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SMOOTH ESTIMATION OF STOCHASTICALLY ORDERED SURVIVAL FUNCTIONS

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Abstract

Let $\bar{F}$ and $\bar{G}$ be two survival functions. Sometimes it is known a prior that $\bar{G}(x) \geq \bar{F}(x)$ for all x. In the one-sample case when $\bar{F}$ is known, the estimator, $\hat{\bar{G}}_n = \max(\bar{G}_n, \bar{F})$ of the survival function $\bar{G}$, has been considered by many researchers in the literature. Here $\bar{G}_n$ denotes the empirical survival function based on a sample of size n from the population with survival function $\bar{G}$. In the two-sample case when both $\bar{F}$ and $\bar{G}$ are unknown, the corresponding estimators are, $\hat{\bar{G}}_{m,n} = \max(\bar{G}_m, \bar{F}_n)$ and $\hat{\bar{F}}_{m,n} = \min(\bar{G}_m, \bar{F}_n)$. These estimators, typically, have jumps at the data points and there is naturally an interest in finding smooth estimators. The popular smoothing methods do not guarantee the stochastic ordering property in the resulting estimators. This paper presents an adaptation of the smoothing technique introduced in Chaubey and Sen [*Statistics and Decisions* **14**, 1996] and investigates the asymptotic properties of the resulting estimators. Some numerical illustrations are also provided.

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1 Introduction

Let $\bar{G}$ and $\bar{F}$ denote the survival functions of two random variables X and Y, respectively. The random variable X is said to be stochastically larger than Y (written as $X \geq_{st} Y$) if $\bar{G}(x) \geq \bar{F}(x)\ \forall x$. This concept was introduced by Lehmann (1955) and has found natural applications in reliability and life testing situations. For example, in an accelerated life-testing situation, it is natural to assume that survival times are larger under lower stress condition than those under the normal stress conditions. Whitt (1980) considered a stronger version of the concept of stochastic ordering known as uniform stochastic ordering which has recently been studied by several authors [see Mukerjee (1996) and references there in], however, we are not going to deal with the latter concept in the present paper. Sometimes it is known a prior that $X \geq_{st} Y$ and we wish to estimate the survival functions under this order restriction. The empirical survival functions (*esf*), in spite of their many good properties, may not preserve this ordering. As a result nonparametric maximum likelihood estimators (NPMLE) have been developed and studied under such order constraints (see Dykstra (1982), Dykstra and Feltz (1989) and Feltz and Dykstra (1985)). These estimators are usually quite complicated and their distributional properties are not fully known. Only recently (see Præstergaard and Huang (1996)) asymptotic theory for NPMLEs has been developed, the limiting distributions, however, are not known in closed form. Some other drawbacks have been pointed out by Dykstra, Kochar and Robertson (1991) and Rojo and Ma (1996). Thus an alternative approach may be preferable. For the one-sample problem when $\bar{F}$ is known and a random sample $(X_1, X_2, ..., X_m)$ from $\bar{G}$ is available, the following estimator has been considered by Ma (1991) and Puri and Singh (1992);

$$\hat{\bar{G}} = \max(\bar{F}, \bar{G}_m) \tag{1}$$

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$$\hat{\bar{G}} = \max(\bar{F}, \bar{G}_m) \tag{1}$$

where $\bar{G}_m$ denotes the empirical survival function based on the random sample $(X_1, X_2, ..., X_m)$, *i.e.*

$$\bar{G}_m(x) = \frac{1}{m}\sum_{i=1}^{m} I(X_i > x). \qquad (2)$$

This estimator is easy to compute in contrast to the non-parametric MLE. Furthermore, as pointed out in Rojo (1995) and Rojo and Ma (1996), it is strongly uniformly consistent and dominates the empirical survival function for a wide class of loss functions. Rojo and Ma (1996) also showed that the above estimator has a uniformly smaller (positive) bias than the corresponding NPMLE and further demonstrated through simulation studies that it has smaller mean-squared error for a variety of distributions. Lo (1987) considered the two-sample problem when both $\bar{F}$ and $\bar{G}$ are unknown and independent random samples $(Y_1, Y_2, ..., Y_n)$ and $(X_1, X_2, ..., X_m)$ are available from the two distributions. The proposed estimators in this case are,

$$\hat{\bar{G}}_{m,n} = \max(\bar{G}_m, \bar{F}_n) \qquad (3)$$

and

$$\hat{\bar{F}}_{m,n} = \min(\bar{G}_m, \bar{F}_n) \qquad (4)$$

Similar considerations have led Rojo and Samaniego (1993) and Mukerjee (1996) to consider alternative estimators in the case of uniform stochastic ordering.

When the distributions are assumed to be continuous, many applied practitioners would prefer to have smooth estimators and as such there is a lot of interest in smooth estimation of survival functions. Kim and Proschan (1991) proposed a piecewise exponential survival function to be preferred over the usual *esf.* There is a lot of literature on smooth estimation of density and distribution functions (see the monographs by Devroye (1989), Härdle (1991), Silverman (1986) and Wertz

(1978)), but there have not been many methods specifically tailored for smooth estimation of survival functions of nonnegative random variables. Bagai and Prakasa Rao (1995) adapt asymmetric kernel method in estimating a density with positive support, where as Chaubey and Sen (1996) propose a new technique based on the so-called Hille's theorem in analysis. The adaptation of Bagai and Prakasa Rao's method in smoothing $\hat{\bar{G}}_m$, $\hat{\bar{G}}_{m,n}$ and $\hat{\bar{F}}_{m,n}$ is not clear as it deals with smooth estimator of a probability density function, however, that of Chaubey and Sen may be readily adapted, because it deals with smoothing of a distribution function directly. However, such smoothing may destroy the stochastic ordering property inherent in the original estimators.

In this paper, we modify the method in Chaubey and Sen (1996) to provide smooth estimators of survival functions that are stochastically ordered. The case of uniform stochastic ordering will be considered separately. The proposed estimators preserve the stochastic ordering property and retain the desirable large sample properties of unsmoothed estimators as given in Rojo (1995) and Rojo and Samaniego (1996). The smooth estimators are introduced in Section 2 and their asymptotic properties are established in Section 3. The last section gives some illustrative examples.

2 Smooth Estimators of Survival Functions

2.1 One-Sample Problem

Let $\bar{G}$ and $\bar{F}$ be absolutely continuous such that $\bar{G}(x) \geq \bar{F}(x)\ \forall x$ and $\bar{F}$ is known. Since $\bar{G}$ and $\bar{F}$ are survival functions, they are assumed to have support $[0, \infty)$. Using the technique in Chaubey and Sen (1996), we present a smooth version of $U_m(x) = \hat{\bar{G}}_m(x) - \bar{F}(x)$ given by

$$\tilde{U}_m(x) = \sum_{k=0}^{\infty} p_k(x\lambda_m) U_m\left(\frac{k}{\lambda_m}\right) \tag{1}$$

where $\{\lambda_m\}_{m=1}^{\infty}$ is a sequence of constants such that $\lambda_m \to \infty$ a.s. as $m \to \infty$ and

$$p_k(\mu) = e^{-\mu}\frac{\mu^k}{k!},\ k = 0, 1, 2, \ldots \tag{2}$$

Note that $\tilde{U}_m(x)$ is nonnegative and infinitely differentiable, hence it is quite smooth. It will be shown that it is strongly consistent for the difference $U = \bar{G} - \bar{F}$, hence, a smooth estimator for $\bar{G}$ is proposed as

$$\tilde{\bar{G}}_m(x) = \tilde{U}_m(x) + \bar{F}(x). \tag{3}$$

Since, $\tilde{U}_m(x)$ is nonnegative, the estimator $\tilde{\bar{G}}_m$ preserves the stochastic ordering condition, *i.e.* $\tilde{\bar{G}}_m(x) \geq \bar{F}(x)\ \forall x$. It will be shown in the next section that it also preserves other asymptotic properties endowed in $\hat{\bar{G}}_m(x)$.

The basic motivation for this estimator comes from the following so-called Hille's lemma (see Feller (1965), pp. 227).

Let $u(x)$ be a bounded continuous function defined on $[0, \infty)$, then the function $\tilde{u}(x)$ defined by

$$\tilde{u}_\lambda(x) = e^{-\lambda x}\sum_{k=0}^{\infty} u\left(\frac{k}{\lambda}\right)\frac{(\lambda x)^j}{j!} \tag{4}$$

converges uniformly to $u(x)$ in any finite sub-interval of $[0, \infty)$, as $\lambda \to \infty$.

Chaubey and Sen (1996) showed that the above convergence holds uniformly almost surely when we replace the bounded continuous function u by the *edf* or *esf*.

2.2 Two-Sample Problem

In this case we have to find a pair of smooth estimators $\tilde{\bar{G}}_{m,n}$ and $\tilde{\bar{F}}_{m,n}$ such that $\tilde{\bar{G}}_{m,n} \geq \tilde{\bar{F}}_{m,n}$. We may first smooth $\hat{\bar{F}}_{m,n}$ and consider it to be fixed in obtaining a smooth estimator of $\bar{G}$ by smoothing $\hat{\bar{G}}_{m,n} - \hat{\bar{F}}_{m,n}$.

Thus the pair of smooth estimators is given by

$$\tilde{\bar{F}}_{m,n}(x) = \sum_{k=0}^{\infty} p_k(x\lambda)\hat{\bar{F}}_{m,n}(\frac{k}{\lambda}) \tag{5}$$

and

$$\tilde{\bar{G}}_{m,n}(x) = \sum_{k=0}^{\infty} p_k(x\lambda)U_{m,n}(\frac{k}{\lambda}) + \tilde{\bar{F}}_{m,n}(x), \tag{6}$$

where $U_{m,n}(x) = \hat{\bar{G}}_{m,n}(x) - \hat{\bar{F}}_{m,n}(x)$.

Alternatively, if we smooth first $\hat{\bar{G}}_{m,n}$, we have the following pair of estimators,

$$\tilde{\bar{G}}^*_{m,n}(x) = \sum_{k=0}^{\infty} p_k(x\lambda)\hat{\bar{G}}_{m,n}\left(\frac{k}{\lambda}\right); \tag{7}$$

$$\tilde{\bar{F}}^*_{m,n}(x) = -\sum_{k=0}^{\infty} p_k(x\lambda)U_{m,n}\left(\frac{k}{\lambda}\right) + \tilde{\bar{G}}^*_{m,n}(x). \tag{8}$$

Asymptotically, the pairs of estimators given in equations (8) and (7) are equivalent to the pair of those given by equations (5) and (6). We may further consider the convex class of distributions with respect to ordering $\bar{G} \geq \bar{F}$

$$C_\alpha = \alpha\bar{F} + (1-\alpha)\bar{G} \tag{9}$$

for $0 \leq \alpha \leq 1$, motivated by Mukerjee (1996). We have for every $\alpha \in [0,1]$, $\bar{F} \leq C_\alpha \leq \bar{G}$. With respect to this class, the pair of estimators $\bar{F}^*_{\alpha(m,n)}, \bar{G}^*_{\alpha(m,n)}$ is given by

$$\bar{F}^*_{\alpha(m,n)} = \min(\bar{F}_n, C_{\alpha(m,n)}) \tag{10}$$

and

$$\bar{G}^*_{\alpha(m,n)} = \max(C_{\alpha(m,n)}, \bar{G}_m), \tag{11}$$

where

$$C_{\alpha(m,n)} = \alpha\bar{F}_n + (1-\alpha)\bar{G}_m.$$

The above pair of estimators may be used in producing a class of smooth estimators by using them in place of the earlier pairs. The value of α may be chosen to be $n/(m+n)$ owing to the consideration in Mukerjee (1996), however, a thorough study on the choice of α is warranted. The estimators of survival distributions originating from the class $\mathcal{C}_\alpha = \{C_\alpha, 0 \leq \alpha \leq 1\}$ were considered in the case of *uniform stochastic ordering*. In later discussions we will only consider the pair given by equations (5) and (6).

3 The Asymptotic Properties of the Estimators

We will use the notation, $\|U\| = \sup_{x\in\mathbf{R}^+} |U(x)|$ for any bounded function U defined on $[0,\infty)$. First, we establish the strong consistency of the smooth estimator in one-sample case as given in the following theorem.

Theorem 3.1. Let $\lambda_m \to \infty$ a.s. as $m \to \infty$. Then

$$\sup_{x\in\mathbf{R}^+} |\tilde{\bar{G}}_m(x) - \bar{G}(x)| \to 0 \text{ a.s. as } m \to \infty$$

PROOF: In order to prove the theorem, it is enough to establish

$$\sup_{x\in\mathbf{R}^+} |\tilde{U}_m(x) - U(x)| \to 0 \text{ a.s. as } m \to \infty \tag{1}$$

where $U = \bar{G} - \bar{F}$. This can be easily proved along the same lines as in Chaubey and Sen (1996)and using the strong uniform consistency of $\hat{\bar{G}}_m$ as established in Rojo and Ma (1996). □

The following theorem gives the weak convergence of the smooth estimator $\tilde{\bar{G}}$ similar to that for $\hat{\bar{G}}$ obtained in Rojo (1995).

Theorem 3.2. If $\lambda \to \infty$ and $m^{-1}\lambda \to 0$ then for $\bar{G}(x) > \bar{F}(x) \forall x$

$$\sqrt{m}(\tilde{\bar{G}}_m - \bar{G}) \to W^0$$

where W^0 denotes a Brownian bridge. However, if $\bar{G}(x_0) = \bar{F}(x_0)$ for some x_0 with $\bar{G} \neq \bar{F}$, then $\tilde{\bar{G}}_m$ does not converge weakly.

PROOF: It is required to establish the following lemma in order to prove the above theorem, which is also of independent interest as it gives the order of closeness of the smooth estimator to the nonsmooth estimator.

Under the conditions on the sequence $\{\lambda_m\}$ given in theorem 3.1, we have

$$\|\tilde{\bar{G}}_m - \hat{\bar{G}}_m\| = O(m^{-3/4} \log m) \ \ a.s. \ \text{as } m \to \infty.$$

PROOF: We may write

$$\begin{aligned} \tilde{\bar{G}}_m(x) - \hat{\bar{G}}_m(x) &= \sum_{k=0}^{\infty} p_k(x\lambda)(\hat{\bar{G}}_m\left(\frac{k}{\lambda}\right) - \bar{G}_m\left(\frac{k}{\lambda}\right) - \hat{\bar{G}}_m(x) + \bar{G}(x)) \\ &\quad + \left\{\sum_{k=0}^{\infty} p_k(x\lambda)(\bar{G}\left(\frac{k}{\lambda}\right) - \bar{G}(x)) \right. \\ &\quad \left. - \sum_{k=0}^{\infty} p_k(x\lambda)(\bar{F}\left(\frac{k}{\lambda}\right) - \bar{F}(x))\right\}. \end{aligned} \tag{2}$$

First, note from Chaubey and Sen (1996) that (see their equation (3.23)) that

$$\sum_{k=0}^{\infty} p_k(x\lambda)[\bar{G}\left(\frac{k}{\lambda}\right) - \bar{G}(x)] = O(m^{-1} \log m) \ a.s. \text{ as } m \to \infty. \tag{3}$$

For the analysis of the first term in equation we break the sum over the regions;

$$N_x = \{k \in N : |k/\lambda - x| \leq (m^{-1} \log m)^{1/2}\}$$

and

$$N_x^c = N \backslash N_x$$

where $N = \{1, 2, 3, ...\}$. Using Lemma 3.1 of Chaubey and Sen (1996) we may claim that

$$|\sum_{k \in N_x^c} p_k(x\lambda)(\hat{\bar{G}}_m\left((\frac{k}{\lambda}\right) - \bar{G}_m\left((\frac{k}{\lambda}\right) - \hat{\bar{G}}_m(x) + \bar{G}(x))| = o(m^{-1}). \quad (4)$$

Now from Rojo and Ma (1996) (see their equation (2.5)), we have

$$\sup_{x \in \mathbf{R}^+} |\hat{\bar{G}}_m(x) - \bar{G}_m(x)| = o(m^{-1/2}) \ a.s.$$

as $m \to \infty$, hence we can write

$$\begin{aligned} &|\sum_{k \in N_x} p_k(x\lambda)[\hat{\bar{G}}_m\left((\frac{k}{\lambda}\right) - \bar{G}_m\left((\frac{k}{\lambda}\right) - \hat{\bar{G}}_m(x) + \bar{G}(x)]| \\ &= |\sum_{k \in N_x} p_k(x\lambda)[\bar{G}_m\left((\frac{k}{\lambda}\right) - \bar{G}_m\left(\frac{k}{\lambda}\right) - \bar{G}_m(x) + G(x)]| \\ &\qquad + o(m^{-1/2}). \end{aligned} \quad (5)$$

Now we use the celebrated Bahadur (1966) representation of quantiles as in Chaubey and Sen (1996) and combine equations (4) and (5) to claim that

$$\sup_{x \in \mathbf{R}^+} |\hat{\bar{G}}_m(\frac{k}{\lambda}) - \bar{G}(\frac{k}{\lambda}) - \hat{\bar{G}}_m(x) + \bar{G}(x)| = O(m^{-3/4} \log m) \ a.s. \text{ as } m \to \infty. \quad (6)$$

Using equations (2), (3) and (6) gives the result. □

Using the above theorem we find that

$$\sup_{x \in \mathbf{R}^+} |\tilde{W}_m(x) - W_m(x)| = O(m^{-1/4} \log m)$$

where

$$W_m(x) = \sqrt{m}(\hat{\bar{G}}_m(x) - \bar{G}(x))$$

and

$$\tilde{W}_m(x) = \sqrt{m}(\tilde{\bar{G}}_m(x) - \bar{G}(x)).$$

Now we use Rojo's Theorem 2.1 for the weak convergence of $W_m(x)$ to prove the desired result. The second part also follows from the same theorem owing to the almost sure equivalence of $\tilde{W}_m(x)$ and $W_m(x)$. The asymptotic results of Rojo (1995) and Rojo and Ma (1996) extend to the smooth estimators in the two sample case also. We merely state them without proof.

Theorem 3.3. Let $m^* = min(m,n)$ and $\{\lambda_{m^*}\}$ be a sequence of constants such that $\lambda_{m^*} \to \infty$ as $m^* \to \infty$, then almost surely,

$$\sup_{x\in\mathbf{R}^+} |\tilde{\bar{G}}_{m,n}(x) - \bar{G}(x)| \to 0.$$

and

$$\sup_{x\in\mathbf{R}^+} |\tilde{\bar{F}}_{m,n}(x) - \bar{F}(x)| \to 0.$$

Theorem 3.4. Let $\{\lambda_{m,n}\}$ be a sequence of constants such that $\lambda_{m,n} \to \infty$, $m^{-1}\lambda_{m,n} \to 0$, and $n^{-1}\lambda_{m,n} \to 0$ as $m,n \to \infty$, then for the case $\bar{G}(x) > \bar{F}(x) \forall x$

$$\sqrt{m}(\tilde{\bar{G}}_{m,n} - \bar{G}) \to W^0$$

and

$$\sqrt{n}(\tilde{\bar{F}}_{m,n} - \bar{F}) \to W^0.$$

If $\bar{G}(x_0) = \bar{F}(x_0)$ for some x_0 with $\bar{G} \neq \bar{F}$, then $\tilde{\bar{G}}_{m,n}$ and $\tilde{\bar{F}}_{m,n}$ do not converge weakly.

Remark 3.1 Chaubey and Sen (1996) found the choice of $\lambda_m = m/\max(X_1, X_2, ..., X_m)$ to be appropriate for the case when we assume $E(X) < \infty$, in the unordered case. This value of λ can still be used for the one-sample case. For the two-sample case, we could use $\lambda_{m,n} = \min(\lambda_m, \lambda_n)$. This value is appropriate asymptotically, however, we have found that $\lambda_m = m/\max(X_1, X_2, ..., X_m)$ should be used for smoothing $\bar{G}_{m,n}$ and $\lambda_n = n/\max(Y_1, Y_2, ..., Y_n)$ should be used for smoothing $\bar{F}_{m,n}$

Remark 3.2 The smoothing technique used here has been extended to the case of random censoring (see Chaubey and Sen (1998)). It can be similarly adapted to the stochastic ordering case using the setup treated in Rojo (1995). The asymptotic results discussed in Rojo (1995) extend to this case also when the supports of the underlying distributions as well as the censoring distribution are infinite.

4 Numerical Illustrations

The following table gives the life times of samples of electrical insulating material subject to a constant stress at 32 Kv and 34 Kv obtained from Nelson (1990, Chapter 3, pp. 128-129). The data has been fitted with Weibull distribution with shape parameter $\alpha = 1/28.94$ and scale parameter $\beta = .561$ for 32 Kv sample and that with $\alpha = 1/11.22$ and $\beta = .771$ for 34 Kv sample. Rojo (1995) points out that the Weibull model used to fit the data on 34 kilovolts does not satisfy the stochastic ordering constraint for the survival functions $\bar{G}$ and $\bar{F}$, where these are respectively the survival distributions for 32 Kv and 34 Kv samples. Physical considerations would require to assume that $\bar{G} \geq \bar{F}$. Hence, considering $\bar{F}$ as a fixed Weibull, he fitted the estimator $\hat{\bar{G}}_m$ to the data in Table 1 for 32kV sample. Figure 1 demonstrates the use of smooth estimation in relation to the Rojo's unsmoothed estimator. In this case $m = 15$ and $\max(X_1, \ldots, X_m) = 215.1$, thus the value of λ chosen for smoothing in this case is taken to be $15/215.1 = .069735$. Closeness of the smooth and nonsmooth graphs is appealing. Figure 2 gives the smooth estimator of the survival distribution for the 34 Kv sample under the stochastic ordering constraint and Figure 3 again demonstrates the use of smooth estimation in relation to the Rojo's unsmoothed estimator in the two-sample case. The value of λ chosen for smoothing the survival distribution for the 34 Kv sample, as indicated in remark 3.2, is taken to be 19/72.89 = .260667, whereas that for getting the smooth estimator in the case of 32 Kv sample is .069735 as it was in the one-sample case. The similarity of the graphs between Figure 1 and Figure 3 basically point out to the superb fit of the Weibull model to the 34 Kv data. We have not shown here the graphs of the other pairs of estimators, but they are qualitatively similar.

Table 1: Ordered times until failure

32 Kv	0.27	0.40	0.69	0.79	2.75	3.19	9.88
(m=15)	13.95	15.93	27.80	53.4	82.85	89.29	100.58
	215.1						
34 Kv	0.18	0.78	0.96	1.31	2.78	3.16	4.15
(n=19)	4.67	4.85	6.50	7.35	8.01	8.27	12.06
	31.75	32.52	33.91	36.71	72.89		

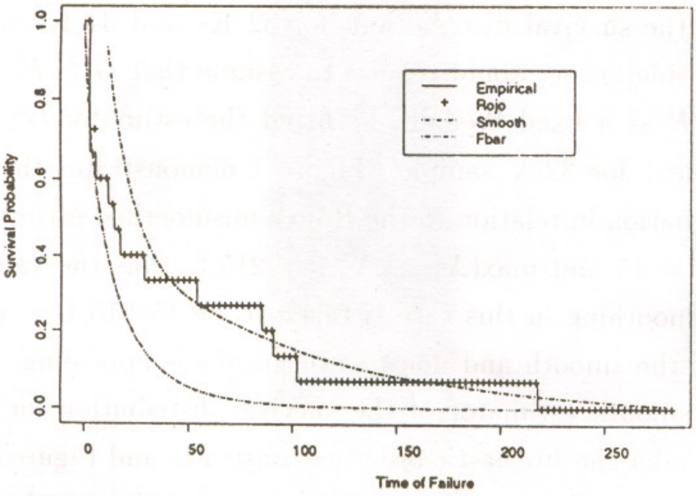

Figure 1: Stochastic order with respect to Weibull (.0871,.771)

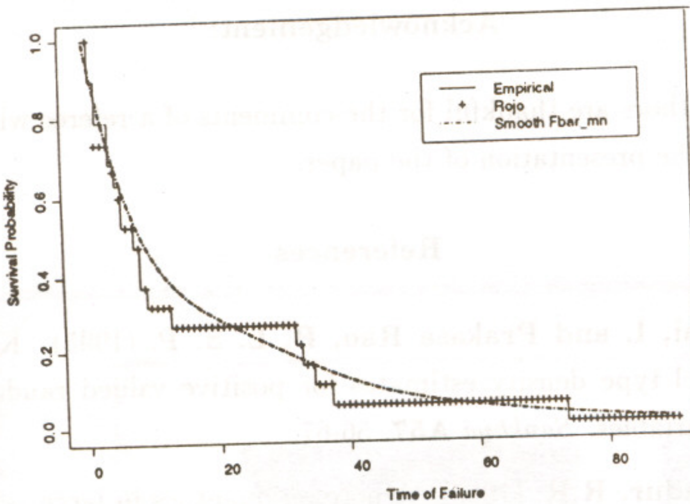

Figure 2: Survival Distribution for 34 Kv sample

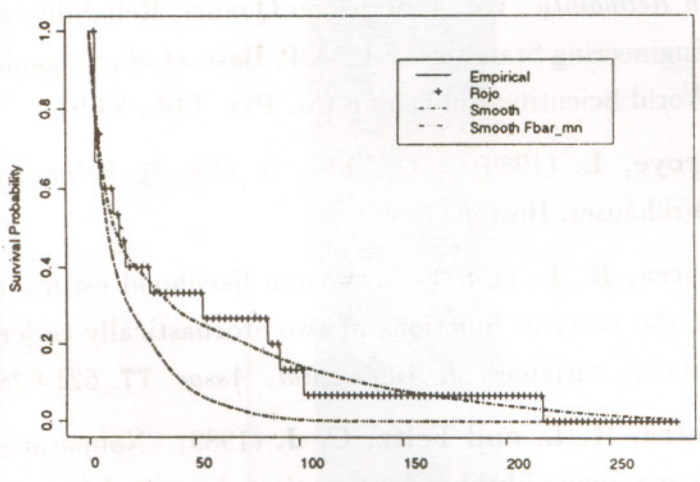

Figure 3: Survival Distribution for 32 Kv sample

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