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Stochastic Orders for Spacings of Heterogeneous Exponential Random Variables

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We obtain some new results on normalized spacings of independent exponential random variables with possibly different scale parameters. It is shown that the density functions of the individual normalized spacings in this case are mixtures of exponential distributions and, as a result, they are log-convex (and, hence, DFR). G. Pledger and F. Proschan (*Optimizing Methods in Statistics* (J. S. Rustagi, Ed.), pp. 89–113, Academic Press, New York, 1971), have shown, with the help of a counterexample, that in a sample of size 3 the survival function of the last spacing is not Schur convex. We show that, however, this is true for the second spacing for all sample sizes. G. Pledger and F. Proschan (ibid.) also prove that the spacings are stochastically larger when the scale parameters are unequal than when they are all equal. We strengthen this result from stochastic ordering to likelihood ratio ordering. Some new results on dispersive ordering between the normalized spacings have also been obtained.

1. Introduction

In reliability theory and life testing, exponential distributions play an important role. They have the property that they never get aged with time and have constant failure rates. There has been a lot of work done in the literature on the stochastic properties of various statistics based on *random samples* from exponential distributions. However, not much attention has

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been given to the case when the underlying random variables are *not* independent or *not* identically distributed. Some interesting results on order statistics from independent random variables with proportional hazard rates have been obtained by Sen [17], Pledger and Proschan [15], Proschan and Sethuraman [16], Bapat and Kochar [4], and Boland, Hollander, Joag-Dev and Kochar [8]. Obviously, the proportional hazard family contains exponential distributions as a special case. Also see, Bapat and Beg [3] for the distribution theory of order statistics from independent, but nonidentically, distributed random variables.

Let $X_1, ..., X_n$ be independent random variables with possibly different probability distributions. We shall denote by $X_{i:n}$ the ith-order statistic of $X_1, ..., X_n$. Let $D_{i:n} = (n-i+1)(X_{i:n} - X_{i-1:n})$ denote the ith normalized spacing, $i = 1, ..., n$, with $X_{0:n} \equiv 0$. To simplify notation, we shall drop the second suffix n in $D_{i:n}$ when there is no ambiguity. It is important to study the stochastic properties of spacings under different models. It is well known that if $X_1, ..., X_n$ is a random sample from an exponential distribution, then $D_1, ..., D_n$ are independent and identically distributed as exponential random variables. But if the random sample comes from a *decreasing failure rate* (DFR) distribution, then the successive normalized spacings are stochastically increasing (cf. Barlow and Proschan [5]). Kochar and Kirmani [12] have strengthened this result from stochastic ordering to *hazard rate ordering*.

If $\bar{F}(\bar{G})$ denotes the survival function of a random variable $X(Y)$, we say that X is greater than Y according to hazard rate ordering (written as $X \overset{\text{hr}}{\geqslant} Y$) if $\bar{F}(x)/\bar{G}(x)$ is nondecreasing in x. In the case of absolutely continuous distributions, this is equivalent to the hazard (or failure) rate of F being uniformly smaller than that of G. If $f(g)$ is the density function of $F(G)$ and $f(x)/g(x)$ is nondecreasing in x, then we say that X is greater than Y according to *likelihood ratio ordering* and write as $X \overset{\text{lr}}{\geqslant} Y$. Likelihood ratio ordering implies hazard rate ordering which in turn implies stochastic ordering. See Shaked, Shanthikumar, and collaborators [18] for detailed discussions on various types of stochastic orders, their interpretations, and their properties.

In this paper we study the stochastic properties of spacings from independent exponential distributions with possibly unequal scale parameters. In Section 2, we obtain the joint, as well as the marginal, distributions of the D_i's. It is shown that the one-dimensional marginals are mixtures of exponential distributions and have DFR distributions. Pledger and Proschan [15] proved that if the scale parameters of the exponential distributions are not all equal then D_i is stochastically smaller than D_{i+1}, $i = 1, ..., n-1$. In Section 3, we explore whether this result can be strengthened. Pledger and Proschan [15] also raised the question whether the survival function of D_i is Schur convex in $(\lambda_1, ..., \lambda_n)$. They show with the help of a counterexample that for $n=3$, the survival function of the last normalized spacing $D_{3:3}$ is not Schur convex. What can we say about the other spacings and for arbitrary n? These and related problems are studied in Section 3. In particular, we prove in that section that the survival function of $D_{2:n}$ is Schur convex in $(\lambda_1, ..., \lambda_n)$ for any n and $D_{1:n} \leqslant^{\mathrm{lr}} D_{i:n}$ for $i=2, ..., n$. Pledger and Proschan [15] also proved that the spacings are stochastically larger when the scale parameters are unequal than when they are all equal. We strengthen this result from stochastic ordering to likelihood ratio ordering. We also establish some dispersive ordering results between spacings in that section. In the last section we show that the vector $(X_{2:n}, ..., X_{n:n})$ is increasing in $X_{1:n}$ according to the *upper orthant order*.

2. The Distributions of Normalized Spacings

In this section we obtain the joint, as well as the marginal, distributions of the normalized spacings from exponential distributions with possibly unequal scale parameters and we study some of their aging properties.

Theorem 2.1. *Let $X_1, ..., X_n$ be independent random variables with X_i having the exponential distribution with survival function $\bar{F}_i(t)=\exp(-\lambda_i t)$, $t \geqslant 0$, for $i=1, ..., n$. Then*

(a) *D_1 has exponential distribution with scale parameter $\sum_{i=1}^n \lambda_i/n$ and D_1 is independent of $(D_2, ..., D_n)$;*

(b) *the joint p.d.f. of $(D_{i_1}, ..., D_{i_k})$, $2 \leqslant i_j \leqslant n$, $j=1, ..., k$, $1 \leqslant k \leqslant n-1$ is*

$$\left(\prod_{i=1}^n \lambda_i\right) \sum_{\mathbf{r}} \frac{1}{\prod_{i=1}^n \left(\sum_{j=i}^n \lambda(r_j)\right)} \prod_{l=1}^k \frac{\sum_{j=i_l}^n \lambda(r_j)}{n-i_l+1} e^{-(d_{i_l}/(n-i_l+1)) \sum_{j=i_l}^n \lambda(r_j)}, \qquad (2.1)$$

for $d_{i_l} \geqslant 0$, $l=1, ..., k$; where $\mathbf{r}$ is a permutation of $(1, 2, ..., n)$ and $\lambda(i)=\lambda_i$;

(c) *for $i \in \{2, ..., n\}$, the distribution of D_i is a mixture of independent exponential random variables with p.d.f.:*

$$\sum_S P(S) \frac{(s-\Lambda(S))}{n-i+1} \exp\{-(s-\Lambda(S))\, d_i/(n-i+1)\}, \qquad d_i \geqslant 0, \quad (2.2)$$

where the $\sum_S$ *is over all subsets* $S \subset \{1, ..., n\}$ *of size* $(i-1)$, $s = \sum_{i=1}^n \lambda_i$, $\Lambda(S) = \sum_{j \in S} \lambda_j$, *and*

$$P(S) = \sum_{\mathbf{r}} \left(\prod_{i \in S} \lambda_i \right) \left[\prod_{l=1}^{i-1} \left\{ \sum_{j=l}^{i-1} \lambda(r(k_j)) + s - \Lambda(S) \right\} \right]^{-1}, \tag{2.3}$$

the $\sum_r$ *is being taken over all permutations* $\mathbf{r} = (r_{k_1}, ..., r_{k_{i-1}})$ *of the elements* $k_j \in S$; $j = 1, ..., i-1$.

Proof. As in Kochar and Kirmani [12], the joint density of the order statistics $X_{1:n}, ..., X_{n:n}$ is

$$\sum_{\mathbf{r}} \left(\prod_{i=1}^n \lambda_i \right) \exp\left\{ -\sum_{i=1}^n \lambda(r_i)\, x_i \right\} \qquad \text{for} \quad 0 \leqslant x_1 \leqslant \cdots \leqslant x_n < \infty,$$

where $\mathbf{r} = (r_1, ..., r_n)$ is a permutation of $(1, ..., n)$.

The linear transformations,

$$D_i = (n-i+1)(X_{i:n} - X_{i-1:n}), \qquad i = 1, ..., n, \quad X_{0:n} \equiv 0,$$

yield the joint p.d.f. of $(D_1, ..., D_n)$ as

$$\begin{aligned} &\frac{\prod_{i=1}^n \lambda_i}{n!} \sum_{\mathbf{r}} \prod_{i=1}^n e^{-\sum_{j=i}^n \lambda(r_j)\, d_i/(n-i+1)} \\ &\quad = \frac{\prod_{i=1}^n \lambda_i}{n!} \sum_{\mathbf{r}} e^{-(s/n)\, d_1} \prod_{i=2}^n e^{-(d_i/(n-i+1)) \sum_{j=i}^n \lambda(r_j)} \\ &\quad = \left\{ \frac{s}{n} e^{-(s/n)\, d_1} \right\} \left\{ \frac{\prod_{i=1}^n \lambda_i}{s(n-1)!} \sum_{\mathbf{r}} \prod_{i=2}^n e^{-(d_i/(n-i+1)) \sum_{j=i}^n \lambda(r_j)} \right\}, \end{aligned} \tag{2.4}$$

where r is a permutation of $(1, 2, ..., n)$ and $\lambda(i) = \lambda_i$. This shows that

(i) D_1 is independent of $(D_2, ..., D_n)$ and it has exponential distribution with parameter s/n.

(ii) the p.d.f.'s of each of $(D_1, ..., D_n)$ and $(D_2, ..., D_n)$ is a mixture of products of exponential densities.

The joint p.d.f. of $(D_{i_1}, ..., D_{i_k})$, $2 \leqslant i_j \leqslant n$, $j = 1, ..., k$, $1 \leqslant k \leqslant n-1$, can be obtained by integrating out d_i, $i \in \{i_1, ..., i_k\}$ from

$$\frac{\prod_{i=1}^n \lambda_i}{s(n-1)!} \sum_{\mathbf{r}} \prod_{i=2}^n e^{-(d_i/(n-i+1)) \sum_{j=i}^n \lambda(r_j)}$$

and is

$$\left(\prod_{i=1}^n \lambda_i \right) \sum_{\mathbf{r}} \frac{1}{\prod_{i=1}^n \left(\sum_{j=i}^n \lambda(r_j) \right)} \prod_{l=1}^k \frac{\sum_{j=i_l}^n \lambda(r_j)}{n - i_l + 1} e^{-(d_{i_l}/(n-i_l+1)) \sum_{j=i_l}^n \lambda(r_j)} \tag{2.5}$$

which is again a mixture of products of exponential densities. In particular, the p.d.f. of D_i alone is

$$\sum_{\mathbf{r}} \frac{(\prod_{i=1}^{n} \lambda_i)}{\prod_{i=1}^{n} (\sum_{j=i}^{n} \lambda(r_j))} \frac{(\sum_{j=i}^{n} \lambda(r_j))\, e^{-(d_i/(n-i+1)) \sum_{j=i}^{n} \lambda(r_j)}}{n-i+1} \tag{2.6}$$

which can be written as

$$\sum_{S} P(S) \frac{(s-\Lambda(S))}{n-i+1} \exp\{-(s-\Lambda(S))\, d_i/(n-i+1)\}, \tag{2.7}$$

where the $\sum_S$ is over all subsets $S \subset \{1, ..., n\}$ of size $(i-1)$, $s = \sum_{i=1}^{n} \lambda_i$, $\Lambda(S) = \sum_{j \in S} \lambda_j$, and

$$P(S) = \sum_{\mathbf{r}} \left(\prod_{i \in S} \lambda_i\right) \left[\prod_{l=1}^{i-1} \left\{\sum_{j=l}^{i-1} \lambda(r(k_j)) + s - \Lambda(S)\right\}\right]^{-1}; \tag{2.8}$$

the $\sum_r$ is being taken over all permutations $\mathbf{r} = (r_{k_1}, ..., r_{k_{i-1}})$ of the elements $k_j \in S$; $j = 1, ..., i-1$. In going from (2.6) to (2.7) (with (2.8)) we used the fact that

$$\sum_{\mathbf{r}} \left(\prod_{i \in S'} \lambda_i\right) \left[\prod_{l=1}^{n-i+1} \left(\sum_{j=l}^{n-i+1} \lambda(r(i_j))\right)\right]^{-1} = 1, \tag{2.9}$$

where $\sum_{\mathbf{r}}$ is taken over all permutations $(r(k_1), ..., r(k_{n-i+1}))$ of the elements $k_j \in S' = \{1, ..., n\} - S$, $j = 1, ..., n-i+1$. The left-hand side of (2.9) is the sum of probabilities of the mixing distribution used in the p.d.f. of $\hat{D}_{n-i+1}$, the last normalized spacing for the set of the exponential random variables $\{X_{k_j}, k_j \in S', j = 1, ..., n-i+1\}$. ▮

Part (a) of the above theorem has also been proved in Gross, Hunt, and Odeh [9] using the complicated theory of permanents. Our proof is much simpler and straightforward. There also is a connection of this to successive sampling from a finite population as follows. The probability $P(S)$ in (2.3) is the probability of obtaining S in successive sampling of size $(i-1)$ from the finite population $\{1, ..., n\}$, where successive draws are made with replacement and at each draw a unit k is chosen with probability $p_k = \lambda_k/(\lambda_1 + \cdots + \lambda_n)$, sampling being continued until $(i-1)$ distinct units are chosen and any multiplicities are discarded. The terms on the left-hand side of (2.9), then, are merely the probabilities of ordered samples $\mathbf{r} = (r(k_1), ..., r(k_{n-i+1}))$ in a successive sample of size $(n-i+1)$ from the finite population S'. The probabilities $P(S)$ in (2.3) have the integral representation (see Andreatta and Kaufman [1])

$$P(S) = (s-\Lambda(S)) \int_0^{\infty} \left\{\prod_{i \in S} (1-e^{-\lambda_i x})\right\} \exp\{-(s-\Lambda(S))\, x\}\, dx. \tag{2.10}$$

Note also that

$$P(S) = P(\max_{i \in S} X_i < \min_{i \in S'} X_i). \tag{2.11}$$

For more information on successive sampling see, for example, Hájek [10].

Barlow and Proschan [5] have shown that if $X_1, ..., X_n$ is a random sample from a DFR distribution, then the spacings also have DFR distributions. It is shown in the next theorem that a similar result holds in the case of independent exponential distributions with unequal scale parameters.

THEOREM 2.2. *Let $D_1, ..., D_n$ be the normalized spacings based on n independent exponential distributions. Then D_i has a log-convex density for $i = 1, ..., n$.*

Proof. The result follows from the fact that mixtures of log-convex densities are log-convex (cf. Marshall and Olkin [13, p. 452]) and from part (c) of Theorem 2.1. ▮

Since probability distributions with log-convex densities have decreasing failure rates we have the following result.

COROLLARY 2.1. *Let $D_1, ..., D_n$ be the normalized spacings based on n independent exponential distributions. Then D_i has a DFR distribution for $i = 1, ..., n$.*

3. STOCHASTIC RELATIONS BETWEEN NORMALIZED SPACINGS

First, we give the definitions of majorization of vectors and Schur convexity of functions on $\mathscr{R}^n$. Let $\{x_{(1)} \leqslant x_{(2)} \leqslant \cdots \leqslant x_{(n)}\}$ denote the increasing arrangement of the components of the vector $\mathbf{x} = (x_1, x_2, ..., x_n)$. The vector $\mathbf{y}$ is said to *majorize* the vector $\mathbf{x}$ (written as $\mathbf{x} \preccurlyeq^{\mathrm{m}} \mathbf{y}$) if $\sum_{i=1}^{j} y_{(i)} \leqslant \sum_{i=1}^{j} x_{(i)}$, $j = 1, ..., n-1$, and $\sum_{i=1}^{n} y_{(i)} = \sum_{i=1}^{n} x_{(i)}$.

DEFINITION 3.1. A real-valued function ϕ defined on a set $\mathscr{A} \subset \mathscr{R}^n$ said to be Schur convex (Schur concave) on $\mathscr{A}$ if $\mathbf{x} \preccurlyeq^{\mathrm{m}} \mathbf{y} \Rightarrow \phi(\mathbf{x}) \leqslant (\geqslant) \, \phi(\mathbf{y})$.

Pledger and Proschan [15] proved the following result.

THEOREM 3.1. *Let $X_1, ..., X_n$ be independent exponential random variables with possibly unequal scale parameters. Then*

$$D_i \preccurlyeq^{\mathrm{st}} D_{i+1}, \qquad i = 1, ..., n-1, \tag{3.1}$$

where $\preccurlyeq^{\mathrm{st}}$ denotes stochastic dominance.

Let $\bar{F}_i(t) = \exp(-\lambda_i t)$ and $\bar{F}_i^*(t) = \exp(-\bar{\lambda} t)$, for $1 = 1, ..., n$, where $\bar{\lambda} = (1/n) \sum_{i=1}^n \lambda_i$. Let $(D_1^*, ..., D_n^*)$ be the normalized spacings associated with $(\bar{F}_1^*, ..., \bar{F}_n^*)$. As discussed earlier, $D_1^*, ..., D_n^*$ are independent and identically distributed each having an exponential distribution with the common scale parameter $\bar{\lambda}$. It follows from Theorem 2.1(a) and (3.1) that

$$D_1^* =^{\text{st}} D_1, \qquad D_i^* \preccurlyeq^{\text{st}} D_i \qquad \text{for} \quad i = 2, ..., n. \tag{3.2}$$

Note that the mean $\bar{\lambda}$ of $(\lambda_1, ..., \lambda_n)$ is used for the comparison. This means that the spacings are comparatively stochastically larger when the scale parameters are unequal than when they are all equal. Pledger and Proschan [15] raised the question whether such a comparison is possible using a vector $\boldsymbol{\lambda}^*$, where $\boldsymbol{\lambda} \succcurlyeq^{\text{m}} \boldsymbol{\lambda}^*$. That is, whether the survival function of D_i is Schur convex in $(\lambda_1, ..., \lambda_n)$. They show with the help of an example that for $n = 3$, the survival function of $D_{3:3}$ is not Schur convex. Does $\boldsymbol{\lambda} \succcurlyeq^{\text{m}} \boldsymbol{\lambda}^*$ imply a weaker ordering between $D_{3:3}$ and $D_{3:3}^*$, namely, $E[D_{3:3}] \geqslant E[D_{3:3}^*]$? The next example shows that even this does not hold.

EXAMPLE 3.1. Let $\boldsymbol{\lambda} = (0.01, 0.1, 0.4)$ and $\boldsymbol{\lambda}^* = (0.01, 0.25, 0.25)$, we find that $E[D_{3:3}] = 495.84 < 498.80 = E[D_{3:3}^*]$ even though $\boldsymbol{\lambda} \succcurlyeq^{\text{m}} \boldsymbol{\lambda}^*$.

However, the survival function of $D_{2:n}$ is Schur convex in $(\lambda_1, ..., \lambda_n)$ for any n as proved in the next theorem.

THEOREM 3.2. *For any n, the survival function of $D_{2:n}$ is Schur convex in $(\lambda_1, ..., \lambda_n)$.*

Proof. The survival function of $D_{2:n}$ at x is

$$\begin{aligned}(1/s) \sum_{i=1}^n \lambda_i \exp\left\{-\frac{x}{n-1}(s - \lambda_i)\right\}& \\ = \frac{1}{s} \exp(-xs/(n-1)) \sum_{i=1}^n \lambda_i \exp\left\{\frac{x\lambda_i}{n-1}\right\}&.\end{aligned} \tag{3.3}$$

Since each term $\lambda_i \exp\{\lambda_i x/(n-1)\}$ in the above sum is convex in λ_i, it follows that the survival function of $D_{2:n}$ is Schur convex in $(\lambda_1, ..., \lambda_n)$ for each x (see Marshall and Olkin [13, p. 64]). ∎

It is interesting to see whether Theorem 3.2 can be strengthened from stochastic ordering to hazard rate ordering. The next example shows that even for $n = 3$, the hazard rate of $D_{2:n}$ is not Schur concave.

EXAMPLE 3.2. Let $\boldsymbol{\lambda} = (40, 10, 1)$ and $\boldsymbol{\lambda}^* = (40, 5.5, 5.5)$. The hazard rate of $D_{2:3}$ at $x = 0.2$ is 11.388, whereas that of $D_{2:3}^*$ at the same value of

x is 11.298, even though $\boldsymbol{\lambda} \succcurlyeq^{\mathrm{m}} \boldsymbol{\lambda}^*$. This proves that the hazard rate of $D_{2:3}$ is not Schur concave.

However, for $n=2$, the hazard rate of $D_{2:2}$ is Schur concave in (λ_1, λ_2).

THEOREM 3.3. *The hazard rate of $D_{2:2}$ is Schur concave in (λ_1, λ_2).*

Proof. From (2.2), the hazard rate $r(x)$ of $D_{2:2}$ is

$$
\begin{aligned}
r(x) &= \frac{\lambda_2 \lambda_1 e^{-\lambda_1 x} + \lambda_1 \lambda_2 e^{-\lambda_2 x}}{\lambda_2 e^{-\lambda_1 x} + \lambda_1 e^{-\lambda_2 x}} \\
&= \frac{\lambda_1 \lambda_2 (e^{\lambda_1 x} + e^{\lambda_2 x})}{\lambda_1 e^{\lambda_1 x} + \lambda_2 e^{\lambda_2 x}} \\
&= \lambda_2 + \frac{\lambda_2(\lambda_1 - \lambda_2)\, e^{\lambda_2 x}}{\lambda_1 e^{\lambda_1 x} + \lambda_2 e^{\lambda_2 x}}.
\end{aligned} \tag{3.4}
$$

On differentiating $r(x)$, we get

$$
\begin{aligned}
\frac{dr(x)}{d\lambda_1} &= \frac{\lambda_2 e^{\lambda_2 x}}{\lambda_1 e^{\lambda_1 x} + \lambda_2 e^{\lambda_2 x}} - \frac{\lambda_2 e^{\lambda_2 x}}{(\lambda_1 e^{\lambda_1 x} + \lambda_2 e^{\lambda_2 x})^2} (e^{\lambda_1 x} + \lambda_1 x e^{\lambda_1 x}) \\
&= \frac{\lambda_2 e^{\lambda_2 x}}{(\lambda_1 e^{\lambda_1 x} + \lambda_2 e^{\lambda_2 x})^2} [\lambda_2 (e^{\lambda_1 x} + e^{\lambda_2 x}) - \lambda_1 (\lambda_1 - \lambda_2)\, x e^{\lambda_1 x}]
\end{aligned}
$$

and by using a result similar to (3.4), obtained by interchanging λ_1 and λ_2, we get

$$
\frac{dr(x)}{d\lambda_2} = \frac{\lambda_1 e^{\lambda_1 x}}{(\lambda_1 e^{\lambda_1 x} + \lambda_2 e^{\lambda_2 x})^2} [\lambda_1 (e^{\lambda_1 x} + e^{\lambda_2 x}) - \lambda_2 (\lambda_2 - \lambda_1)\, x e^{\lambda_2 x}].
$$

Hence,

$$
\begin{aligned}
&\left(\frac{dr(x)}{d\lambda_1} - \frac{dr(x)}{d\lambda_2} \right) (\lambda_1 - \lambda_2) \\
&\quad = \frac{(\lambda_1 - \lambda_2)}{(\lambda_1 e^{\lambda_1 x} + \lambda_2 e^{\lambda_2 x})^2} [(e^{\lambda_1 x} + e^{\lambda_2 x})(\lambda_2^2 e^{\lambda_2 x} - \lambda_1^2 e^{\lambda_1 x}) \\
&\quad\quad + 2\lambda_1 \lambda_2 (\lambda_2 - \lambda_1)\, e^{(\lambda_1 + \lambda_2) x}] \\
&\quad \leqslant 0.
\end{aligned}
$$

Hence $r(x)$, the hazard rate of $D_{2:2}$ is Schur concave in (λ_1, λ_2) by Theorem A.4, page 57, of Marshall and Olkin [13]. ∎

This result is related to a result of Boland, El-Neweihi, and Proschan [7] which says that the hazard rate of a parallel system of two independent exponential random variables with parameters λ_1 and λ_2 is Schur concave in (λ_1, λ_2). The life of the parallel system can be expressed as $\frac{1}{2} D_{1:2} + D_{2:2}$. Now the distribution of $D_{1:2}$ which is exponential with parameter $(\lambda_1 + \lambda_2)/2$, is IFR. Since $D_{1:2}$ is independent of $D_{2:2}$, the above result of Boland, El-Neweihi, and Proschan [7] also follows from Theorem 3.3 and Lemma 1B.5, page 16, of Shaked, Shanthikumar, and collaborators [18].

In the case of dependent random variables, studying the stochastic ordering between the marginal distributions may not be very useful in revealing monotone tendencies between dependent variables because the dependence information is being ignored. Realizing this, Shanthikumar and Yao [19] introduced some new stochastic orders for comparing the components of a random vector. We focus our discussion on the extension of the idea of likelihood ratio ordering. For two *independent* random variables X_1 and X_2, it is known that $X_1 \preccurlyeq^{\mathrm{lr}} X_2$ if and only if

$$E[\phi(X_1, X_2)] \geqslant E[\phi(X_2, X_1)] \qquad \forall \phi \in \mathscr{G}_{\mathrm{lr}}, \tag{3.5}$$

where

$$\mathscr{G}_{\mathrm{lr}}\colon \{\phi\colon \phi(x_2, x_1) \leqslant \phi(x_1, x_2)\ \forall x_1 \leqslant x_2\}. \tag{3.6}$$

Motivated by the above characterization of likelihood ratio ordering, Shanthikumar and Yao [19] extended this concept to the bivariate case as follows.

DEFINITION 3.2. For a bivariate random variable (X_1, X_2), X_1 is said to be smaller than X_2 according to joint likelihood ratio ordering $(X_1 \preccurlyeq^{\mathrm{lr}:j} X_2)$ if and only if (3.5) holds.

It can be seen that

$$X_1 \preccurlyeq^{\mathrm{lr}:j} X_2 \Leftrightarrow f \in \mathscr{G}_{\mathrm{lr}},$$

where $f(\cdot, \cdot)$ denotes the joint density of (X_1, X_2).

A bivariate function $\phi \in \mathscr{G}_{\mathrm{lr}}$ is called *arrangement increasing* ($\mathscr{AI}$). In their seminal work on order relations between the components of a bivariate random vector, Yanagimoto and Sibuya [20] also considered this ordering, although they did not relate it to the notion of likelihood ratio ordering.

As pointed out by Shanthikumar and Yao [19] joint likelihood ratio ordering between two dependent random variables may not imply likelihood ratio ordering between their marginal distributions, but it does imply stochastic ordering between them (that is, $X_1 \preccurlyeq^{\mathrm{lr}:j} X_2 \Rightarrow X_1 \preccurlyeq^{\mathrm{st}} X_2$). Obviously, in case of independent random variables, $X_1 \preccurlyeq^{\mathrm{lr}:j} X_2 \Leftrightarrow X_1 \preccurlyeq^{\mathrm{lr}} X_2$.

Shanthikumar and Yao [19] also extend the concept of joint likelihood ratio ordering to compare the components of an n-dimensional random vector. Let $\mathbf{x} = (x_1, ..., x_n)$ and $\mathbf{y} = (y_1, ..., y_n)$ be two vectors. We say that $\mathbf{x}$ *is better arranged than* $\mathbf{y}(\mathbf{x} \succcurlyeq^{\mathrm{a}} \mathbf{y})$ if $\mathbf{x}$ can be obtained from $\mathbf{y}$ through successive pairwise interchanges of its components, with each interchange resulting in an increasing order of the two interchanged components, e.g., $(4, 1, 5, 3) \succcurlyeq^{\mathrm{a}} (4, 3, 5, 1) \succcurlyeq^{\mathrm{a}} (4, 5, 3, 1)$. A function $g: \mathscr{R}^n \to \mathscr{R}$ that preserves the ordering $\succcurlyeq^{\mathrm{a}}$ is called an *arrangement increasing* function denoted by $g \in \mathscr{A}I$ if $\mathbf{x} \succcurlyeq^{\mathrm{a}} \mathbf{y} \Rightarrow g(\mathbf{x}) \geqslant g(\mathbf{y})$ (cf. [13, p. 160] for the definition of an arrangement increasing function on $\mathscr{R}^n$).

DEFINITION 3.3. Let $f(x_1, ..., x_n)$ denote the joint density of $\mathbf{X}$. Then

$$X_1 \preccurlyeq^{\mathrm{lr}:j} X_2 \preccurlyeq^{\mathrm{lr}:j} \cdots \preccurlyeq^{\mathrm{lr}:j} X_n \Leftrightarrow f(x_1, ..., x_n) \in \mathscr{A}I. \tag{3.7}$$

Hollander, Proschan, and Sethuraman [11] call such a function a *decreasing in transposition* (DT) function. They also discuss many properties of such functions and give an extensive list of multivariate densities which are DT (or arrangement increasing).

Kochar and Kirmani [12] proved the following result on spacings from independent random variables with log-convex densities.

THEOREM 3.4. *Let $X_1, X_2, ..., X_n$ be independent random variables with log-convex densities. Then*

$$D_{1:n} \preccurlyeq^{\mathrm{lr}:j} \cdots \preccurlyeq^{\mathrm{lr}:j} D_{n:n}. \tag{3.8}$$

Theorem 3.1, originally proved by Pledger and Proschan [15], trivially follows from this since joint likelihood ratio ordering implies stochastic ordering between the marginal distributions. However, as discussed above, in general, (3.8) may not imply likelihood ratio ordering between the successive normalized spacings. Because of the independence between D_1 and $(D_2, ..., D_n)$ in the exponential case, we have the following result on likelihood ratio ordering between $D_{1:n}$ and $D_{i:n}$ for $1 < i \leqslant n$.

THEOREM 3.5. *Let $\bar{F}_i(x) = \exp(-\lambda_i x)$ and $\bar{F}_i^*(x) = \exp(-\bar{\lambda} x)$, for $i = 1, ..., n$, where $\bar{\lambda} = s/n$. Let $(D_1^*, ..., D_n^*)$ be the normalized spacings associated with $(\bar{F}_1^*, ..., \bar{F}_n^*)$. Then for $i = 1, ..., n$,*

$$D_i^* \preccurlyeq^{\mathrm{lr}} D_i. \tag{3.9}$$

Or, equivalently,

$$D_1 \preccurlyeq^{\mathrm{lr}} D_i \qquad \text{for} \quad i = 2, ..., n. \tag{3.10}$$

Proof. As seen in Theorem 2.1, $D_1^* =^{\text{st}} D_1$ and D_1 is independent of $(D_2, ..., D_n)$. The required result follows from Theorem 3.4 and the fact that for independent random variables joint likelihood ratio ordering is the same as ordinary likelihood ratio ordering. ∎

This result strengthens Theorem 3.1 of Pledger and Proschan [15] from stochastic ordering to likelihood ratio ordering.

Another natural question to ask is whether (3.1) can be strengthened to establish hazard rate or likelihood ratio ordering between consecutive normalized spacings. Theorem 3.5 establishes likelihood ratio ordering only between the first normalized spacing and the others. We make the following conjecture.

CONJECTURE. *Let $X_1, ..., X_n$ be independent exponential random variables. Then*

$$D_i \leqslant^{\text{hr}} D_{i+1}, \qquad i = 1, ..., n-1. \tag{3.11}$$

We give below the proof of this conjecture for $n = 3$. First we prove the following lemmas.

LEMMA 3.1. *Let $P(S)$ be as defined in Theorem* 2.1. *Suppose that S_1 and S_2 are two subsets of $\{1, ..., n\}$ of size i $(1 \leqslant i \leqslant n-1)$ and that they have all but one element in common. Denote the uncommon element in S_1 by a_1 and that in S_2 by a_2. Then*

$$\lambda(a_2)\, P(S_1) \geqslant \lambda(a_1)\, P(S_2) \qquad \textit{if} \quad \lambda(a_1) \geqslant \lambda(a_2).$$

Proof. Let $c_1, ..., c_{i-1}$ be the common elements and $s' = s - \sum_{j=1}^{i-1} \lambda(c_j) - \lambda(a_1) - \lambda(a_2)$. Let $\mathbf{r}$ be a permutation of the elements of the set $(c_1, ..., c_{i-1}, a_1)$ and let k $(1 \leqslant k \leqslant i)$ be the position of a_1 in $\mathbf{r}$. Replace a_1 in $\mathbf{r}$ by a_2 and denote this permutation of the elements of S_2 by $\mathbf{r}'$. Then

$$\begin{aligned}
\lambda(a_2)\, P(S_1) &= \sum_{\mathbf{r}} \left\{ \lambda(a_1)\, \lambda(a_2) \prod_{j=1}^{i-1} \lambda(c_j) \right\} \left\{ \prod_{l=k+1}^{i} \left(s' + \sum_{j=l}^{i} \lambda(r_j) + \lambda(a_2) \right) \right\}^{-1} \\
&\quad \times \left\{ \prod_{l=1}^{k} \left(s' + \sum_{j=l}^{i} \lambda(r_j) + \lambda(a_2) \right) \right\}^{-1} \\
&\geqslant \sum_{\mathbf{r}'} \left\{ \lambda(a_1)\, \lambda(a_2) \prod_{j=1}^{i-1} \lambda(c_j) \right\} \left\{ \prod_{l=k+1}^{i} \left(s' + \sum_{j=l}^{i} \lambda(r_j) + \lambda(a_1) \right) \right\}^{-1} \\
&\quad \times \left\{ \prod_{l=1}^{k} \left(s' + \sum_{j=l}^{i} \lambda(r_j) + \lambda(a_1) \right) \right\}^{-1} \\
&= \lambda(a_1)\, P(S_2). \quad ∎
\end{aligned}$$

LEMMA 3.2 (Čebyšev's inequality, Theorem 1, page 36 of Mitrinović [14]). *Let* $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$ *and* $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n$ *be two increasing sequences of real numbers. Then*

$$n \sum_{i=1}^{n} a_i b_i \geqslant \left(\sum_{i=1}^{n} a_i \right) \left(\sum_{i=1}^{n} b_i \right).$$

With the help of the above lemmas, we prove the next theorem.

THEOREM 3.6. $D_{2:3} \preccurlyeq^{\text{hr}} D_{3:3}$.

Proof. By Theorem 2.1(ii), the ratio of the survival function of $D_{3:3}$ at x to that of $D_{2:3}$ at x is

$$\frac{\sum_{i=1}^{3} P(S_i) \exp\{-x(s - \Lambda(S_i))\}}{\sum_{i=1}^{3} (\lambda_i/s) \exp\{-(x/2)(s - \lambda_i)\}} = g(x), \qquad \text{say},$$

where the λ_i's are ordered from the smallest to the largest and where $S_i = \{\lambda_1, \lambda_2, \lambda_3\} - \{\lambda_i\}$, $i = 1, 2, 3$. Now $g(x)$ is increasing in x if its derivative $g'(x) \geqslant 0$ for all $x \geqslant 0$, and $g'(x) \geqslant 0$ if

$$\begin{aligned}
&\text{numerator of } \frac{g'(x)}{s} \\
&\quad = \sum_{i'=1}^{3} \lambda_{i'} \exp\left\{ -\frac{x}{2}(s - \lambda_{i'}) \right\} \\
&\qquad \times \left[\sum_{i=1}^{3} \exp\{-x(s - \Lambda(S_i))\} \, P(S_i) \left\{ \frac{s - \lambda_{i'}}{2} - (s - \Lambda(S_i)) \right\} \right] \qquad (3.12)
\end{aligned}$$

is $\geqslant 0$, as the denominator of $g'(x)$ is positive. Now by the definition of S_i and the ordering of the λ_i's, it follows from Lemma 3.1, or otherwise that the $P(S_i)$'s are decreasing in i. Hence,

$$a_i = P(S_i) \exp\{-x(s - \Lambda(S_i)\}, \qquad i = 1, 2, 3,$$

are decreasing in i. We next consider the values of $(s - \lambda_{i'})/2 - (s - \Lambda(S_i))$ for each i'. They are respectively given by the rows of

$$\begin{pmatrix} (\lambda_2 + \lambda_3 - 2\lambda_1)/2 & (\lambda_3 - \lambda_2)/2 & (\lambda_2 - \lambda_3)/2 \\ (\lambda_3 - \lambda_1)/2 & (\lambda_3 + \lambda_1 - 2\lambda_2)/2 & (\lambda_1 - \lambda_3)/2 \\ (\lambda_2 - \lambda_1)/2 & (\lambda_1 - \lambda_2)/2 & (\lambda_1 + \lambda_2 - 2\lambda_3)/2 \end{pmatrix}.$$

The terms within the square brackets in (3.12) are respectively the inner products of the rows of the above matrix with (a_1, a_2, a_3). Since the a_i's are decreasing, the three inner products are respectively greater than or equal to

$$\{(\lambda_2+\lambda_3-2\lambda_1)/2\}\ a_1, \quad \{(\lambda_3+\lambda_1-2\lambda_2)/2\}\ a_2, \quad \{(\lambda_1+\lambda_2-2\lambda_3)/2\}\ a_3.$$

Notice that $b_1=(\lambda_2+\lambda_3-2\lambda_1)/2$, $b_2=(\lambda_3+\lambda_1-2\lambda_2)/2$, $b_3=(\lambda_1+\lambda_2-2\lambda_3)/2$ are decreasing in i. So are the c_i's, where

$$c_i=a_i\lambda_i \exp\left\{-\frac{x}{2}(s-\lambda_i)\right\}=\lambda_i P(S_i)\exp\left\{-\frac{x}{2}(s+\lambda_i)\right\}$$

(again by Lemma 3.1). Finally, by Lemma 3.2 and (3.12),

$$\text{numerator of } \frac{g'(x)}{s} \geqslant \sum_{i=1}^{3} b_i c_i \geqslant \left(\sum_{i=1}^{3} b_i\right)\left(\sum_{i=1}^{3} c_i\right)\Big/ 3=0,$$

since $\sum_{i=1}^{3} b_i=0$. This proves the required result. ∎

As mentioned in Kochar and Kirmani [12], in case $X_1, ..., X_n$ is a random sample from a distribution with log-convex density, then

$$D_i \preccurlyeq^{\mathrm{lr}} D_{i+1}, \qquad i=1, ..., n-1. \tag{3.13}$$

It will be interesting to know whether (3.13) holds in the case of independent exponential random variables with unequal scale parameters.

Dispersive Ordering between Normalized Spacings. Now we study dispersive (or variability) ordering between normalized spacings. Let X and Y be two random variables with distribution functions F and G, respectively. We say that distribution G is less dispersed than $F (G \preccurlyeq^{\mathrm{disp}} F)$ if

$$G^{-1}(v)-G^{-1}(u) \leqslant F^{-1}(v)-F^{-1}(u) \qquad \text{for} \quad 0<u \leqslant v<1.$$

This means that the difference between any two quantiles of G is smaller than the difference between the corresponding quantiles of F. One of the consequences of $G \preccurlyeq^{\mathrm{disp}} F$ is that $\mathrm{var}(Y) \leqslant \mathrm{var}(X)$. For other properties of dispersive ordering, see Chapter 2 of Shaked, Shanthikumar, and collaborators [18].

Bagai and Kochar [2] proved that if $G \preccurlyeq^{\mathrm{hr}} F$ and if either F or G is DFR than $G \preccurlyeq^{\mathrm{disp}} F$. Since likelihood ratio ordering implies hazard rate ordering, the proof of the next theorem follows from the above result, Theorem 3.3, and the DFR properties of spacings as established in Corollary 2.1.

THEOREM 3.7. *Let* $\bar{F}_i(x) = \exp(-\lambda_i x)$ *and* $\bar{F}_i^*(x) = \exp(-\bar{\lambda} x)$ *for* $i = 1, ..., n$, *where* $\bar{\lambda} = s/n$. *Let* $(D_1^*, ..., D_n^*)$ *be the normalized spacings associated with* $(\bar{F}_1^*, ..., \bar{F}_n^*)$. *Then*

(a)

$$D_{i:n}^* \preccurlyeq^{\text{disp}} D_{i:n} \qquad \textit{for} \quad i = 2, ..., n,$$

(b) *for* $n = 2$,

$$(\lambda_1, \lambda_2) \succcurlyeq^{\text{m}} (\lambda_1^*, \lambda_2^*) \Rightarrow D_{2:2}(\lambda_1, \lambda_2) \succcurlyeq^{\text{disp}} D_{2:2}(\lambda_1^*, \lambda_2^*).$$

4. DEPENDENCE BETWEEN ORDER STATISTICS

Boland, Hollander, Joag-Dev, and Kochar [8] have studied in detail the different kinds of dependence that hold between order statistics from independent, but nonidentical, distributions. In particular, it follows from their Theorem 2.2 that $X_{i:n}$ is stochastically increasing in $X_{1:n}$ for any $i > 1$. In the next theorem, we strengthen this result. First, we give the definition of *upper orthant ordering*.

DEFINITION 4.1. Let $\mathbf{X} = (X_1, ..., X_n)$ be an n-dimensional random vector with joint survival function $\bar{F}(x_1, ..., x_n)$ and let $\mathbf{Y}$ be another n-dimensional random vector with joint survival function $\bar{G}(x_1, ..., x_n)$. If

$$\bar{F}(x_1, ..., x_n) \leqslant \bar{G}(x_1, ..., x_n) \qquad \text{for all } \mathbf{x} \tag{4.1}$$

then we say that $\mathbf{X}$ is smaller than $\mathbf{Y}$ in the upper orthant order.

For properties and implications of this ordering, see Section 4.G of Shaked, Shanthikumar, and collaborators [18]. In the next theorem, we prove that $(X_{2:n}, ..., X_{n:n})$ is increasing in $X_{1:n}$ in the sense of upper orthant order.

THEOREM 4.1. *Let* $X_1, ..., X_n$ *be independent random variables with* X_i *having the exponential distribution with survival* $\bar{F}_i(t) = \exp(-\lambda_i t)$, $t \geqslant 0$, *for* $i = 1, ..., n$. *Then* $(X_{2:n}, ..., X_{n:n})$ *is increasing in* $X_{1:n}$ *in the sense of upper orthant order.*

Proof. It follows from Theorem 2.1(i) that under the above assumptions $X_{1:n}$ is independent of $(X_{2:n} - X_{1:n}, ..., X_{n:n} - X_{1:n})$. Now

$$\begin{aligned} &P[X_{2:n} \geqslant x_2, ..., X_{n:n} \geqslant x_n \mid X_{1:n} = x_1] \\ &\quad = P[X_{2:n} - X_{1:n} \geqslant x_2 - x_1, ..., X_{n:n} - X_{1:n} \geqslant x_n - x_1 \mid X_{1:n} = x_1] \\ &\quad = P[X_{2:n} - X_{1:n} \geqslant x_2 - x_1, ..., X_{n:n} - X_{1:n} \geqslant x_n - x_1] \end{aligned}$$

which is obviously nondecreasing in x_1. This proves the required result. ∎

The above result can be easily extended from exponential distributions to the case of distributions with proportional hazards. It may be noted that the present proof is much simpler and more general than the one originally given by Boland, Hollander, Joag-Dev, and Kochar [8, Theorem 2.2].

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