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Some New Results on Stochastic Comparisons of Spacings from Heterogeneous Exponential Distributions

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Some new results are obtained on stochastic orderings between random vectors of spacings from heterogeneous exponential distributions and homogeneous ones. Let $D_1, ..., D_n$ be the normalized spacings associated with independent exponential random variables $X_1, ..., X_n$, where X_i has hazard rate λ_i, $i = 1, 2, ..., n$. Let $D_1^*, ..., D_n^*$ be the normalized spacings of a random sample $Y_1, ..., Y_n$ of size n from an exponential distribution with hazard rate $\bar{\lambda} = \sum_{i=1}^{n} \lambda_i / n$. It is shown that for any $n \geqslant 2$, the random vector $(D_1, ..., D_n)$ is greater than the random vector $(D_1^*, ..., D_n^*)$ in the sense of multivariate likelihood ratio ordering. It also follows from the results proved in this paper that for any j between 2 and n, the survival function of $X_{j:n} - X_{1:n}$ is Schur convex. © 1996 Academic Press, Inc.

1. INTRODUCTION

There is an extensive literature on order statistics and spacings from a single underlying distribution. However, not much attention has been given to the case when the underlying random variables are not independent or not identically distributed. Some interesting partial ordering results on

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order statistics and spacings from independent but nonidentical random variables have been obtained by Pledger and Proschan [10], Proschan and Sethuraman [11], Bapat and Kochar [1], Boland, El-Neweihi, and Proschan [2], Kochar and Kirmani [6], Boland, Hollander, Joag-Dev, and Kochar [3], and Kochar and Korwar [7].

Let $X_1, ..., X_n$ be independent random variables with possibly different probability distributions. Let $X_{i:n}$ denote the ith order statistic of $X_1, ..., X_n$. Let $D_{i:n} = (n-i+1)(X_{i:n} - X_{i-1:n})$ denote the ith normalized spacing, $i = 1, ..., n$, with $X_{0:n} \equiv 0$. To simplify notation, we shall drop the second suffix n in $D_{i:n}$ when there is no ambiguity. Pledger and Proschan [10] considered the problem of stochastically comparing the order statistics and the spacings of nonidentical independent exponential random variables with those corresponding to stochastically comparable independent and identically distributed exponential random variables. Kochar and Korwar [7] pursued this topic further in their paper and strengthened some of the results of Pledger and Proschan [10]. In this paper some new results on this problem are obtained.

There are many ways in which stochastic comparisons between a random variable X and another random variable Y can be made. In the *usual* stochastic ordering case, one says that a random variable X with distribution function F is stochastically smaller than a random variable Y with distribution function G (and write it, as $X \overset{st}{\leqslant} Y$) if $F(t) \geqslant G(t)$ for all t. That is, $X \overset{st}{\leqslant} Y$ if the survival function of X is everywhere dominated by that of Y. In some cases, a pair of distributions may satisfy a stronger condition called *likelihood ratio ordering*. If distributions F and G possess densities (or probability mass functions) f and g, respectively, and $f(x)/g(x)$ is nonincreasing in x, then we say that X is smaller than Y according to likelihood ratio ordering. This is denoted by $X \overset{\ell r}{\leqslant} Y$. It is known that $X \overset{\ell r}{\leqslant} Y$ implies $\bar{F}(x)/\bar{G}(x)$ is nonincreasing in x, where $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$ denote the survival functions of X and Y, respectively. This latter condition defines *hazard rate ordering*. In the case of absolutely continuous distributions, this is equivalent to the hazard rate of F, $r_F(x) = f(x)/\bar{F}(x)$, being uniformly greater than $r_G(x) = g(x)/\bar{G}(x)$, the hazard rate of G. If this happens, we say that X is smaller than Y according to *hazard rate ordering* and write it as $X \overset{hr}{\leqslant} Y$. Note that hazard rate ordering implies stochastic ordering. Lehmann and Rojo [8] characterize these orderings in terms of maximal invariants with respect to the group of monotone transformations.

The above notions of stochastic dominance among univariate random variables can be extended to the multivariate case. A random vector $\mathbf{X} = (X_1, ..., X_n)$ is *smaller than another random vector* $\mathbf{Y} = (Y_1, ..., Y_n)$ *in the multivariate stochastic order* (and written as $\mathbf{X} \overset{st}{\leqslant} \mathbf{Y}$) if $E[\phi(\mathbf{X})] \leqslant E[\phi(\mathbf{Y})]$

for all increasing functions ϕ whenever the expectations exist. To define multivariate likelihood ratio ordering, let us denote by f and g the density functions of $\mathbf{X}$ and $\mathbf{Y}$, respectively. Then $\mathbf{X}$ *is smaller than* $\mathbf{Y}$ *in the multivariate likelihood ratio order* (written as $\mathbf{X} \overset{\ell r}{\preccurlyeq} \mathbf{Y}$) if

$$f(\mathbf{x})\, g(\mathbf{y}) \leqslant f(\mathbf{x} \wedge \mathbf{y})\, g(\mathbf{x} \vee \mathbf{y}) \qquad \text{for every } \mathbf{x} \text{ and } \mathbf{y} \text{ in } \mathscr{R}^n, \tag{1.1}$$

where $\mathbf{x} \wedge \mathbf{y} = (\min(x_1, y_1), ..., \min(x_n, y_n))$ and $\mathbf{x} \vee \mathbf{y} = (\max(x_1, y_1), ..., \max(x_n, y_n))$.

It is known that multivariate likelihood ratio ordering implies multivariate stochastic ordering, but the converse is not true. Also if two random vectors are ordered according to multivariate likelihood ratio (stochastic) ordering, then their corresponding subsets of components are also ordered accordingly. See Chapters 1 and 4 of Shaked and Shanthikumar [12] for more details on various kinds of stochastic orders, their interrelationships and their properties.

The concepts of majorization of vectors and Schur convexity of functions will also be needed. Let $\{x_{(1)} \leqslant x_{(2)} \leqslant \cdots \leqslant x_{(n)}\}$ denote the increasing arrangement of the components of the vector $\mathbf{x} = (x_1, x_2, ..., x_n)$. The vector $\mathbf{y}$ is said to *majorize* the vector $\mathbf{x}$ (written as $\mathbf{x} \overset{\mathrm{m}}{\preccurlyeq} \mathbf{y}$) if $\sum_{1=1}^{j} y_{(i)} \leqslant \sum_{i=1}^{j} x_{(i)}$, for $j = 1, ..., n-1$ and $\sum_{i=1}^{n} y_{(i)} = \sum_{i=1}^{n} x_{(i)}$.

Definition 1.1. A real-valued function ϕ defined on a set $\mathscr{A} \subset \mathscr{R}^n$ is said to be Schur convex (Schur concave) on $\mathscr{A}$ if $\mathbf{x} \overset{\mathrm{m}}{\preccurlyeq} \mathbf{y} \Rightarrow \phi(\mathbf{x}) \leqslant (\geqslant)\, \phi(\mathbf{y})$.

In this paper some new stochastic relations between spacings of independent but nonidentically distributed exponential random variables are established. Let $D_1, ..., D_n$ be the normalized spacings associated with independent exponential random variables $X_1, ..., X_n$, with X_i having hazard rate λ_i, $i = 1, ..., n$. Let $Y_1, ..., Y_n$ be a random sample of size n from an exponential distribution with common hazard rate $\bar{\lambda} = \sum_{i=1}^{n} \lambda_i / n$. Let $D_1^*, ..., D_n^*$ be their associated normalized spacings. Pledger and Proschan [10] proved that in this case

$$D_1^* \overset{st}{=} D_1, \qquad D_i^* \overset{st}{\preccurlyeq} D_i, \qquad \text{for} \quad i = 2, ..., n. \tag{1.2}$$

Kochar and Korwar [7] strengthened this result from stochastic ordering to likelihood ratio ordering. In the next section this result is further strengthened to establish multivariate likelihood ratio ordering between the vectors of spacings $(D_1, ..., D_n)$ and $(D_1^*, ..., D_n^*)$. This result is analogous to Theorem 1.2 of Proschan and Sethuraman [11] on multivariate stochastic ordering between the corresponding vectors of order statistics. In our case the stochastic comparison is in terms of *multivariate likelihood ratio ordering*, an ordering which is stronger than the multivariate stochastic ordering. A consequence of this result is that $X_{j:n} - X_{i:n}$ is stochastically greater than $Y_{j:n} - Y_{i:n}$ for $1 \leqslant i < j \leqslant n$. In the case of the sample range, a stronger result is proved. It is shown that its survival function is Schur convex in $(\lambda_1, ..., \lambda_n)$. This and some other related results are discussed in Section 3.

2. COMPARISONS WITH I.I.D. EXPONENTIALS

Now we prove the main theorem of this section.

THEOREM 2.1. *Let $D_1, ..., D_n$ be the normalized spacings associated with independent exponential random variables $X_1, ..., X_n$, where X_i has hazard rate λ_i, $i = 1, ..., n$. Let $D_1^*, ..., D_n^*$ be the normalized spacings of a random sample $Y_1, ..., Y_n$ of size n from an exponential distribution with hazard rate $\bar{\lambda} = \sum_{i=1}^n \lambda_i / n$. Then for any $n \geqslant 2$,*

$$(D_1^*, ..., D_n^*) \overset{\ell r}{\leqslant} (D_1, ..., D_n). \tag{2.1}$$

Proof. Let

$$u_j = \sum_{i=1}^{j} \frac{y_i}{n-i+1}, \qquad j = 1, ..., n.$$

Then as seen in Theorem 3.1 of Kochar and Kirmani [6], the joint density of $(D_1, ..., D_n)$ is

$$g(\mathbf{y}) = \frac{\prod_{i=1}^n \lambda_i}{n!} \sum_{\mathbf{r}} e^{-\sum_{i=1}^n \lambda_i u_{j_i}}, \tag{2.2}$$

where $\sum_{\mathbf{r}}$ denotes summation over all permutations $\{j_1, ..., j_n\}$ of n integers $\{1, ..., n\}$.

Since $D_1^*, ..., D_n^*$ are i.i.d. exponentials each with hazard rate $\bar{\lambda}$, the joint density of $D_1^*, ..., D_n^*$ is

$$\begin{aligned} f(\mathbf{x}) &= \bar{\lambda}^n e^{-\bar{\lambda}(x_1 + \cdots + x_n)} \\ &= \bar{\lambda}^n e^{-\bar{x}(\lambda_1 + \cdots + \lambda_n)}. \end{aligned}$$

Therefore,

$$f(\mathbf{x})\, g(\mathbf{y}) = \frac{\prod_{i=1}^n \lambda_i}{n!} \bar{\lambda}^n \sum_{\mathbf{r}} e^{-\sum_{i=1}^n \lambda_i (u_{j_i} + \bar{x})}$$

and

$$f(\mathbf{x} \wedge \mathbf{y})\, g(\mathbf{x} \vee \mathbf{y}) = \frac{\prod_{i=1}^{n} \lambda_i}{n!} \bar{\lambda}^n \sum_{\mathbf{r}} e^{-\sum_{i=1}^{n} \lambda_i (v_{j_i} + \overline{x \wedge y})},$$

where $(j_1, ..., j_n)$ is a permutation of n integers $\{1, ..., n\}$ and

$$v_j = \sum_{i=1}^{j} \frac{x_i \vee y_i}{n-i+1}.$$

It is enough to prove that under the given conditions,

$$f(\mathbf{x})\, g(\mathbf{y}) \leqslant f(\mathbf{x} \wedge \mathbf{y})\, g(\mathbf{x} \vee \mathbf{y}) \qquad \text{for every } \mathbf{x} \text{ and } \mathbf{y} \text{ in } \mathscr{R}^n. \tag{2.3}$$

Since the exponential density is log-convex, it follows from Marshall and Olkin [9, p. 85] that the function

$$h(\mathbf{z}) = \sum_{\mathbf{r}} e^{-\sum_{i=1}^{n} \lambda_i z_{j_i}}$$

is Schur-convex in $\mathbf{z} = (\mathbf{z_1}, ..., \mathbf{z_n})$. The required result will follow from this if we can show that

$$(u_1 + \bar{x}, ..., u_n + \bar{x}) \overset{m}{\leqslant} (v_1 + \overline{x \wedge y}, ..., v_n + \overline{x \wedge y}). \tag{2.4}$$

Clearly the components of the two vectors in (2.4) are nondecreasing and the sum of the elements of each vector is $n(\bar{x} + \bar{y})$.

We have to prove that

$$\sum_{\ell=1}^{j} (v_\ell + \overline{x \wedge y}) \leqslant \sum_{\ell=1}^{j} (u_\ell + \bar{x}) \qquad \text{for} \quad j = 1, ..., n. \tag{2.5}$$

The right-hand side of (2.5) is

$$\begin{aligned} \sum_{\ell=1}^{j} (u_\ell + \bar{x}) &= \sum_{\ell=1}^{j} \left\{ \sum_{i=1}^{\ell} \frac{y_i}{n-i+1} + \bar{x} \right\} \\ &= \frac{j}{n} y_1 + \frac{j-1}{n-1} y_2 + \cdots + \frac{1}{n-j+1} y_j + j\bar{x} \\ &= \frac{j}{n} y_1 + \frac{j-1}{n-1} y_2 + \cdots + \frac{1}{n-j+1} y_j + \frac{j}{n}(x_1 + \cdots + x_n) \end{aligned}$$

$$\sum_{\ell=1}^{j} (u_\ell + \bar{x}) = \frac{j}{n}(y_1 + x_1) + \frac{j-1}{n-1}(y_2 + x_2) + \cdots + \frac{1}{n-j+1}(y_j + x_j)$$
$$+ \left(\frac{j}{n} - \frac{j-1}{n-1}\right) x_2 + \left(\frac{j}{n} - \frac{j-2}{n-2}\right) x_3 + \cdots$$
$$+ \left(\frac{j}{n} - \frac{1}{n-j+1}\right) x_j. \tag{2.6}$$

A similar argument applied to the left-hand side of (2.5) yields the following result:

$$\sum_{\ell=1}^{j} (v_\ell + \overline{x \wedge y}) = \frac{j}{n}(y_1 + x_1) + \frac{j-1}{n-1}(y_2 + x_2) + \cdots + \frac{1}{n-j+1}(y_j + x_j)$$
$$+ \left(\frac{j}{n} - \frac{j-1}{n-1}\right)(x_2 \wedge y_2) + \left(\frac{j}{n} - \frac{j-2}{n-2}\right)(x_3 \wedge y_3) + \cdots$$
$$+ \left(\frac{j}{n} - \frac{1}{n-j+1}\right)(x_j \wedge y_j). \tag{2.7}$$

The required result (2.5) then follows by comparing (2.6) and (2.7) and noting that $j/n - (j-k)/(n-k) > 0$ for $j < n$. The relationship (2.3) then follows immediately. ▮

Theorem 3.5 of Kochar and Korwar [7] which establishes likelihood ratio ordering between D_i and D_i^* immediately follows from this since if two random vectors are ordered according to multivariate likelihood ratio ordering, then so are their marginals. As is well known, multivariate likelihood ratio ordering implies multivariate stochastic ordering, and the latter is invariant under monotone transformations. This is the context of the following corollary.

COROLLARY 2.1. *Let $D_1, ..., D_n$ be the normalized spacings associated with independent exponential random variables $X_1, ..., X_n$, where X_i has hazard rate λ_i, $i = 1, ..., n$. Let $D_1^*, ..., D_n^*$ be the normalized spacings of a random sample $Y_1, ..., Y_n$ of size n from an exponential distribution with hazard rate $\bar{\lambda}$. Then*

(a) *for any $n \geqslant 2$*

$$(D_1^*, ..., D_n^*) \overset{st}{\leqslant} (D_1, ..., D_n), \tag{2.8}$$

(b) *for $1 \leqslant i < j \leqslant n$,*

$$(Y_{j:n} - Y_{i:n}) \overset{st}{\leqslant} (X_{j:n} - X_{i:n}). \tag{2.9}$$

Proof. (a) Since multivariate likelihood ratio ordering implies multivariate stochastic ordering, the result follows.

(b) It follows from the property of multivariate stochastic ordering that if $\mathbf{X} \overset{st}{\leqslant} \mathbf{Y}$ then $E[\phi(\mathbf{X})] \leqslant E[\phi(\mathbf{Y})]$ for all increasing functions ϕ whenever the expectations exist. The proof follows after observing that $X_{j:n} - X_{i:n} = \sum_{r=i+1}^{j} D_r/(n-r+1)$ is a nondecreasing function of $(D_1, ..., D_n)$. ▮

In particular, the above result gives a lower bound on the survival function of the sample range of heterogeneous exponential random variables in terms of that of the sample range of a random sample of the same size form an exponential distribution with common hazard rate $\bar{\lambda}$. The distribution of the sample range of a random sample from a distribution F is well known. (See, e.g., Eq. (2.3.3), page 12 of David [4]). Taking F to be the exponential distribution, it is then easy to see that the distribution of the sample range is the same as that of the largest order statistic in a sample of size $(n-1)$ from the exponential distribution with hazard rate $\bar{\lambda}$. Using these results we get the following corollary.

COROLLARY 2.2. *Let $X_1, ..., X_n$ be independent exponential random variables with X_i having hazard rate λ_i, for $i = 1, ..., n$. Then for $x > 0$,*

$$P[X_{n:n} - X_{1:n} \leqslant x] \leqslant [1 - \exp(-\bar{\lambda}x)]^{n-1}. \tag{2.10}$$

3. SOME SCHUR TYPE RESULTS

The sample range and the generalized spacings of the type $X_{j:n} - X_{1:n}$ are of special interest in statistics. Kochar and Korwar [7] have proved in their Theorem 3.2 that for any $n > 1$, the survival function of $X_{2:n} - X_{1:n}$ is Schur convex in $\boldsymbol{\lambda}$. The next result generalizes this result and strengthens Corollary 2.2 above. It is shown that the vector $(X_{2:n} - X_{1:n}, ..., X_{n:n} - X_{1:n})$ is stochastically larger when the λ_i's are more dispersed in the sense of majorization.

THEOREM 3.1. *Let $X_1, ..., X_n$ be independent exponential random variables with X_i having hazard rate λ_i, for $i = 1, ..., n$. Let $Y_1, ..., Y_n$ be another set of independent exponential random variables with λ_i^* as the hazard rate of Y_i, $i = 1, ..., n$. Let $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_n)$ and $\boldsymbol{\lambda}^* = (\lambda_i^*, ..., \lambda_n^*)$, then $\boldsymbol{\lambda} \overset{m}{\geqslant} \boldsymbol{\lambda}^*$ implies*

$$(X_{2:n} - X_{1:n}, ..., X_{n:n} - X_{1:n}) \overset{st}{\geqslant} (Y_{2:n} - Y_{1:n}, ..., Y_{n:n} - Y_{1:n}). \tag{3.1}$$

Proof. Proschan and Sethuraman [11] have shown that under the conditions of this theorem,

$$(X_{1:n}, ..., X_{n:n}) \overset{st}{\succcurlyeq} (Y_{1:n}, ..., Y_{n:n}) \tag{3.2}$$

and $X_{1:n} \overset{st}{=} Y_{1:n}$.

It follows from Kamae, Krengel, and O'Brien [5] (also see Theorem 4.B.1 of Shaked and Shanthikumar [12]) that there exist random variables $\hat{Z}$ and $(\hat{X}_{i:n}, \hat{Y}_{i:n})$, $i=2, ..., n$ on the same probability space such that

$$X_{1:n} \overset{st}{=} Y_{1:n} \overset{st}{=} \hat{Z},$$

$$\hat{X}_{i:n} \overset{st}{=} X_{i:n}, \qquad \hat{Y}_{i:n} \overset{st}{=} Y_{i:n}, \qquad i=2, ..., n;$$

and with probability one,

$$\hat{X}_{i:n} \geqslant \hat{Y}_{i:n}, \qquad i=2, ..., n.$$

Hence with probability one,

$$(\hat{X}_{2:n} - \hat{Z}, ..., \hat{X}_{n:n} - \hat{Z}) \geqslant (\hat{Y}_{2:n} - \hat{Z}, ..., Y_{n:n} - \hat{Z}). \tag{3.3}$$

The required proof follows from this. ∎

An important consequence of this result is that the sample range $X_{n:n} - X_{1:n}$ is stochastically larger when the λ_i's are more dispersed, a result more general than the one given by Corollary 2.2.

Boland, Hollander, Joag-Dev, and Kochar [3] have studied different kinds of dependence relations between order statistics from independent, but otherwise arbitrary, distributions. In particular, it follows from their Theorem 2.2 that in the case of independent exponential random variables, for any $i > 1$, $X_{i:n}$ is stochastically increasing in $X_{1:n}$ in the sense that $P[X_{i:n} > y \mid X_{1:n} = x]$ is nondecreasing in x for $y \geqslant x$. It will be interesting to study the properties of this conditional probability as a function of the λ_i's. An important consequence of the next corollary is that this conditional probability is Schur convex in $\boldsymbol{\lambda}$.

COROLLARY 3.1. *Let $X_1, ..., X_n$ be independent exponential random variables with X_i having hazard rate λ_i, for $i=1, ..., n$. Then for $0 \leqslant x_1 \leqslant \cdots \leqslant x_n$,*

$$P[X_{2:n} > x_2, ..., X_{n:n} > x_n \mid X_{1:n} = x_1]$$

is Schur convex in $(\lambda_1, ..., \lambda_n)$.

Proof.

$$\begin{aligned} P[X_{2:n} > x_2, ..., X_{n:n} > x_n \mid X_{1:n} = x_1] \\ = P[X_{2:n} - X_{1:n} > x_2 - x_1, ..., X_{n:n} - X_{1:n} > x_n - x_1 \mid X_{1:n} = x_1] \\ = P[X_{2:n} - X_{1:n} > x_2 - x_1, ..., X_{n:n} - X_{1:n} > x_n - x_1]. \end{aligned} \tag{3.4}$$

The last equality follows from Theorem 4.1 of Kochar and Korwar [7]. The fact that (3.4) is Schur convex in $(\lambda_1, ..., \lambda_n)$ follows from Theorem 3.1. ∎

One may wonder whether one can extend Theorem 3.1 to other spacings. Pledger and Proschan [10] have shown with the help of an example that for $n = 3$, the survival function of the last spacing $D_{3:3}$ is not Schur convex. However, we have the following positive and even stronger result for $n = 2$.

THEOREM 3.2. *Let X_1 and X_2 be two independent exponential random variables with hazard rates λ_1 and λ_2, respectively. Let Y_1 and Y_2 be another set of independent exponential random variables with respective hazard rates λ_1^* and λ_2^*. Then for $(\lambda_1, \lambda_2) \overset{m}{\succcurlyeq} (\lambda_1^*, \lambda_2^*)$,*

$$X_{2:2} - X_{1:2} \overset{\ell r}{\succcurlyeq} Y_{2:2} - Y_{1,2}. \tag{3.5}$$

Proof. Let $\lambda_1 + \lambda_2 = \lambda_1^* + \lambda_2^* = s$. Assume without loss of generality that $\lambda_1 \leqslant \lambda_2$, $\lambda_1^* \leqslant \lambda_2^*$. Then for $\lambda_1 < \lambda_1^*$ (and consequently for $\lambda_2 > \lambda_2^*$), $(\lambda_1, \lambda_2) \underset{m}{\succcurlyeq} (\lambda_1^*, \lambda_2^*)$. Now the p.d.f. of $X_{2:2} - X_{1:2}$ is $(\lambda_1 \lambda_2 / s)[e^{-\lambda_1 x} + e^{-\lambda_2 x}]$ (see, Theorem 2.1 of Kochar and Korwar [7]).

We have to prove that under the above constraints on the parameters, the ratio of the densities

$$g(x) = \frac{e^{-\lambda_1 x} + e^{-\lambda_2 x}}{e^{-\lambda_1^* x} + e^{-\lambda_2^* x}}$$

is nondecreasing in x for $x > 0$.

The numerator of $g'(x)$ is

$$\begin{aligned} -[e^{-\lambda_1^* x} + e^{-\lambda_2^* x}][\lambda_1 e^{-\lambda_1 x} + \lambda_2 e^{-\lambda_2 x}] + [e^{-\lambda_1 x} + e^{-\lambda_2} x] \\ \times [\lambda_1^* e^{-\lambda_1^* x} + \lambda_2^* e^{-\lambda_2^* x}] \\ = (\lambda_1^* - \lambda_1)\, e^{-(\lambda_1 + \lambda_1^*)\, x} + (\lambda_2^* - \lambda_2)\, e^{-(\lambda_2 + \lambda_2^*)\, x} + (\lambda_2^* - \lambda_1)\, e^{-(\lambda_1 + \lambda_2^*)\, x} \\ + (\lambda_1^* - \lambda_2)\, e^{-(\lambda_1^* + \lambda_2)\, x} \\ = (\lambda_1^* - \lambda_1)[e^{-(\lambda_1 + \lambda_1^*)\, x} - e^{-(\lambda_2 + \lambda_2^*)\, x}] + (\lambda_2^* - \lambda_1) \\ \times [e^{-(\lambda_1 + \lambda_2^*)\, x} - e^{-(\lambda_1^* + \lambda_2)\, x}]. \end{aligned} \tag{3.6}$$

Here we have used the fact that $\lambda_2^* - \lambda_2 = -(\lambda_1^* - \lambda_1)$ and $\lambda_1^* - \lambda_2 = \lambda_1 - \lambda_2^*$.

As $\lambda_1+\lambda_1^* \leqslant \lambda_2+\lambda_2^*$, for each $x \geqslant 0$, $e^{-(\lambda_1+\lambda_1^*)x}-e^{-(\lambda_2+\lambda_2^*)x} \geqslant 0$. Also $\lambda_1^* > \lambda_1$. Therefore the first term in (3.6) is nonnegative.

Now consider the second term in (3.6). Since $\lambda_1 < \lambda_1^*$ and $\lambda_2^* < \lambda_2$, $\lambda_1+\lambda_2^* < \lambda_1^*+\lambda_2$. Therefore, for each $x \geqslant 0$, $e^{-(\lambda_1+\lambda_2^*)x}-e^{-(\lambda_1^*+\lambda_2)x} \geqslant 0$. Also $\lambda_2^* \geqslant \lambda_1^* > \lambda_1 \Rightarrow \lambda_2^* - \lambda_1 \geqslant 0$. Therefore, the second term of (3.6) is also nonnegative. Hence $g(x)$ is nondecreasing in x for $x \geqslant 0$. This proves the required result. ∎

This result strengthens Theorem 3.3 of Kochar and Korwar [7] from hazard rate ordering to likelihood ratio ordering.

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REFERENCES

[1] Bapat, R. B., and Kochar, S. C. (1994). On likelihood ratio ordering of order statistics, *Linear Algebra Appl.* **199** 281–291.

[2] Boland, P. J., El-Neweihi, E., and Proschan, F. (1994). Applications of the hazard rate ordering in reliability and order statistics, *J. Appl. Probab.* **31** 180–192.

[3] Boland, P. J., Hollander, M., and Joag-Dev, K., and Kochar, S. (1996). Bivariate dependence properties of order statistics. *J. Multivariate Anal.* **56** 75–89.

[4] David, H. A. (1981). Order Statistics. Wiley, New York.

[5] Kamae, T., Krengel, U., and O'Brien, G. L. (1975). Stochastic inequalities on partially ordered spaces. *Ann. Probab.* **5** 899–912.

[6] Kochar, S. C., and Kirmani, S. N. U. A. (1995). Some results on normalized spacings from restricted families of distributions. *J. Statist. Plan. Inf.* **46** 47–57.

[7] Kochar, S. C., and Korwar, R. (1996). Stochastic orders for spacings of heterogeneous exponential random variables, *J. Multivariate Anal.* **57** 69–73.

[8] Lehmann, E. L., and Rojo, J. (1992). Invariant directional orderings. *Ann. Statist.* **20** 2100–2110.

[9] Marshall, A. W., and Olkin, I. (1979). *Inequalities: Theory of Majorization and Its Applications*. Academic Press, New York.

[10] Pledger, G., and Proschan, F. (1971). Comparisons of order statistics and spacings from heterogeneous distributions. In *Optimizing Methods in Statistics* (J. S. Rustagi, Ed.), pp. 89–113. Academic Press, New York.

[11] Proschan, F., and Sethuraman, J. (1976). Stochastic comparisons of order statistics from heterogeneous populations, with applications in reliability. *J. Multivariate Anal.* **6** 608–616.

[12] Shaked, M., and Shanthikumar, J. G. (1994). *Stochastic Orders and Their Applications*. Academic Press, San Diego, CA.