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13

A NEW CLASS OF TESTS FOR TESTING EXPONENTIALITY AGAINST POSITIVE AGING

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Abstract

In this paper a new class of tests is proposed to test the hypothesis of exponentiality against various alternatives involving positive aging, viz., IFRA, NBU and HNBUE classes of distributions. Typically the test statistic is the sample version of a functional which discriminates between exponentiality and the above hypotheses. It is seen to be a linear function of the order statistics of the random sample. Its exact and asymptotic distributions are discussed and the asymptotic relative efficiency of some selected members of this class are calculated as compared to the cumulative total time on test statistic for some wellknown distributions belonging to the above positive aging classes.

1. Introduction :

A central question in reliability theory pertains to the modelling of the probability distributions of random variables representing life time of units, whether humans, animals, microbes, radioactive substances, components andt systems of components, etc. A popular model is the exponential distribution which is useful wherever the 'no aging' phenomenon is evident. In probabilistic terms this phenomenon is represented by the lack of memory aspect of the exponential distribution. Translated into reliability terms it simply means that the probability distribution of the life time of a unit does not change with the knoweldege that the unit has already survived for a given time. In other words if X is the random variable representing the life time then the conditional survival function $P[X > s + t \mid X > t]$ is a function of s alone and does not depend on t, the time for which the unit has already survived. As agains this 'no aging' phenomenon, many units exhibit the positive aging phenomena This term is used to denote the situation where the preformance of a unit in some probabilistic sense, deteriorates with its age, that is, with the time for which it has already survived. Many probabilistic concepts have been introduced to describe various notions of positive againg. They are based on successively weaker properties of the conditional survival function. A popular chain of these notions is given by

Exponential distribution

↓

Increasing failure rate (IFR)

↘

Increasing feailure rate average (IFRA)

↙ ↓

Decreasing mean residual life (DMRL)

New better than used (NBU)

↘ ↓

New better than used in expectation (NBUE)

↓

Harmonic new better than used in expectation (HNBUE)

The implication signs mean that the higher property implies the lower property, thus leading to a larger class of probability distributions exhibiting positive aging in some sense. Detailed discussions of these and related concepts are available in Barlow and Praschan (1975), Klefsjö (1983b), Hollander and Proschan (1975), etc. Often in practical situations the life time of units have been noticed to possess one or the other of the above aging properties. Hence it has been the concern of statisticians to propose tests for the null hypothesis of exponentiality against the alternative hypotheses specifying one of the above positive aging classes. Such tests have been proposed by Proschan and Pyke (1967), Bickel and Doksum (1969), Hollander and Proschan (1972), Deshpande (1983) Deshpande and Kochar (1983), Klefsjö (1983a) and many others. A comprehensive review is available in the paper by Hollander and Proschan (1983). The most widely investigated and perhapas the most widely used test is based on the celebrated cumulative total time on test statistic. Recently Klefsjö has demonstrated that the test is consistent for testing exponentiality against the entire HNBUE class, which is the largest class of probability distributions proposed so far posessing the positive aging property in some sense.

In the second section we define the functional which discriminates between exponential distribution and a positive aging distribution. We also introduce its sample version and show that it is a linear function of the order statistics of the random sample. The class of tests is then introduced on heuristic grounds, The cumulative total time on test statistic is demonstrated to be a member of this class. In the third section we consider the asymptotic distributions of members of this class. It is shown that being linear functions of order stastistics, these statistics are asymptotically normally distributed under certain sufficient conditions. The asymptotic means and variances of these statisties are explicitely calculated under the null hypothesis. A comment is made on the consistency of members of this class of tests for the various classes of positive aging distributions.

The fourth section is concerning the asymptotic relative efficiencies (in the Pitman sense) of members of this class. The endevour is to pick the tets from the class with the maximum ARE with respect to the cumulative total time on test statistic for three commonly used positive aging distributions, vis., Weibull, Makeham and the linear increasing failure rate distributions.

2. Motivation for the Proposed Class of Tests

Let $X_1, \cdots\cdots, X_n$ be a random sample of size n from an absolutely continuous life distribution with distribution function F, density function f, survival function $\bar{F} = 1 - F$ and the failure rate $r_F = f/\bar{F}$, whenever $\bar{F}$ is not zero. The problem considered in this paper is to test the null hypothesis H_0:F is negative exponential, that is, $\bar{F}(x) = e^{-\lambda x}$, $x \geqslant 0$, where λ (> 0) is the unknown scale parameter. The alternatives being considered are the various hypotheses which describe the phenomenon of positive aging.

It is wellknown that F belongs to the IFRA class of life distributions if and only if

$$\bar{F}^{1/\alpha}(x) \geqslant \bar{F}\left(\frac{x}{\alpha}\right) \tag{2·1}$$

for every $0 < \alpha < 1$ and for every $x \geqslant 0$. Equality in (2·1) for all x obtains only for the exponential distribution. That is to say, in (2·1) there must be strict inequality for some x (hence with probability greater than zero) for all IFRA distributions which are not exponential.

An equivalent version of the above inequality is

$$\bar{F}^{\beta/\alpha}(x) \geqslant \bar{F}^{\beta}\left(\frac{x}{\alpha}\right) \tag{2·2}$$

for $0 < \alpha < 1$, $\beta > 0$ an d for every $x \geqslant 0$. Integrating both sides of (2·2) we obtain

$$\int_0^\infty \bar{F}^{\beta/\alpha}(x)\, dx \geqslant \int_0^\infty \bar{F}^{\beta}\left(\frac{x}{\alpha}\right) dx = \alpha \int_0^\infty \bar{F}^{\beta}(x)\, dx$$

Let us define

$$\triangle = \int_0^\infty \bar{F}^{\beta/\alpha}(x) - \alpha \int_0^\infty \bar{F}^{\beta}(x)\, dx \tag{2·3}$$

It is easily seen that $\triangle_F = 0$ if F is exponential, whereas

(i) under H_1 : F is IFRA and not exponential, $\triangle_F > 0$ for $0 < \alpha < 1$ and $\beta > 0$.

(ii) under H_2 : F is NBU and not exponential, $\triangle_F > 0$ for $\alpha = 1/K$, $K = 2, 3, \cdots$ and $\beta > 0$. and

(iii) under H_3 : F is HNBUE and not exponential, $\triangle_F > 0$ for $\alpha = 1/K$, $K = 2, 3, \cdots$ and $\beta = 1$.

The inequalities in (i) and (ii) above follow under the respective hypothesis H_1 and H_2 since (2·2) is true under these alternatives under the stated conditions on α and β. Inequlity in (iii) was established by Klefsjö (1983b).

The value of F may, therefore, be taken as a measure of deviation (it is not a metric) of F from the null hypothesis of expenentiality in the sense of positive aging. The tests for testing H_0 against the various positive aging hypotheses are proposed to be based on the sample analogue of $\triangle_F$ which is seen to be a consistent estimator of $\triangle_F$.

Let F_n be the empirical distributinn function based on the random sample $X_1, \cdots\cdots, X_n$. Then the sample analogue of $\triangle_F$ is

$$\triangle_{F_n} = \left\{ \bar{F}_n^{\beta/\alpha}(x) - \alpha \bar{F}_n^{\beta}(x) \right\} dx$$

$$= \sum_{i=1}^{n} \left[\bar{F}_n^{\beta/\alpha}\{X_{(i)}\} - \alpha \bar{F}_n^{\beta}\{X_{(i)}\} \right] [X_{(i)} - X_{(i-1)}]$$

$$= \frac{1}{n} \sum_{i=1}^{n} a_{ni} X_{(i)}, \qquad (2\cdot4)$$

where $X_{(i)}$, $i = 1, 2, \cdots\cdots, n$ are the order statistic from the random sample and

$$a_{ni} = n \left[\left\{ \left(1 - \frac{i}{n}\right)^{\beta/\alpha} - \alpha \left(1 - \frac{i}{n}\right)^{\beta} \right\} - \left\{ \left(1 - \frac{i+1}{n}\right)^{\beta/\alpha} \alpha \left(1 - \frac{i+1}{n}\right)^{\beta} \right\} \right] \qquad (2\cdot5)$$

However, it is seen that $\triangle_{F_n}$ is not scale invariant. To make it scale invariant, we modify $\triangle_{F_n}$ to

$$T_n = \frac{\triangle_{F_n}}{n\bar{X}} = \sum_{i=1}^{n} a_{ni} X_{(i)} / n\bar{X} \qquad (2\cdot6)$$

where X is the sample mean.

The special case of T_n obtained by taking $\alpha = 1/2$ and $\beta = 1$ is a linear function of the celebrated cumulative total time on test statistic originally proposed for teting H_0 against IFR alternative. And the special case obtained by taking a $\alpha = 1/K$ $K = 2, 3, \cdots$ and $\beta = 1$ lead to the statistic considered by Klefsjö (1983b) for testing H_0 against the much wider positive aging alternative of HNBUE distributions.

The test consists of :

Reject H_0 against H_i, $i = 1, 2, 3$ if $T_n > T_{n, \alpha}$

where T_n is based on suitable values of α and β and $T_{n, \alpha}$ is the critical point with the required significance level α.

The critical points will be obtained from the null distribution of the stastistic T_n. Box (1954) has derived the distibution of the statisic of the type T_n, when X_i's are independent and identically distriuted random variables from a negative exponential distribution. But the null distribution is complicated. In the next section we show that asymptotically the statistic T_n is normally distributed. Hence the critial points from the normal distribution may be used as approximation for the exact values.

3. Asympotic Distribution of the Test Statistic

Let us consider the scores

$$a_{ni} = n\left[\left(1 - \frac{i}{n}\right)^{\beta/\alpha} - \left(1 - \frac{i+1}{n}\right)^{\beta/\alpha}\right] - n\alpha\left[\left(1 - \frac{i}{n}\right)^{\beta} - \left(1 - \frac{i+1}{n}\right)^{\beta}\right] \quad (3\cdot1)$$

It is seen that if we define

$$J(u) = \alpha\beta\,(1-u)^{\beta-1} - \frac{\beta}{\alpha}\,(1-u)^{\beta/\alpha - 1} \quad (3\cdot2)$$

then $a_{ni} \to J(u)$ if $n \to \infty$ in such a say that $i/n \to u$, $0 < u < 1$.

Let

$$\mu(F) = \int_0^\infty x\,dF(x)$$

$$\mu(J, F) = \int_0^\infty xJ\,[\,F(x)]\;dF(x), \text{ and} \quad (3\cdot3)$$

$$\sigma^2\,(J^*,F) = \int_0^\infty \int_0^\infty J^*\,(F(x)]\;J^*\,[F(y)]\;[F\,(\min\,(x, y)) - F(x)\,F(y)]\;dx\,dy \quad (3\cdot4)$$

where $J^*(u) = J(u) - \dfrac{\mu\,(J, F)}{\mu(F)}$. $(3\cdot5)$

Then it follows from the results of Stigler (1974) on the asymptotic distribution of linear functions of order statistics used in conjuction with Slutsky's theorem, that

$$\frac{\sqrt{n}\left[T_n - \frac{\mu(J,F)}{\mu(F)}\right]}{\sigma(J^*,F)/\mu(F)} \rightarrow N(\mu^*(F),\ 1) \tag{3·6}$$

where

$$\mu^*(F) = \mu(J^*,F)/\mu(F) = 0 \quad \text{provided} \quad 0 < \sigma(J^*,F) < \infty.$$

It is seen that under H_0, $\bar{F}_0(x) = e^{-\lambda x}$, $\mu(J, F_0) = 0$

and

$$\frac{\sigma^2(J^*,F_0)}{\mu^2(F_0)} = 2\beta\left[\frac{1}{2\beta-\alpha} - \frac{\alpha(\alpha+1)}{\beta+\beta\alpha-\alpha} - \frac{\alpha^2+1}{2\beta} + \frac{\alpha(\alpha+1)}{\beta+\beta\alpha} + \frac{\alpha^2}{2\beta-1}\right]$$

$$= 2\,\beta\, C_{\alpha,\beta}$$

The above results imply that the tests are consistent for alternatives whenever $\mu^*(F) > 0$ and $0 < \sigma(J^*, F) < \infty$. In particular, the tests are consistent for $\alpha > 0, \beta > \frac{1}{2}$ for H_1 for $\alpha = 1/K$, $\beta > \frac{1}{2}$ for H_2 and for $\alpha = 1/K$, $\beta = 1$ for H_3. It is seen that $\sigma^2(J^*, F)$ may not be a well defined function if $0 < \beta \leqslant \frac{1}{2}$.

4. Asympotic Relative Efficiency

These tests are proposed on heuristic grounds for testing exponentiality against the nonparametric classes of distributions specified by the hypotheses H_1, H_2 aud H_3. As pointed out earlier some tests belonging to this class are already well-known whereas others are new. We calculate the Pitman ARE's of some selected new tests belonging to this class with respect to the cumulative total time on test statistic, which also belongs to this class. The parametric alternatives considered are

(i) *Weibull distribution :*

$$\bar{F}_1(x) = \exp\{-x^{\theta}\},\ x \geqslant 0,\ \theta \geqslant 1, \tag{4·1}$$

(ii) *Makeham distribution* :

$$\bar{F}_2(x) = \exp\left\{-(x + \theta(x + e^{-x} - 1))\right\},\ x \geqslant 0,\ \theta \geqslant 0, \tag{4.2}$$

(iii) *Linearly increasing failure rate (LIFR) distribution :*

$$\bar{F}_3(x) = \exp\left\{-\left(x + \frac{\theta}{2}x^2\right)\right\},\ x \geqslant 0,\ \theta \geqslant 0. \tag{4·3}$$

In case of F_1, $\theta = 1$ and in case of F_2 and F_3, $\theta = 0$ give distributions in the null hypothesis H_0 and the distributions in the remaining range of θ are known to possess increasing failure rates and hence belong to the larger classes defined by H_1, H_2, H_3 as well.

Let $e_{\alpha,\beta}(F)$ denote the efficiency of the T_n statistic against the alternative F. It can be shown after some simple algebraic manupulations that

$$e_{\alpha,\beta}(F_1) = \frac{\alpha^2 (\log \alpha)^2}{2\beta^3 C_{\alpha,\beta}} \tag{4·4}$$

$$e_{\alpha,\beta}(F_2) = \Big[\frac{\alpha\beta}{(\alpha+\beta)^2} - \frac{\beta^2}{(\alpha+\beta)^2} + \frac{\alpha\beta}{(\beta+1)^2} + \frac{\alpha\beta^2}{(\beta+1)^2} - \alpha + 1\Big]^2 \Big/ 2\beta\, C_{\alpha,\beta} \tag{4·5}$$

$$e_{\alpha,\beta}(F_3) = \frac{\alpha^2 (1-\alpha)^2}{2\beta\, C_{\alpha,\beta}} \tag{4·6}$$

where

$$C_{\alpha,\beta} = \frac{1}{2\beta - \alpha} - \frac{\alpha(\alpha+1)}{\beta + \beta\alpha - \alpha} - \frac{\alpha^2+1}{2\beta} + \frac{\alpha(\alpha+1)}{\beta+\beta\alpha} + \frac{\alpha^2}{2\beta-1} \tag{4·7}$$

We first consider the case $\beta = 1$ and choose α such that the efficacy of the test based on T_n is maximized. The results are presented in the first row of Table 4·1.

Next we consider the case when $\alpha = 1/2$ and β varying. This case is being considered because with $\alpha = 1/2$ and $\beta = 1.$, statistic T_n reduces to a linear function of the cumulative total time on test statistic which is best for testing exponentiality against Makeham distribution. We fix $\alpha = 1/2$ and choose β such that the efficacy of the test based on T_n is maximized. The results are presented in the second row of Table 4·1. The last row in this table gives the efficacy $e_2(P)$ of the cumulative total time on test statistic, which also belongs to this class of tests, for the corresponding alternatives.

Table 4·1

F = Distribution	Weibull	Makeham	LIFR
$\max_{\alpha} e_{\alpha,1}(F)$	1·5095 at $\cdot 31 \leq \alpha \leq \cdot 32$	1/12 at $\alpha = 1/2$	1 as $\alpha \to 1$
$\max_{\beta} e_{1/2,\beta}(F)$	1·4615 at $1\cdot 22 \leq \beta \leq 1\cdot 225$	1/12 at $\beta = 1$	80/81 at $\beta = 3/4$
$e_2(F)$	1·44136	1/12	3/4

It is seen from the above table that the membes of the newly proposed class of tests are quite efficient. In particular, for Weibull and linear increasing failure rate alternatives it is possible to choose tests which are more efficient than the cumulative total time on test statistic. Similarly, it should be possible for many other alternatives which depict positive aging.

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