Biometrika (1979), **66**, 3, *pp*. 437-41
Printed in Great Britain

Distribution-free comparison of two probability distributions with reference to their hazard rates

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SUMMARY

Let $F(x)$ and $G(x)$ be the absolutely continuous distribution functions of two life distributions with $r_F(x)$ and $r_G(x)$ as their respective hazard rates. In the class of increasing failure rate distributions, one-sided scale as well as one-sided location alternatives imply that one hazard rate is uniformly smaller than the other. A new distribution-free test for testing $H_0: r_F(x) = r_G(x)$ against $H_A: r_F(x) \leqslant r_G(x)$ has been proposed. The test is seen to possess robust asymptotic efficiency properties.

Some key words: Asymptotic relative efficiency; Increasing failure rate; Location-scale alternative; U-statistic.

1. INTRODUCTION

The lifetimes of physical, biological and many other systems, that is the times for which they perform their defined purpose adequately, are random variables. By ageing we mean the phenomenon whereby an older system has a shorter remaining lifetime, in some statistical sense, than a newer or a younger one. This concept of ageing has been considered in many aspects by Bryson & Siddiqui (1969), among others.

In the present paper we consider the problem of comparing the lifetimes of two systems. Let $\mathscr{F}$ be the class of all absolutely continuous distribution functions H with $H(x) = 0$ for $x \leqslant 0$. Let X and Y be random variables denoting the lifetimes of the two systems with distribution functions $F(x)$ and $G(x)$, respectively, both belonging to $\mathscr{F}$. Let f and g be their probability density functions; and $\bar{F} = 1-F$ and $\bar{G} = 1-G$ the corresponding survival functions. Let $r_F(t)$ and $r_G(t)$ be the hazard or failure rates of the two systems defined by $r_F(t) = f(t)/\bar{F}(t)$ and $r_G(t) = g(t)/\bar{G}(t)$, whenever $\bar{F}(t) > 0$ and $\bar{G}(t) > 0$, respectively. We consider the problem of testing the null hypothesis

$$H_0: r_F(t) = r_G(t)$$

or, equivalently

$$F(t) = G(t) \tag{1·1}$$

against the alternative

$$H_A: r_F(t) \leqslant r_G(t) \quad (t \geqslant 0) \tag{1·2}$$

with strict inequality over a set of nonzero probability.

Chikkagoudar & Shuster (1974) have considered this problem of testing H_0 against H_A. They have provided the locally most powerful rank tests for some specific Lehmann type alternatives belonging to H_A.

The alternative H_A may appear to be too restrictive since one hazard rate is required to be uniformly smaller than the other. But this is not the case. In the following theorem we show that for increasing failure rate distributions, a one-sided location-scale alternative implies that one hazard rate is uniformly smaller than the other.

THEOREM 1·1. *Let F belong to $\mathscr{F}$ and have increasing failure rate. If $G(x) = F(\sigma x+\theta)$ for $\sigma \geqslant 1$, $\theta \geqslant 0$, then $r_F(x) \leqslant r_G(x)$ for every x.*

Proof. We have that

$$\begin{aligned} r_G(x) = g(x)/\bar{G}(x) &= \sigma f(\sigma x+\theta)/\bar{F}(\sigma x+\theta) \\ &= \sigma r_F(\sigma x+\theta) \\ &\geqslant r_F(\sigma x+\theta) \\ &\geqslant r_F(x) \end{aligned}$$

because $\sigma \geqslant 1$ and $r_F(x)$ is nondecreasing.

It can be seen that H_A holds if and only if $\bar{F}(t)/\bar{G}(t)$ is nondecreasing in t for those $t \geqslant 0$ such that $\bar{F}(t)$ and $\bar{G}(t)$ are both greater than zero, that is, H_A holds if and only if, for $s \geqslant t \geqslant 0$,

$$\delta(s,t) = \bar{F}(s)\,\bar{G}(t) - \bar{F}(t)\,\bar{G}(s) \geqslant 0 \tag{1·3}$$

with strict inequality over a set of nonzero probability. Taking $t = 0$ in (1·3), we see that $r_F(s) \leqslant r_G(s)$ for $s \geqslant 0$ implies that $F(s) \leqslant G(s)$ for $s \geqslant 0$. Thus H_A is a subhypothesis of the more general slippage alternative H_B: $F(x) \leqslant G(x)$ for $x \geqslant 0$ and with strict inequality over a set of nonzero probability.

In § 2, we propose a distribution-free test for testing H_0 against H_A based on a generalized U-statistic and discuss its distribution. In § 3, some specific alternatives belonging to H_A are considered and comparisons of Pitman asymptotic relative efficiency are made. It is shown that the proposed test is a good competitor to both the Savage and the Wilcoxon tests.

2. The proposed test and its distribution

Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ be independent random samples from the two distributions F and G, respectively. On the basis of these samples, we want to test H_0 against H_A. We have seen that H_A is equivalent to: $\delta(s,t) \geqslant 0$ for $s \geqslant t \geqslant 0$. Define

$$\begin{aligned} \eta(F,G) &= E[\delta\{\max(X,Y), \min(X,Y)\}] \\ &= \iint_{0\leqslant y\leqslant x} \delta(x,y)\,\{dF(x)\,dG(y) + dF(y)\,dG(x)\} \\ &= \mathrm{pr}\,(Y_1 \leqslant Y_2 \leqslant X_1 \leqslant X_2) + \mathrm{pr}\,(X_1 \leqslant Y_1 \leqslant Y_2 \leqslant X_2) \\ &\qquad - \mathrm{pr}\,(X_1 \leqslant X_2 \leqslant Y_1 \leqslant Y_2) - \mathrm{pr}\,(Y_1 \leqslant X_1 \leqslant X_2 \leqslant Y_2), \end{aligned} \tag{2·1}$$

where X_1 and X_2 are independent observations from F and Y_1 and Y_2 are two independent observations from G. Also the X's are independent of the Y's.

Under H_0, $\eta(F,G) = 0$ but under H_A, $\eta(F,G) > 0$. The quantity $\eta(F,G)$ can be taken as a measure of deviation between the distributions F and G, in the failure rate sense.

Let F_n and G_m be the empirical distribution functions based on the random samples $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$, respectively. Then $\eta(F_n, G_m)$ is a possible test statistic. We, however, consider a generalized U-statistic W which is asymptotically equivalent to $\eta(F_n, G_m)$. A U-statistic is a minimum variance unbiased estimator of its expectation in the class of all absolutely continuous distributions (Puri & Sen, 1971, p. 55). Below we construct a U-statistic with expectation $4\eta(F,G)$ to be used as a test statistic in this problem.

Let

$$\phi(x_1, x_2; y_1, y_2) = \begin{cases} 1 & \text{for } yyxx \text{ or } xyyx, \\ 0 & \text{for } xyxy \text{ or } yxyx, \\ -1 & \text{for } xxyy \text{ or } yxxy. \end{cases} \tag{2·2}$$

The arrangement $yyxx$ represents

$$\{y_1 \leqslant y_2 \leqslant x_1 \leqslant x_2\} \cup \{y_2 \leqslant y_1 \leqslant x_1 \leqslant x_2\} \cup \{y_1 \leqslant y_2 \leqslant x_2 \leqslant x_1\} \cup \{y_2 \leqslant y_1 \leqslant x_2 \leqslant x_1\}$$

and similarly we interpret the other arrangements of x's and y's. Then the U-statistic W is defined by

$$W = \left\{\binom{n}{2}\binom{m}{2}\right\}^{-1} \Sigma\, \phi(X_{i_1}, X_{i_2}; Y_{j_1}, Y_{j_2}), \tag{2·3}$$

where the sum is over $1 \leqslant i_1 < i_2 \leqslant n$, $1 \leqslant j_1 < j_2 \leqslant m$.

The test procedure is to reject the null hypothesis H_0 in favour of H_A if the value of the statistic W is significantly large.

Now $E(W) = 4\eta(F, G)$ and

$$\operatorname{var}(W) = \left\{\binom{n}{2}\binom{m}{2}\right\}^{-1} \sum_{c=0}^{2} \sum_{d=0}^{2} \binom{2}{c}\binom{n-2}{2-c}\binom{2}{d}\binom{m-2}{2-d} \zeta_{c,d}. \tag{2·4}$$

Here

$$\zeta_{c,d} = \operatorname{cov}\{\phi(X_1, X_2; Y_1, Y_2), \phi(X_3, X_4; Y_3, Y_4)\},$$

where c of the X's and d of the Y's are common in the two terms in the covariance $c, d = 0, 1, 2$.

It can be seen that under H_0, $E(W) = 0$ and

$$\zeta_{10} = \zeta_{01} = \tfrac{8}{105}, \quad \zeta_{11} = \tfrac{11}{60}, \quad \zeta_{12} = \zeta_{21} = \tfrac{11}{30}, \quad \zeta_{20} = \zeta_{02} = \tfrac{1}{5}, \quad \zeta_{22} = \tfrac{2}{3}.$$

Substituting these values in (2·4), we find that the variance of W, under H_0, is

$$\operatorname{var}(W) = \left\{210\binom{n}{2}\binom{m}{2}\right\}^{-1} \{16nm(n+m) - (11m^2 + 11n^2 + 6mn) - 3(m+n) + 8\}.$$

Since the kernel ϕ is square integrable, the proof of the following theorem follows from the well-known properties of generalized U-statistics (Lehmann, 1951; Puri & Sen, 1971, p. 62).

THEOREM 2·1. *Let $N = n+m$. The asymptotic distribution of $N^{\frac{1}{2}}(W - 4\eta)$ as $N \to \infty$ in such a way that $p_N = n/N$ tends to p, $0 < p < 1$, is normal with mean zero and variance σ^2 given by*

$$\sigma^2 = 4p^{-1}\zeta_{10} + 4q^{-1}\zeta_{01}.$$

Under H_0, $\eta = 0$ and $\sigma^2 = 32/(105pq)$.

For large sample sizes the distribution of the standardized version of the statistic W may be approximated by the standard normal distribution. The small sample null distribution of the test statistic W may be obtained by enumeration.

3. Asymptotic relative efficiencies

To compare the asymptotic efficiencies we parameterize the problem in the following way. Let $F(x) = F_0(x)$ and $G(x) = F_\theta(x)$, where θ is a positive real number such that $F_0(x) \leqslant F_\theta(x)$ for all $x \geqslant 0$ and with strict inequality over a set of nonzero probability for every $\theta > 0$.

We study the Pitman asymptotic relative efficiency of the W test relative to the Savage test (1956), the Wilcoxon test (1945) and the locally most powerful rank tests for the following alternatives belonging to H_A: H_1: $r_{F_\theta}(x) = (\theta+1)\, r_F(x)$, i.e. $\bar{F}_\theta(x) = \{\bar{F}(x)\}^{1+\theta}$.

Now $\bar{F}(x) = e^{-x}$ $(x>0)$, the exponential survival function, leads to $\bar{F}_\theta(x) = e^{-(\theta+1)x}$ which is the scale alternative in the exponential case. We consider the following

$$H_2\colon \bar{F}_\theta(x) = \bar{F}(x)\left[1-\theta\left\{\sum_{i=1}^{k} F^i(x)\right\}\right],$$

i.e. for $k \geqslant 1$, $0<\theta<1/k$

$$F_\theta(x) = F(x)+\theta F(x)\{1-F^k(x)\};$$

$$H_3\colon \bar{F}_\theta(x) = (1-\theta)\,\bar{F}(x)+\theta\bar{F}(x)\{1-F^k(x)\},$$

i.e. for $k>\frac{1}{2}$

$$F_\theta(x) = F(x)+\theta F^k(x)\{1-F(x)\};$$

$$H_4\colon r_{F_\theta}(x) = 1+\theta(1-e^{-x}),$$

i.e. the Makeham distribution

$$F_\theta(x) = 1-\exp[-\{x+\theta(x+e^{-x}-1)\}];$$

$$H_5\colon r_{F_\theta}(x) = r_F(x)\{1+\theta\log\bar{F}(x)\},$$

i.e.

$$\bar{F}_\theta(x) = \bar{F}(x)\exp[-\tfrac{1}{2}\theta\{\log\bar{F}(x)\}^2].$$

Now $\bar{F}(x) = e^{-\lambda x}$ gives $r_{F_\theta}(x) = \lambda+\lambda^2\theta x$, the linearly increasing hazard rate.

The alternatives H_1, H_3 and H_5 have been considered by Chikkagoudar & Shuster (1974). They have obtained the locally most powerful rank tests for these alternatives. The locally most powerful rank tests for H_2 and H_4 can be obtained similarly.

Table 1 gives the Pitman asymptotic relative efficiency of the Wilcoxon test, the Savage test and the W test with respect to the corresponding locally most powerful rank tests for the above five alternatives. The asymptotic relative efficiency of the W test usually lies in between those of the Wilcoxon and Savage tests. For H_2 and H_3 the asymptotic relative efficiency results are rather diverse, the W test being less efficient than the other two for $k=1$ and being more efficient than both for $k = 2$, 3 and 4. Thus the W test is fairly efficiency-robust.

Table 1. *Pitman asymptotic relative efficiencies with respect to the locally most powerful rank tests*

Alternative hypotheses		Wilcoxon	Savage	W
H_1		0·75	1	0·8203
H_2	$k=1$	1	0·75	0·70
	$k=2$	0·4166	0·8681	0·8933
	$k=3$	0·2593	0·9128	0·9481
	$k=4$	0·1458	0·9264	0·9417
H_3	$k=1$	1	0·75	0·70
	$k=2$	0·625	0·8333	0·9843
	$k=3$	0·35	0·7292	0·7813
H_4		0·25	0·75	0·5353
H_5		0·0938	0·5	0·2307

The consistency of the W test for the alternative H_A follows from Theorem 2·1 and the fact that its expectation under H_A is greater than its expectation under H_0.

The author is indebted to Dr Jayant V. Deshpandé for his guidance and encouragement throughout this work. Thanks are due to the referee for suggesting modifications to the paper and a simpler proof of Theorem 1·1.

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[*Received February* 1979. *Revised May* 1979]