) holds, then

$$(X_1|V_1) \prec_{SI} (X_2|V_2),$$

which is equivalent to, for $0 < p \le q \le 1$,

$$(1-u)^{\xi_{2q}/\xi_{2p}} \ge (1-u)^{\xi_{1q}/\xi_{1p}},$$

i.e.,

$$\frac{\xi_{2q}}{\xi_{2p}} \le \frac{\xi_{1q}}{\xi_{1p}},$$

which implies

$$V_2 \le_* V_1.$$

The required result follows immediately. ■

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