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A new distribution-free test for the equality of two failure rates

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SUMMARY

Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ be two independent random samples from two life distributions with absolutely continuous distribution functions $F(x)$ and $G(x)$, respectively. Let $r_F(x)$ and $r_G(x)$ be their respective failure rates. A new distribution-free test for testing H_0: $r_F(x) = r_G(x)$ against H_A: $r_F(x) \leqslant r_G(x)$ is proposed. The properties of the test are investigated; it possesses high efficiency for many specific alternatives belonging to H_A.

Some key words: Asymptotic relative efficiency; Failure rate; Lehmann-type alternative.

1. INTRODUCTION

In this paper we consider the comparison of the lifetimes of two systems. Let $\mathscr{F}$ be the class of all absolutely continuous distribution functions H with $H(x) = 0$ for $x \leqslant 0$. Let X and Y be random variables denoting the lifetimes of two systems with distribution functions $F(x)$ and $G(x)$, respectively, both belonging to $\mathscr{F}$. Let $\bar{F} = 1-F$ and $\bar{G} = 1-G$ be their survival functions, and $r_F(t)$ and $r_G(t)$ the corresponding failure or hazard rates. We consider testing the null hypothesis

$$H_0\colon r_F(t) = r_G(t) \tag{1·1}$$

against the alternative

$$H_A\colon r_F(t) \leqslant r_G(t) \quad (t \geqslant 0) \tag{1·2}$$

with strict inequality over a set of nonzero probability.

This problem has been considered by Chikkagoudar & Shuster (1974) and Kochar (1979). The importance and various implications of such alternatives have been discussed in the latter paper. Whereas Chikkagoudar & Shuster have provided locally most powerful rank tests for some specific Lehmann-type alternatives belonging to H_A, Kochar (1979) proposed a test based on a U-statistic for testing H_0 against H_A.

In § 2, we propose a new distribution-free test for testing H_0 against H_A and discuss its distribution. In § 3 some specific alternatives belonging to H_A are considered and the asymptotic relative efficiency found with respect to relevant locally most powerful rank tests. An alternative belonging to H_A has been identified for which the proposed test is asymptotically optimal.

2. THE PROPOSED TEST AND ITS DISTRIBUTION

Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ be independent random samples from the distributions F and G, respectively. On the basis of these samples we want to test H_0 against H_A.

It can be shown that H_A holds if and only if $\bar{F}(x)/\bar{G}(x)$ is nondecreasing in x for those $x \geqslant 0$ such that $\bar{F}(x)$ and $\bar{G}(x)$ are both greater than zero; that is, H_A is true if and only if, for $s \geqslant t \geqslant 0$,

$$\delta(s,t) = \frac{\bar{F}(s)}{\bar{F}(t)} - \frac{\bar{G}(s)}{\bar{G}(t)} \geqslant 0. \tag{2·1}$$

Taking $t = 0$ in (2·1), we find that H_A implies the stochastic ordering alternative $\bar{F}(s) \geqslant \bar{G}(s)$ for every $s \geqslant 0$.

Define as a measure of deviation between F and G

$$\begin{aligned}\Delta(F,G) &= E\{\delta(X,Y)\,|\,X \geqslant Y\}\\ &= \iint_{s \geqslant t} \delta(s,t)\,dF(s)\,dG(t)\\ &= \iint_{s \geqslant t} \{\bar{F}(s)/\bar{F}(t)\}\,dF(s)\,dG(t) - \iint_{s \geqslant t} \{\bar{G}(s)/\bar{G}(t)\}\,dF(s)\,dG(t)\\ &= \tfrac{1}{2} + \int_0^\infty F(x)\,\{\tfrac{1}{2} + \log \bar{G}(x)\}\,dG(x). \end{aligned} \tag{2·2}$$

Under H_0, $\Delta(F,G) = 0$ but under H_A, $\Delta(F,G) > 0$.

Let F_n and G_m be the empirical distribution functions based on the given random samples $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$, respectively. We base our test on the statistic

$$S = \int_0^\infty F_n(x)\left[\tfrac{1}{2} + \log\left\{1 - \frac{m}{m+1} G_m(x)\right\}\right] dG_m(x).$$

The test is to reject H_0 for large values of S.

Let $Y_{(1)}, \ldots, Y_{(m)}$ be the order statistics of the Y sample and let $R_{(j)}$ denote the rank of $Y_{(j)}$ in the combined increasing arrangement of X's and Y's. Then S can also be expressed in the form

$$S = \frac{1}{m}\sum_{j=1}^m \left(\frac{R_{(j)} - j}{n}\right)\left\{\tfrac{1}{2} + \log\left(1 - \frac{j}{m+1}\right)\right\} = \frac{1}{nm}\left(\sum_{j=1}^m a_j R_{(j)} - \sum_{j=1}^m j a_j\right), \tag{2·3}$$

where $a_j = \frac{1}{2} + \log\{1 - j/(m+1)\}$.

Statistics of the above type have been considered by Sen (1964), Sen & Govindarajulu (1966), Govindarajulu (1966) and Deshpandé (1972). Deshpandé calls them linear ordered rank statistics.

Let $N = n+m$. It follows from the results of Deshpandé (1972) that under H_0

$$E(S) = \frac{1}{m(m+1)}\sum_{j=1}^m j a_j, \quad \operatorname{var}(S) = \frac{(N+1)}{n(m+2)\,m^2(m+1)^2} a'\Omega a,$$

where the matrix $\Omega = ((\omega_{ij}))$ $(i,j = 1, \ldots, m)$, has $\omega_{ij} = i(m+1-j) = \omega_{ji}$ for $i \leqslant j$ and $a' = (a_1, \ldots, a_m)$.

Govindarajulu (1966) has studied the asymptotic distribution of statistics of the type

$$\tau = \int_{-\infty}^\infty F_n(x)\,J_m\left\{\frac{m}{m+1} G_m(x)\right\} dG_m(x).$$

Our test statistic S falls in this class and in our case the J_m function is

$$J_m\left\{\frac{m}{m+1}G_m(x)\right\} = \tfrac{1}{2}+\log\left\{1-\frac{m}{m+1}G_m(x)\right\},$$

the limiting case being $J(u) = \frac{1}{2}+\log(1-u)$. It follows from his results that the limiting distribution of $\sigma^{-1}(S-\mu)$ is standard normal, where

$$\mu = \int_{-\infty}^{\infty} F(x)\,J\{G(x)\}\,dG(x),$$

$$m\sigma^2 = \frac{2}{\lambda_m}\iint_{x<y} F(x)\,\bar{F}(y)\,J\{G(x)\}\,J\{G(y)\}\,dG(x)\,dG(y)$$

$$+2\iint_{x<y} G(x)\,\bar{G}(y)\,J\{G(x)\}\,J\{G(y)\}\,dF(x)\,dF(y),$$

provided that $\sigma \neq 0$ and $\lambda_m = n/m$ is such that $0 < \lambda_0 \leqslant \lambda_m$.

Under H_0

$$\mu = \int_0^1 u\{\tfrac{1}{2}+\log(1-u)\}\,du = -\tfrac{1}{2}, \qquad m\sigma^2 = \frac{N}{54n}.$$

3. Asymptotic efficiencies

To compare the Pitman asymptotic relative efficiencies we parameterize the problem as follows. Let $F(x) = F_0(x)$ and $G(x) = F_\theta(x)$, where θ is a positive real number such that $\bar{F}_0(x)/\bar{F}_\theta(x)$ is nondecreasing in x for those $x \geqslant 0$ for which $\bar{F}_0(x)$ and $\bar{F}_\theta(x)$ are both greater than zero.

We study the Pitman asymptotic relative efficiency of the S test relative to the tests of Savage (1956), Wilcoxon (1945) and the W test of Kochar (1979). The following alternatives are considered for efficiency comparisons:

$$H_1\colon \bar{F}_\theta(x) = \{\bar{F}(x)\}^{1+\theta},$$

$$H_2\colon \bar{F}_\theta(x) = \bar{F}(x)\left[1-\theta\left\{\sum_{i=1}^{k} F^i(x)\right\}\right] \quad (k \geqslant 1, 0 < \theta < 1/k),$$

$$H_3\colon \bar{F}_\theta(x) = (1-\theta)\,\bar{F}(x)+\theta\bar{F}(x)\,\{1-F^k(x)\} \quad (k > \tfrac{1}{2}),$$

$$H_4\colon \bar{F}_\theta(x) = \exp[-\{x+\theta(x+e^{-x}-1)\}],$$

$$H_5\colon \bar{F}_\theta(x) = \bar{F}(x)\exp[-\tfrac{1}{2}\theta\{\log\bar{F}(x)\}^2],$$

$$H_6\colon \bar{F}_\theta(x) = (1-\theta)\,\bar{F}(x)+\theta\bar{F}^2(x)\,\{1-\log\bar{F}(x)\} \quad (0 \leqslant \theta \leqslant 1).$$

All these alternatives belong to H_A since $\bar{F}_\theta(x)/\bar{F}_0(x)$ is nonincreasing in x in each case. The first five alternatives were considered also by Kochar (1979). Locally most powerful rank tests can be obtained for testing H_0 against the above alternatives (Hájek & Šidák, 1967, Chapter 2; Chikkagoudar & Shuster, 1974).

Table 1 gives the Pitman asymptotic relative efficiencies of the Wilcoxon test, the Savage test, the W test earlier proposed by the author and the S test with respect to the

Table 1. *Pitman asymptotic relative efficiencies with respect to the locally most powerful rank tests*

	Alternative hypothesis	Wilcoxon	Savage	W	S
H_1		0·75	1	0·8203	0·8438
H_2	$k = 1$	1	0·75	0·70	0·5
	$k = 2$	0·4166	0·8681	0·8933	0·732
	$k = 3$	0·2593	0·9128	0·9481	0·8629
	$k = 4$	0·1458	0·9264	0·9417	0·9302
H_3	$k = 1$	1	0·75	0·70	0·5
	$k = 2$	0·625	0·8333	0·9843	0·957
	$k = 3$	0·35	0·7292	0·7813	0·9663
H_4		0·25	0·75	0·5353	0·7812
H_5		0·0938	0·5	0·2307	0·4219
H_6		0·5	0·8438	0·896	1

corresponding locally most powerful rank tests for the above alternatives. The S test is asymptotically optimal in the sense of Pitman efficiency for testing H_0 against H_6. No test is uniformly best against the given set of alternatives. The Wilcoxon test has minimum efficiency in all cases except the Lehmann alternative defined by H_2 with $k = 1$ or H_3 with $k = 1$. Therefore it is not recommended for testing H_0 against H_A.

The newly proposed S test performs better than the W test in all cases except H_3 with $k = 1$ and 2, and H_2. All these alternatives except H_4 are Lehmann-type alternatives. Therefore the efficiencies of these tests are independent of the underlying distribution F for these alternatives. The S test gives less emphasis to extremely large observations than does the W test. The table shows that the S test is more efficient than the W test over a broad spectrum of alternatives belonging to H_A.

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