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Competitors of the Kendall-tau test for testing independence against positive quadrant dependence

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Summary

A class of distribution-free tests has been proposed for testing independence against positive quadrant dependence. The Kendall-tau test belongs to this class and the newly proposed tests are seen to be quite efficient.

Some key words: Asymptotic relative efficiency; U-statistic.

1. Introduction

Let (X, Y) be an absolutely continuous random variable with joint probability distribution function $F(x, y)$ and survival function $\bar{F}(x, y) = \text{pr}(X > x, Y > y)$. Let F_X and F_Y denote the distribution functions of the marginal distributions of X and Y, respectively with the corresponding survival functions $\bar{F}_X$ and $\bar{F}_Y$. We say that the pair (X, Y) or its distribution is positively quadrant dependent if, for all x, y,

$$F(x, y) \geqslant F_X(x)F_Y(y). \tag{1·1}$$

This is also equivalent to requiring for all x, y,

$$\bar{F}(x, y) \geqslant \bar{F}_X(x)\bar{F}_Y(y). \tag{1·2}$$

The dependence is strict if inequality holds for at least one pair (x, y).

The above inequalities can be interpreted more clearly in reliability terms. Given that the component Y has not failed up to time y, the conditional probability of survival, or reliability, for the component X up to time x is greater than its marginal reliability up to time x for all x and y. The concept of positive quadrant dependence is symmetric in x and y. If inequalities are reversed in (1·1) and (1·2), then X and Y are said to be negatively quadrant dependent.

In this paper we consider the problem of testing the null hypothesis of independence H_0: $F(x, y) = F_X(x)F_Y(y)$ for all x, y, against the alternative H_A of strict positive quadrant dependence.

There are many tests available in the literature for testing independence against positive dependence. Mention may be made of the Kendall-tau test and Spearman's rank correlation test. For details see Lehmann (1966) and Joag-Dev (1984).

It is felt that most of the tests available in the literature are suitable when the distributions are symmetric. For skew distributions, which arise particularly when the random variables are nonnegative as in the case of reliability problems, we propose below, a class of distribution-free tests for testing H_0 against H_A. The last section is devoted to asymptotic relative efficiency comparisons and simulation work to estimate the powers of the tests for small sample sizes.

2. The proposed class of tests

Let k be a fixed integer and consider

$$\delta_k(x, y) = F^k(x, y) - F_X^k(x)F_Y^k(y).$$

Under H_0, $\delta_k(x, y) = 0$ for all (x, y) but, under H_A, $\delta_k(x, y) \geq 0$ for all (x, y) and with a strict inequality over a set of nonzero probability. We consider the following measure of deviation between H_0 and H_A:

$$\Delta_k = \iint_{\mathbb{R}^2} \delta_k(x, y)\, dF(x, y) = \pi_{1k} - \pi_{2k},$$

where

$$\pi_{1k} = \iint_{\mathbb{R}^2} F^k(x, y)\, dF(x, y) = \mathrm{pr}\,\{\max(X_1, \ldots, X_k) \leq X_{k+1}, \max(Y_1, \ldots, Y_k) \leq Y_{k+1}\},$$

$$\pi_{2k} = \iint_{\mathbb{R}^2} F_X^k(x) F_Y^k(y)\, dF(x, y) = \iint_{\mathbb{R}^2} \bar{F}(x, y)\, dF_X^k(x)\, dF_Y^k(y),$$

(X_i, Y_i), for $i = 1, \ldots, k+1$, being independent copies of (X, Y).

Under H_0, $\pi_{1k} = \pi_{2k} = (k+1)^{-2}$ but, under H_A, $\pi_{1k} > \pi_{2k} > (k+1)^{-2}$. Given a random sample (X_i, Y_i), for $i = 1, \ldots, n$, from F, we base our test on the U-statistic estimator of π_{1k}. Let $\phi_{k+1}\{(X_1, Y_1), \ldots, (X_{k+1}, Y_{k+1})\}$ equal one if $\{\max(X_1, \ldots, X_{k+1}), \max(Y_1, \ldots, Y_{k+1})\}$ belongs to the same pair (X, Y), and zero otherwise. The corresponding U-statistic is

$$U_{k+1} = \left\{\binom{n}{k+1}\right\}^{-1} \sum \phi_{k+1}\{(X_{i_1}, Y_{i_1}), \ldots, (X_{i_{k+1}}, Y_{i_{k+1}})\},$$

where the sum is extended over all combinations of $(k+1)$ integers $(i_1, \ldots, i_{k+1})$ chosen out of $(1, \ldots, n)$.

Large values of U_{k+1} are significant for testing H_0 against H_A. Clearly, U_2 is the well-known Kendall's tau statistic.

To find the exact distribution of U_{k+1}, let

$$p_{n,k+1}(v) = \mathrm{pr}_{H_0}\left[U_{k+1} = \left\{\binom{n}{k+1}\right\}^{-1} v\right].$$

Then it can easily be seen that

$$p_{n+1,k+1}(v) = \frac{1}{n+1}\left[p_{n,k+1}\left\{v - \binom{n}{k}\right\} + \ldots + p_{n,k+1}\left\{v - \binom{k+1}{k}\right\} + p_{n,k+1}(v-1) + k p_{n,k+1}(v)\right],$$

for $n > k$ and $p_{k,k}(0) = (k-1)/k$, $p_{k,k}(1) = 1/k$.

For the asymptotic distribution of U_{k+1}, the proof of the following theorem follows easily from the general properties of U-statistics; see, for example, Puri & Sen (1971, p. 62).

THEOREM 2·1. *The asymptotic distribution of $n^{\frac{1}{2}}\{U_{k+1} - E(U_{k+1})\}$ as $n \to \infty$, is normal with mean zero and variance $\sigma_{k+1}^2 = (k+1)^2 \xi_1$, where*

$$\xi_1 = E\{\psi_1^2(X_1, Y_1)\} - E^2(U_{k+1}), \quad \psi_1(x_1, y_1) = E[\phi_{k+1}\{(x_1, y_1), \ldots, (X_{k+1}, Y_{k+1})\}].$$

Under H_0, $E(U_{k+1}) = (k+1)^{-1}$ and $\sigma_{k+1}^2 = k^2(2k+1)^{-2}$.

3. Asymptotic relative efficiencies and empirical power comparisons

The U_{k+1} tests are designed for testing independence against positive quadrant dependence for skewed distributions. For efficiency and power comparisons, we consider the absolutely continuous bivariate exponential distribution of Block & Basu (1974) with survival function

$$\bar{F}_\theta(x, y) = \tfrac{1}{2}(2+\theta)\exp[-\{x + y + \theta \max(x, y)\}] - \tfrac{1}{2}\theta \exp\{-(\theta+2)\max(x, y)\}.$$

This distribution is strictly positive quadrant dependent when $\theta > 0$ and (X, Y) are independent when $\theta = 0$. The asymptotic relative efficiencies of U_3, U_4 and U_5 with respect to the Kendall-tau test U_2 for this distribution are 1·1728, 1·1776 and 1·1232 respectively.

Monte Carlo studies have been made to compare the powers of the tests U_2, U_3 and U_4 for sample sizes 8 and 12 for the absolutely continuous bivariate exponential distribution with various θ values at 5% level of significance. Four thousand random samples from the absolutely continuous bivariate exponential distribution were generated in each case. These results are given in Table 1. The last column in this table gives the estimated critical points. It is clear from the table that the U_3 and U_4 tests are good competitors of the Kendall test U_2 even for small sample sizes.

It is easy to show that the proposed tests are distribution-free, unbiased and consistent for testing H_0 against H_A.

Table 1. *Estimates of power of the* U_{k+1} *tests based on* 4000 *repetitions for the absolutely continuous bivariate exponential distribution*

	Tests	$\theta = 1\cdot0$	$\theta = 5\cdot0$	$\theta = 10\cdot0$	5% critical point
$n = 8$	U_2	0·06725	0·12100	0·14475	0·75000
	U_3	0·09450	0·16075	0·18500	0·66071
	U_4	0·0950	0·15825	0·18200	0·64286
$n = 12$	U_2	0·10900	0·21200	0·24400	0·68182
	U_3	0·11575	0·21925	0·26450	0·57727
	U_4	0·1185	0·22225	0·25875	0·53535

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