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TESTING EXPONENTIALITY AGAINST IDMRL DISTRIBUTIONS WITH UNKNOWN CHANGE POINT

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Guess, Hollander and Proschan proposed tests for exponentiality versus IDMRL (increasing initially and then decreasing mean residual life) distributions when the change point, or corresponding quantile, is known. In this paper we propose two tests which do not require such knowledge of the change point. The tests are based on estimates of functionals of the cdf which discriminate between the exponential and IDMRL families.

1. Introduction and summary. Let $\mathscr{F}$ denote the set of absolutely continuous cdf's F on $\mathbb{R}$ such that $F(0) = 0$, $\int_0^\infty x\bar{F}(x)\,dx < \infty$ and $\bar{F}(t) > 0$ for all $t \geq 0$, where $\bar{F}(x) = 1 - F(x)$. Distributions in $\mathscr{F}$ are called life distributions.

For each $F \in \mathscr{F}$, the mean residual life (MRL) function $m_F(t) = E_F(X - t|X \geq t) = \{\bar{F}(t)\}^{-1}\int_t^\infty \bar{F}(x)\,dx$ is defined and finite for each $t \geq 0$. Each $F \in \mathscr{F}$ is uniquely determined by m_F, via the relation

$$\bar{F}(x) = m_F(0)\{m_F(x)\}^{-1}\exp\left\{-\int_0^x [m_F(u)]^{-1}\,du\right\}, \qquad x \geq 0.$$

Theoretical properties of the MRL function are given in Cox (1962), Kotz and Shambhag (1980), Hall and Wellner (1981) and Bhattacharjee (1982). Applications of it are surveyed in Guess and Proschan (1988), where it is seen that various families of life distributions defined in terms of the MRL (e.g., increasing MRL, decreasing MRL) have been used as models for lifetimes for which such prior information is available.

One such family of distributions is $\mathscr{G} = \{F \in \mathscr{F}$: there exists a unique $t^* > 0$ such that $m_F(t)$ is strictly increasing (decreasing) for $t < t^*(t > t^*)\}$, the so-called IDMRL (increasing initially then decreasing mean residual life) family. t^* is called the change point [see Guess, Hollander and Proschan (1986), henceforth GHP]. IDMRL distributions model lifetimes in which, in terms of residual life, aging initially is beneficial but eventually is detrimental. Such lifetimes are exemplified by: (i) human lifetimes: High infant mortality causes the initially increasing MRL and deterioration with advancing age causes the subsequently decreasing MRL. (ii) Employment time with a given company: The remaining employment time (residual life) of an employee with several years with a company is likely (due to time investment, career value,

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etc.) to exceed that of an employee with the company only several months. This results in increasing MRL with years of employment up to a certain point (t^*), after which, due to retirement, MRL decreases. See Guess and Proschan (1988) and the references therein for further applications of the IDMRL family.

Let $\mathscr{E}$ denote the class of exponential distributions [i.e., $F \in \mathscr{E}$ means that $F'(x) = \lambda e^{-\lambda x} I(x \geq 0)$ for some $\lambda > 0$]. Then $\mathscr{E} \subset \mathscr{F}$ and $m_F(t)$ is constant for all $t \geq 0$ if and only if $F \in \mathscr{E}$. Due to this "no-aging" property of $F \in \mathscr{E}$, it is of practical interest to know whether a given life distribution F is in $\mathscr{E}$. Alternatively, one may ask if $F \in \mathscr{G}$.

Therefore, in this paper we consider the problem of testing H_0: $F \in \mathscr{E}$ versus H_1: $F \in \mathscr{G}$, based on a random sample $X_1, \dots, X_n$ from $F \in \mathscr{F}$ (F unknown). This problem was considered by GHP, who obtained tests assuming either (a) t^* is known or (b) $\rho = F(t^*)$ is known. In practice, however, such information is usually lacking, as was noted by GHP, who left as an open problem that of devising a test not requiring (a) or (b). In this paper, we address this open problem by devising two tests which do not require these assumptions. (However, our tests require more restrictions on F than do GHP's tests, in particular that $E_F X^2 < \infty$.) We propose two new tests for purposes of power comparison, since presently we do not know how to go about finding optimal tests for these hypotheses.

Like the tests of GHP, our tests are based on estimates of functionals which distinguish $F \in \mathscr{E}$ from $F \in \mathscr{G}$. These functionals are, for $F \in \mathscr{F}$ and tests 1 and 2, respectively,

$$\phi_1(F) = \sup\{\psi_t^{(1)}(F) : 0 \leq t \leq F^{-1}(1 - \varepsilon)\},$$

$$\phi_2(F) = \sup\{\psi_t^{(2)}(F) : t \geq 0\},$$

where $\varepsilon > 0$ is a small fixed number,

$$\psi_t^{(1)}(F) = m_F(t) - m_F(0),$$

$$\psi_t^{(2)}(F) = \int_0^t \{m_F(s) f(s) - \bar{F}(s)\}\bar{F}(s)\, ds - \int_t^\infty \{m_F(s) f(s) - \bar{F}(s)\}\bar{F}(s)\, ds$$

and $f(s) = F'(s)$. The functional $\phi_1(F)$ is clearly 0 for $F \in \mathscr{E}$ and strictly positive for $F \in \mathscr{G}$. Although not so obviously, $\phi_2(F)$ has the same properties and may also be written in terms of F only (see Theorem 1). The functional ϕ_1 is the more natural of the two, but requires the somewhat arbitrary choice of ε. The functional ϕ_2 avoids this problem.

Our test statistics are appropriately normalized versions of $\phi_i(F_n)$, where $F_n(x) = n^{-1} \sum_{i=1}^n I(x_i \leq x)$ is the empirical cdf. Using statistical differentials, we show (Theorem 2) that these statistics have limit distributions which coincide with the distributions of the suprema of certain Gaussian processes. Using these limit results, critical values are obtained. Monte Carlo power comparisons of our tests with those of GHP indicate that test 2 generally dominates test 1 and compares well with the GHP tests when t^* occurs below the 75th quantile of F. When t^* exceeds the 75th quantile, neither test 1 nor test 2 clearly dominates the other, and neither compares well with GHP's tests. Both our tests and GHP's tests apparently lose power as t^* increases into the right tail of F.

The rest of the paper is organized as follows. In Section 2 the test statistics and their limiting null distributions are given. Section 3 contains the power comparisons. Section 4 contains the derivations of the asymptotic null distributions of the statistics.

2. The test statistics and their limiting null distributions. We first motivate the functional $\phi_2(F)$ via the following result, parts (i) and (ii) of which show that ϕ_2 distinguishes $F \in \mathscr{E}$ from $F \in \mathscr{G}$ like ϕ_1 does.

THEOREM 1.

(i) *If* $F \in \mathscr{E}$, *then* $\phi_2(F) = 0$.
(ii) *If* $F \in \mathscr{G}$, *then* $\psi_t^{(2)}(F)$ *is strictly increasing (decreasing) for* $t < t^*(t > t^*)$ *and* $\phi_2(F) = \psi_{t^*}^{(2)}(F) > 0$.
(iii) $\psi_t^{(2)}(F) = \int_0^\infty \overline{F} - 2\int_0^\infty \overline{F}^2 - 2\overline{F}(t)\int_t^\infty \overline{F} + 4\int_t^\infty \overline{F}^2$.

PROOF. Since $m_F(t)\overline{F}(t) = \int_t^\infty \overline{F}(x)\,dx$, one may check that $m_F(t)$ is differentiable in t where $\overline{F}(t)$ is in $(0, \infty)$ with

$$m'_F(t) = \frac{m_F(t)\,f(t) - \overline{F}(t)}{\overline{F}(t)}, \qquad t > 0.$$

Further, $\phi_t^{(2)}(F)$ is differentiable in $t > 0$ and

$$\frac{d}{dt}\psi_t^{(2)}(F) = 2\{m_F(t)\,f(t) - \overline{F}(t)\}\overline{F}(t) = 2\overline{F}^2(t)\,m'_F(t)$$

clearly has the same sign as does $m'_F(t)$. Thus, since $F \in \mathscr{G}$ implies that $m'_F(t) > 0$, $m'_F = 0$ or $m'_F < 0$ as $t < t^*$, $t = t^*$ or $t > t^*$, the same holds for $\psi_t^{(2)}(F)$. This gives (ii). Result (i) holds since $m_F(t)$ is constant for $F \in \mathscr{E}$, so that $m'_F(t) = 0$ for all $t > 0$ and hence the integrand of $\psi_t^{(2)}(F)$ is zero for all $t > 0$. Result (iii) holds by a straightforward calculation. □

Now let $\overline{X}_n$ denote the sample mean. Then our test statistics are

$$T_n^{(1)} = n^{1/2}\overline{X}_n^{-1}\phi_1(F_n), \tag{2.1}$$

$$T_n^{(2)} = n^{1/2}\overline{X}_n^{-1}\phi_2(F_n). \tag{2.2}$$

Since it may be shown that for $F \in \mathscr{F}$,

$$\sup\{|\psi_t^{(1)}(F_n) - \psi_t^{(1)}(F)| : 0 \le t \le F^{-1}(1 - \varepsilon)\} = O_p(n^{-1/2}) \tag{2.3}$$

and

$$\sup\{|\psi_t^{(2)}(F_n) - \psi_t^{(2)}(F)| : t \ge 0\} = O_p(n^{-1/2}), \tag{2.4}$$

each of $T_n^{(1)}$ and $T_n^{(2)}$ will be near zero under H_0 and large positive under H_1, making large values of these statistics significant for testing H_0 versus H_1.

The following computational formulae are easily derived, where $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ denote the order statistics ($X_{(0)} \triangleq 0$) and $D_j = X_{(j+1)} - X_{(j)}$:

$$T_n^{(1)} = n^{1/2}\overline{X}_n^{-1} \max_{0 \le k \le n^*} \left\{ \frac{n}{n-k} \sum_{j=k}^{n-1} \left(1 - \frac{j}{n}\right) D_j - \overline{X}_n \right\},$$

$$T_n^{(2)} = n^{1/2}\overline{X}_n^{-1} \max_{0 \le k \le n} \xi_{nk},$$

where

$$\xi_{nk} = A_n - 2\left(1 - \frac{k}{n}\right) \sum_{j=k}^{n-1} \left(1 - \frac{j}{n}\right) D_j + 4 \sum_{j=k}^{n-1} \left(1 - \frac{j}{n}\right)^2 D_j,$$

$$A_n = -X_{(1)} + \sum_{j=1}^{n-1} c_{nj} D_j, \qquad c_{nj} = 1 - \frac{j}{n} - 2\left(1 - \frac{j}{n}\right)^2,$$

$$n^* = [n(1-\varepsilon)].$$

(Here $[s]$ denotes integer part of s.) The quantities ξ_{nk} can be computed recursively in k. A FORTRAN program for computation of these statistics is available from the first author.

Both of these statistics are distribution-free over $\mathscr{E}$ since the maximands are ratios of linear functions of order statistics. (This is the motivation for the $\overline{X}_n^{-1}$ factor in the definition of each statistic.) Also, it is interesting to observe that the statistic $\psi_0^{(2)}(F_n)/\overline{X}_n$ is asymptotically equivalent to the celebrated cumulative total-time-on-test statistic, which is asymptotically optimal for testing $F \in \mathscr{E}$ versus F having the Makeham form

$$\overline{F}_\theta(x) = \exp\{-[x + \theta(x + e^{-x} - 1)]\} I(x \ge 0), \qquad \theta \ge 0;$$

see Klefsjö (1983), Kochar (1985), Hollander and Proschan (1984) and Barlow and Doksum (1972).

The asymptotic null distributions of $T_n^{(1)}$ and $T_n^{(2)}$ are given in Theorem 2, which is proved in Section 4. In this direction, let $Z_{1\varepsilon} = \{Z_{1\varepsilon}(p): 0 \le p \le 1 - \varepsilon\}$ denote a mean-zero Gaussian process with covariance $E\{Z_{1\varepsilon}(p)Z_{1\varepsilon}(q)\} = p/(1-p)$ for $p \le q$ and let $Z_2 = \{Z_2(p): 0 \le p \le 1\}$ denote another mean-zero Gaussian process with covariance $E\{Z_2(p)Z_2(q)\} = (1/3) + 2(p - q) - 2(p^2 - q^2) + 2(p^3 - q^3)/3$, $p \le q$.

THEOREM 2. *Under* H_0: $F \in \mathscr{E}$:

(i) $T_n^{(1)} \to_{\mathscr{L}} Z_{1\varepsilon}^* \triangleq \sup\{Z_{1\varepsilon}(p): 0 \le p \le 1 - \varepsilon\}$.
(ii) $T_n^{(2)} \to_{\mathscr{L}} Z_2^* \triangleq \sup\{Z_2(p): 0 \le p \le 1\}$.

Asymptotic critical values based on the distribution of $Z_{1\varepsilon}^*$ may be obtained exactly, since the process $Y(p) = (1-p)Z_{1\varepsilon}(p)$, $0 \le p \le 1 - \varepsilon$, has covariance

TABLE 1
Exact quantiles of $Z^*_{1\varepsilon}$ ($\varepsilon = 0.10$) and estimated quantiles of Z^*_2

α	**0.90**	**0.95**	**0.99**
$Z^*_{1\varepsilon}$ quantile	4.94	5.88	7.73
Z^*_2 quantile	1.41	1.59	1.93

function $p(1-q)$ for $p \le q$ and hence is a Brownian bridge. Thus,

$$
\begin{aligned}
P\{Z^*_{1\varepsilon} > c\} &= P\left\{\sup\left[\frac{Y(p)}{(1-p)} : 0 \le p \le 1-\varepsilon\right] > c\right\} \\
&= P\left\{\sup\left[(1+u)Y\left(\frac{u}{1+u}\right) : 0 \le u \le \frac{1-\varepsilon}{\varepsilon}\right] > c\right\} \\
&= P\left\{\sup\left[W(u) : 0 \le u \le \frac{1-\varepsilon}{\varepsilon}\right] > c\right\} \\
&= 2\left\{1 - \Phi\left(c\sqrt{\frac{\varepsilon}{1-\varepsilon}}\right)\right\},
\end{aligned}
\tag{2.5}
$$

where $W(\cdot)$ is the Wiener process on $[0, 1]$, $\Phi(x)$ is the $N(0,1)$ cdf and the last equality follows by the reflection principle for the Wiener process. Table 1 contains selected quantiles of the distribution of $Z^*_{1\varepsilon}$ for $\varepsilon = 0.10$, computed from (2.5).

Asymptotic critical values based on the distribution of Z^*_2 may be obtained from Durbin's (1985) approximation [using his delete the indicator function method with ϕ denoting the $N(0,1)$ pdf]

$$
P\{Z^*_2 > c\} = \{2\sqrt{3}c + O(c^{-1})\}\phi(\sqrt{3}c) \quad \text{as } c \to \infty. \tag{2.6}
$$

Table 1 contains selected quantiles of the distribution of Z^*_2, computed from (2.6).

3. Power comparison. Since our tests do not require knowledge of t^* or of $\rho = F(t^*)$, which GHP's tests do require, it is natural to compare the powers of our tests with that of GHP's tests. This was done for a parametric subfamily $\mathscr{G}_s$ of $\mathscr{G}$, whose typical member (indexed by $\alpha > 0, \beta > 0, \gamma > 0$) is

$$
\begin{aligned}
\bar{F}_{\alpha,\beta,\gamma}(x) &= \left\{\frac{\beta}{\beta + \gamma e^{-\alpha x}(1 - e^{-\alpha x})}\right\}\left\{\frac{[1+d]^2 - c^2}{[e^{\alpha x} + d]^2 - c^2}\right\}^{1/2\alpha\beta} \\
&\quad \times \left\{\frac{e^{\alpha x} + d - c}{e^{\alpha x} + d + c}\,\frac{1 + d + c}{1 + d - c}\right\}^{\gamma/4\alpha\beta^2 c}, \qquad x \ge 0
\end{aligned}
$$

where $d = (2\beta/\gamma)^{-1}$, $c^2 = [4(\beta/\gamma) + 1]/[4(\beta/\gamma)^2]$. This distribution has MRL

TABLE 2
Monte Carlo power comparison ($n = 100$, $\beta = 1$)

α	ρ for γ = 0.5	1.0	2.0	STAT	$\gamma = 0.5$ Size = 0.10	0.05	$\gamma = 1.0$ 0.10	0.05	$\gamma = 2.0$ 0.10	0.05
2.0	0.35	0.40	0.48	T_1	0.084	0.049	0.143	0.070	0.652	0.419
				T_2	0.705	0.558	0.990	0.967	1.000	1.000
				GHP_1	0.605	0.475	0.961	0.912	1.000	1.000
				GHP_2	0.589	0.462	0.966	0.927	0.999	0.994
1.0	0.53	0.56	0.60	T_1	0.138	0.081	0.280	0.174	0.751	0.567
				T_2	0.449	0.312	0.874	0.791	1.000	0.998
				GHP_1	0.408	0.271	0.831	0.717	0.997	0.996
				GHP_2	0.521	0.384	0.888	0.812	0.996	0.991
0.5	0.75	0.75	0.76	T_1	0.184	0.117	0.384	0.251	0.779	0.684
				T_2	0.254	0.165	0.576	0.425	0.951	0.916
				GHP_1	0.215	0.116	0.512	0.361	0.915	0.864
				GHP_2	0.383	0.259	0.701	0.592	0.969	0.943
0.25	0.93	0.92	0.92	T_1	0.174	0.112	0.330	0.236	0.687	0.567
				T_2	0.151	0.075	0.275	0.181	0.658	0.534
				GHP_1	0.112	0.042	0.245	0.146	0.595	0.478
				GHP_2	0.265	0.187	0.490	0.367	0.851	0.766

function

$$m_{\alpha,\beta,\gamma}(t) = \beta + \gamma e^{-\alpha t}(1 - e^{-\alpha t}), \qquad t \geq 0.$$

The motivation for choosing $\mathscr{G}_s$ is best seen through the MRL function, which can represent $F \in \mathscr{E}$ (let $\gamma \downarrow 0$) and which, for any choice of (α, β, γ) has the IDMRL structure with change point $t^* = \alpha^{-1} \ln 2$. Of course, $\mathscr{G}_s$ contains only a curve in $\mathscr{G}$, so our results should be interpreted accordingly.

Table 2 contains Monte Carlo estimated powers based on 1000 realizations of samples of size $n = 100$ from $F_{\alpha,\beta,\gamma}$ for $\beta = 1$ and a selection of (α, γ). Included are our tests (T_1, T_2) (using $\varepsilon = 0.10$ for T_1), the test (GHP_1) based on GHP's T_n [see GHP immediately following (2.2)], which requires knowledge of t^* and the test (GHP_2) based on GHP's V_n [see GHP at (3.2)], which requires knowledge of $\rho = F(t^*)$. Asymptotic critical values are used for all tests. The size in the table heading refers to *nominal* size.

The power results should be viewed in light of the sizes. Here Monte Carlo size estimates (1000 replications) for nominal sizes $(0.10, 0.05, 0.01)$ are $(0.070, 0.042, 0.008)$ for T_1, $(0.110, 0.060, 0.017)$ for T_2, $(0.070, 0.033, 0.007)$ for GHP_1 and $(0.109, 0.056, 0.013)$ for GHP_2. Thus, T_1 and GHP_1 are slightly conservative. However, the power estimates in Table 2 are indicative of those to be obtained in practice if the asymptotic critical values are used.

Looking at Table 2, first note that GHP_2 generally dominates both our tests (except when $\rho \leq 0.5$, where T_2 seems to dominate). This is generally to be expected since our tests do not use information about t^* required by GHP_2.

(GHP_2 dominates GHP_1 presumably because GHP_1, with asymptotic critical values, is conservative.) When $\rho = F(t^*) \leq 0.75$, T_2 dominates T_1 and compares well with GHP's tests. For ρ in the neighborhood of 0.90, T_1 apparently dominates T_2, but both are considerably dominated by GHP_2. Of course, all of these comments strictly apply only to the parametric family $\mathscr{G}_s$.

Another feature of Table 2 is that, for fixed γ, the powers of all the tests decrease rapidly as α decreases. It is difficult, however, to tell whether this decrease is related to the corresponding increase in t^* or to the changes in slope, shape and so on which occur when α decreases. Therefore, based on our study of the family $\mathscr{G}_s$, we cannot definitely say that the powers of these tests will decrease as t^* increases in general nonparametric IDMRL families, although this behavior in $\mathscr{G}_s$ is suggested by our results.

To gain some insight into why the natural T_1 is dominated by the less intuitive T_2, a referee suggested that we consider the nonnull distributions of these statistics by computing standardized noncentrality parameters. This is possible for T_2 (similar things are true for T_1) since one may write, for any $F \in \mathscr{F} - \mathscr{E}$,

$$n^{1/2}\psi_t^{(2)}(F_n) = n^{1/2}\{\psi_t^{(2)}(F_n) - \psi_t^{(2)}(F)\} + n^{1/2}\psi_t^{(2)}(F), \tag{3.1}$$

with the first term in (3.1) weakly converging (via the same proof as in Section 4) to a mean-zero Gaussian process $\overline{Z}_2(F) = \{\overline{Z}_2(p; F): 0 \leq p \leq 1\}$ [with a covariance structure different from that of Z_2^* in Theorem 2(ii)] and the second term nonzero (but of course unbounded as $n \to \infty$). Insofar as the supremum of the left side of (3.1) may be roughly approximated by that of the second term on the right side, the noncentrality $\psi_t^{(2)}(F)$, suitably standardized, might explain the power properties of T_2 to some extent.

At the referee's suggestion, we compare the standardized noncentralities [in the notation implied by (3.1)] $\overline{\psi}^{(i)}(F) \triangleq \psi_{t^*}^{(i)}(F)/\sqrt{\operatorname{var}(\overline{Z}_i(F(t^*); F))}$, $i = 1, 2$ for $F = F_{\alpha,\beta,\gamma} \in \mathscr{G}_s$. The comparison is made at the change point t^* since the supremum in each of the functionals ϕ_1 and ϕ_2 is attained at t^*. [For T_1 this is true only if $t^* \leq F^{-1}(1 - \varepsilon)$.] Since the variance expressions are extremely complicated for $\overline{Z}_2$, the details are omitted here but are available from the first author, who has written a FORTRAN program to compute $\overline{\psi}^{(i)}(F)$ for any $F \in \mathscr{F}$.

The results of such computations appear in Table 3 for the same choices of (α, β, γ) as in Table 2. [In Table 3, the column headed S.D. gives the standard deviation in the denominator of $\overline{\psi}^{(i)}(F)$.] Generally the values of $\overline{\psi}^{(1)}$ and $\overline{\psi}^{(2)}$ corroborate the power results in Table 2. However, the severe domination of T_1 by T_2 for $\alpha = 2$ does not seem to be reflected by the noncentralities, nor does the apparent domination of T_2 by T_1 for ρ near 0.90 ($\alpha = 0.25$ in Table 3). The general (though apparently nonuniform) domination of T_1 by T_2 seems, from Table 3, to derive from fact that the standard deviation of $\overline{Z}_1(F(t^*), F)$ increases faster relative to the value of $\psi_{t^*}^{(1)}(F)$ then occurs for the corresponding quantities for T_2. Of course, all of these comments are subject to the rather crude approximation being used here.

TABLE 3
Standardized noncentralities for alternative $F_{\alpha,1,\gamma}$

		Functional ϕ_1			Functional ϕ_2		
α	γ	$\psi_{t^*}^{(1)}$	SD_1	$\bar{\psi}^{(1)}$	$\psi_{t^*}^{(2)}$	SD_2	$\bar{\psi}^{(2)}$
2.0	0.5	0.125	0.790	0.158	0.114	0.593	0.193
	1.0	0.250	0.926	0.270	0.211	0.600	0.352
	2.0	0.500	1.172	0.427	0.365	0.618	0.591
1.0	0.5	0.125	1.144	0.109	0.087	0.588	0.147
	1.0	0.250	1.286	0.194	0.161	0.598	0.269
	2.0	0.500	1.560	0.321	0.283	0.636	0.445
0.5	0.5	0.125	1.901	0.066	0.063	0.585	0.108
	1.0	0.250	2.072	0.121	0.120	0.599	0.210
	2.0	0.500	2.414	0.207	0.214	0.644	0.332
0.25	0.5	0.125	4.062	0.031	0.043	0.592	0.072
	1.0	0.250	4.263	0.059	0.082	0.612	0.133
	2.0	0.500	4.689	0.107	0.151	0.666	0.226

4. Proof of Theorem 2. The method of proof is the same for both (i) and (ii). For brevity, only the proof of (ii) is given here. [The proof of (i) may be found in the technical report by Hawkins, Kochar and Loader (1991).] Letting $\psi_t(F)$ denote either $\psi_t^{(1)}(F)$ or $\psi_t^{(2)}(F)$, define the Gateaux differential, for each $t \geq 0$, by

$$D\psi_t(F)(G-F) \triangleq \lim_{\varepsilon \to 0} \frac{\psi_t(F+\varepsilon(G-F)) - \psi_t(F)}{\varepsilon} \tag{4.1}$$

for each $F, G \in \mathscr{F}$ for which the limit exists. We shall use a statistical differential approximation to show, for $F \in \mathscr{F}$ and $t \geq 0$, that

$$\psi_t(F_n) - \psi_t(F) = D\psi_t(F)(F_n - F) + R_n(F,t), \tag{4.2}$$

where $R_n(F,t)$ is a remainder term satisfying

$$n^{1/2} \sup_{t \geq 0} |R_n(F,t)| \to_p 0. \tag{4.3}$$

This implies that the asymptotic distribution of $n^{1/2}$ times the left side of (4.2) is the same as that of $n^{1/2}$ times the first term on the right side, which can be obtained by standard methods since it is a linear functional of $F_n - F$. We use the notion of functional differentiation only formally to suggest the approximation (4.2) to $\psi_t(F_n)$. The usefulness of the approximation (4.2) derives from (4.3), which can be shown by standard methods without functional analysis. In what follows, we have written $m_t(F)$ rather than $m_F(t)$ to emphasize our present view of the MRL as a functional of F for fixed t. This should cause no confusion.

PROOF OF THEOREM 2(ii). We require the following result, proved at the end of the section.

LEMMA. *If* $F \in \mathscr{F}$, *then* $\int_0^\infty |\bar{F}_n(x) - \bar{F}(x)| dx \to_p 0$ *as* $n \to \infty$.

By straightforward calculation, for $F \in \mathscr{F}$ we have the Gateaux differential

$$D\psi_t^{(2)}(F)(F_n - F) = \int_0^\infty (\bar{F}_n - \bar{F}) + 4\int_0^t \bar{F}(\bar{F} - \bar{F}_n) - 4\int_t^\infty \bar{F}(\bar{F} - \bar{F}_n) \tag{4.4}$$
$$- 2\bar{F}(t)\int_t^\infty (\bar{F}_n - \bar{F}) + 2\{\bar{F}(t) - \bar{F}_n(t)\}\int_t^\infty \bar{F},$$

whence

$$R_n^{(2)}(F,t) \triangleq \psi_t^{(2)}(F_n) - \psi_t^{(2)}(F) - D\psi_t^{(2)}(F)(F_n - F)$$
$$= -2\int_0^t (\bar{F}_n - \bar{F})^2 + 2\int_t^\infty (\bar{F}_n - \bar{F})^2 + 2\{\bar{F}(t) - \bar{F}_n(t)\}\int_t^\infty (\bar{F}_n - \bar{F}). \tag{4.5}$$

Thus, for any $t \geq 0$ and $F \in \mathscr{F}$,

$$n^{1/2}|R_n^{(2)}(F,t)| \leq 4n^{1/2}\int_0^\infty (\bar{F}_n - F)^2 + 2n^{1/2}|\bar{F}(t) - \bar{F}_n(t)|\int_0^\infty |\bar{F}_n - F| \tag{4.6}$$
$$\leq 6n^{1/2}\sup_{s\geq 0}|\bar{F}_n(s) - \bar{F}(s)|\int_0^\infty |\bar{F}_n - F| = o_p(1)$$

by the lemma and the classical weak convergence of the empirical process.

Now define for each $n \geq 1$ and $F \in \mathscr{F}$ the stochastic process $Z_n(F) = \{Z_n(p; F): 0 \leq p \leq 1\}$ by

$$Z_n(p;F) = n^{1/2} D\psi_{t(p)}^{(2)}(F)(F_n - F), \qquad t(p) = F^{-1}(p).$$

Observe that $Z_n(F) \in D([0,1])$ for each n and that $\{t(p): 0 \leq p \leq 1\} = [0,\infty)$ since $F \in \mathscr{F}$ is continuous. By (4.5) and Theorem 1(ii), we have for $F \in \mathscr{E}$ that

$$n^{1/2}\psi_{t(p)}^{(2)}(F_n) = Z_n(p;F) + n^{1/2}R_n^{(2)}(F, t(p)), \qquad 0 \leq p \leq 1,$$

so in view of (4.6) the result follows if we show that for $F \in \mathscr{E}$,

$$\bar{X}_n^{-1} Z_n(F) \to_w Z_2 \quad \text{as } n \to \infty. \tag{4.7}$$

In this direction, we have by (4.4), for $F \in \mathscr{F}$,

$$n^{-1/2}Z_n(p;F) = \int_0^\infty (\bar{F}_n - \bar{F}) + 4\int_0^{F^{-1}(p)} \bar{F}(\bar{F} - \bar{F}_n) - 4\int_{F^{-1}(p)}^\infty \bar{F}(\bar{F} - \bar{F}_n) \tag{4.8}$$
$$- 2(1-p)\int_{F^{-1}(p)}^\infty (\bar{F}_n - \bar{F}) + 2\{F_n(F^{-1}(p)) - p\}\int_{F^{-1}(p)}^\infty \bar{F}.$$

Making the transformation $u = F(x)$ and defining $W_n(u) = n^{1/2}\{F_n(F^{-1}(u)) - u\}$, $0 \le u \le 1$, we have, since $(d/du)F^{-1}(u) = \lambda^{-1}(1-u)^{-1}$ for $F \in \mathscr{E}$,

$$\lambda^{-1}Z_n(p;F) = -\int_0^1 \frac{W_n(u)}{1-u}\,du + 4\int_0^p W_n(u)\,du - 4\int_p^1 W_n(u)\,du \tag{4.9}$$
$$+ 2(1-p)\int_p^1 \frac{W_n(u)}{1-u}\,du + 2(1-p)W_n(p), \qquad F \in \mathscr{E}.$$

Expression (4.7) follows from (4.9), the fact that $\bar{X}_n \to_{\text{a.s.}} \lambda^{-1}$ and the fact that the functional of W_n in (4.9) converges weakly to the same functional of the Brownian bridge process, whose distribution is that of Z_2. □

PROOF OF THE LEMMA.

$$\int_0^\infty |\bar{F}_n(x) - \bar{F}(x)|\,dx = \int_0^{X_{(n)}} |\bar{F}_n(x) - \bar{F}(x)|\,dx + \int_{X_{(n)}}^\infty |0 - \bar{F}(x)|\,dx$$
$$\le \left\{\sup_{x \in \mathbb{R}} n^{1/2}|\bar{F}_n(x) - \bar{F}(x)|\right\}\{n^{-1/2}X_{(n)}\} + \int_{X_{(n)}}^\infty \bar{F}(x)\,dx. \tag{4.10}$$

Since $\bar{F}(x) > 0$ for all $x \ge 0$, $X_{(n)} \to_p \infty$, so the second term in (4.10) is $o_p(1)$. For the first term in (4.10), the first factor is $O_p(1)$ by the classical weak convergence of the empirical process. The second factor is $o_p(1)$ since each $F \in \mathscr{F}$ has a finite second moment. □

REMARKS. A finite second moment is critical in the proof of this lemma [and hence for Theorem 2(ii)]. To see this let $X_1, \dots, X_n$ have pdf $f_\delta(x) = (\delta + 1)x^{-2-\delta}I(x \ge 1)$, where $0 < \delta < 1$. Then $EX_1^{1+\delta} < \infty$, but $n^{-1/2}X_{(n)} \to_p \infty$ as $n \to \infty$. However, $EX^2 = \infty$. A finite second moment is also critical for Theorem 2(i) since it is required by Yang's (1978) Theorem 1 (used in the proof not presented here). Of course, any $F \in \mathscr{E}$ has a finite second moment. However, the proofs of (2.3) and (2.4), which justify the proposed tests and involve $F \in \mathscr{F} - \mathscr{E}$, require finite second moments, explaining this specification in our definition of $\mathscr{F}$.

REFERENCES

BARLOW, R. E. and DOKSUM, K. (1972). Isotonic tests for convex orderings. *Proc. Sixth Berkeley Symp. Math. Statist. Probab.* **1** 293–323. Univ. California Press, Berkeley.

BHATTACHARJEE, M. C. (1982). The class of mean residual lives and some consequences. *SIAM J. Algebraic Discrete Methods* **3** 56–65.

BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York.

COX, D. R. (1962). *Renewal Theory*. Methuen, London.

DURBIN, J. (1985). The first-passage density of a continuous Gaussian process to a general boundary. *J. Appl. Probab.* **22** 99–122.

GUESS, F. and PROSCHAN, F. (1988). Mean residual life: Theory and applications. In *Handbook of Statistics: Quality Control and Reliability* (P. R. Krishnaiah and C. R. Rao, eds.) **7** 215–224. North-Holland, Amsterdam.

GUESS, F., HOLLANDER, M. and PROSCHAN, F. (1986). Testing exponentiality versus a trend change in mean residual life. *Ann. Statist.* **14** 1388–1398.

HALL, W. J. and WELLNER, J. A. (1981). Mean residual life. In *Statistics and Related Topics* (M. Csörgö, D. A. Dawson, J. N. K. Rao and A. K. Md. E. Saleh, eds.) 169–184. North-Holland, Amsterdam.

HAWKINS, D. L., KOCHAR, S. and LOADER, C. (1991). Testing exponentiality against IDMRL distributions with unknown change point. Technical Report 279, Univ. Texas, Arlington.

HOLLANDER, M. and PROSCHAN, F. (1984). Nonparametric concepts and methods in reliability. In *Handbook of Statistics*: *Nonparametric Methods* (P. R. Krishnaiah and P. K. Sen, eds.) **4** 613–655. North-Holland, Amsterdam.

KLEFSJÖ, B. (1983). Testing exponentiality against HNBUE. *Scand. J. Statist.* **10** 65–75.

KOCHAR, S. C. (1985). Testing exponentiality against monotone failure rate average. *Comm. Statist. Theory Methods* **14** 381–392.

KOTZ, S. and SHAMBHAG, D. N. (1980). Some new approaches to probability distributions. *Adv. in Appl. Probab.* **12** 903–921.

YANG, G. L. (1978). Estimation of a biometric function. *Ann. Statist.* **6** 112–116.

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