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# Stochastic comparisons of order statistics in the scale model

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ABSTRACT

Independent random variables $X_{\lambda_1},\ldots,X_{\lambda_n}$ are said to belong to the scale family of distributions if $X_{\lambda_i} \sim F(\lambda_i x)$, for $i=1,\ldots,n$, where $F$ is an absolutely continuous distribution function with hazard rate $r$ and reverse hazard rate $\tilde{r}$. We show that the hazard rate (reverse hazard rate) of a series (parallel) system consisting of components with lifetimes $X_{\lambda_1},\ldots,X_{\lambda_n}$ is Schur concave (convex) with respect to the vector $\boldsymbol{\lambda}$, if $x^2 r'(x)$ ($x^2\tilde{r}'(x)$) is decreasing (increasing). We also show that if $xr(x)$ is increasing in $x$, then the survival function of the parallel system is increasing in the vector $\boldsymbol{\lambda}$ with respect to $p$-larger order, an order weaker than majorization. We prove that all these new results hold for the scaled generalized gamma family as well as the power-generalized Weibull family of distributions. We also show that in the case of generalized gamma and power generalized Weibull distribution, under some conditions on the shape parameters, the vector of order statistics corresponding to $X_{\lambda_i}$'s is stochastically increasing in the vector $\boldsymbol{\lambda}$ with respect to majorization thus generalizing the main results in Sun and Zhang (2005) and Khaledi and Kochar (2006).





## 1. Introduction

Independent random variables $X_{\lambda_1},\ldots,X_{\lambda_n}$ are said to belong to the scale family of distributions if $X_{\lambda_i} \sim F(\lambda_i x)$, for $i=1,\ldots,n$, where $F$ is an absolutely continuous distribution function with density function $f$. It means that the random variables $\lambda_1 X_{\lambda_1},\ldots,\lambda_n X_{\lambda_n}$ are independent and identically distributed with common c.d.f. $F$. $F$ is called the baseline distribution and the $\lambda_i$'s are the scale parameters. It includes many important distributions like normal, exponential, Weibull, gamma as special cases. The scale model is of theoretical as well as practical importance in various fields of probability and statistics.

There is an extensive literature on stochastic orderings among order statistics when the observations follow the exponential distribution with different scale parameters. Important contributions in this area have been made by Pledger and Proschan (1971), Proschan and Sethuraman (1976), Boland et al. (1994 a, b), Dykstra et al. (1997), Khaledi and Kochar (2000), Bon and Paltanea (2006), among others. Also see a review paper by Kochar and Xu (2007) on this topic. Sun and Zhang (2005) considered the case of gamma distribution whereas Khaledi and Kochar (2006) considered the case of Weibull distribution.

Pledger and Proschan (1971) and Hu (1995) considered the general scale model and obtained stochastic ordering results between two vectors of order statistics when one set of scale parameters majorizes the other. In this paper we revisit this

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problem and obtain some new results and apply them when the baseline distributions are generalized gamma and power-generalized Weibull distributions. Our results extend some of the existing results.

We first introduce some notations and give definitions. Throughout this paper *increasing* means *nondecreasing* and *decreasing* means *nonincreasing*; and we shall be assuming that all distributions under study are absolutely continuous.

Let $X$ and $Y$ be univariate random variables with distribution functions $F$ and $G$, survival functions $\overline{F}$ and $\overline{G}$, hazard rate functions $r_F$ and $r_G$, reverse hazard rate functions $\tilde{r}_F$ and $\tilde{r}_G$, respectively.

**Definition 1.1.**

(a) $X$ is said to be *stochastically* smaller than $Y$ (denoted by $X \le_{st} Y$) if $\overline{F}(x) \le \overline{G}(x)$ for all $x$. This is equivalent to saying that $Eg(X) \le Eg(Y)$ for any increasing function $g$ for which expectations exist.
(b) $X$ is said to be smaller than $Y$ according to hazard rate ordering (denoted by $X \le_{hr} Y$) if $r_F(x) \ge r_G(x)$ for all $x$.
(c) $X$ is said to be smaller than $Y$ according to reverse hazard rate ordering (denoted by $X \le_{rh} Y$) if $\tilde{r}_F(x) \le \tilde{r}_G(x)$ for all $x$.
(d) $X$ is said to be smaller than $Y$ according to dispersive ordering (denoted by $X \le_{disp} Y$) if $G^{-1}(v) - G^{-1}(u) \ge F^{-1}(v) - F^{-1}(u)$, for $0 < u < v < 1$.

**Definition 1.2.** A random vector $\mathbf{X}=(X_1,\ldots,X_n)$ is said to be smaller than another random vector $\mathbf{Y}=(Y_1,\ldots,Y_n)$ according to multivariate stochastic ordering (denoted by $\mathbf{X} \overset{st}{\preceq} \mathbf{Y}$) if $h(\mathbf{X}) \le_{st} h(\mathbf{Y})$ for all increasing functions $h$.

It is easy to see that multivariate stochastic ordering implies component-wise stochastic ordering. For more details see Chapters 1 and 6 of Shaked and Shanthikumar (2007).

Next we introduce the notion of majorization which is one of the basic tools in establishing various inequalities in statistics and probability.

**Definition 1.3.** Let $\{x_{(1)} \le \cdots \le x_{(n)}\}$ denote the increasing arrangement of the components of a vector $\mathbf{x}=(x_1,\ldots,x_n)$. A vector $\mathbf{x}$ is said to majorize another vector $\mathbf{y}$ (written $\mathbf{x} \overset{m}{\succcurlyeq} \mathbf{y}$) if $\sum_{i=1}^{j} x_{(i)} \le \sum_{i=1}^{j} y_{(i)}$ for $j=1,\ldots,n-1$ and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$.

Functions that preserve the majorization ordering are called Schur-convex functions. Marshall and Olkin (1979) provides extensive and comprehensive details on the theory of majorization and its applications in statistics.

In the following, the $i$th order statistic corresponding to a sample of size $n$ from a random variable $X$ is denoted by $X_{i:n}$, $i=1,\ldots,n$. Pledger and Proschan (1971) proved the following general result for the scale model.

**Theorem 1.1.** *Let $X_1,\ldots,X_n$; $Y_1,\ldots,Y_n$ be independent nonnegative random variables with $X_i \sim F(\lambda_i x)$, $Y_i \sim F(\mu_i x)$, $\lambda_i > 0$, $\mu_i > 0$, $i=1,\ldots, n$ where $F$ is an absolutely continuous distribution. If $r(x)$, the hazard rate of $F$, is decreasing, then*

$$(\lambda_1,\ldots,\lambda_n) \overset{m}{\succcurlyeq} (\mu_1,\ldots,\mu_n) \Longrightarrow X_{k:n} \ge_{st} Y_{k:n}$$

*for $k=1,\ldots,n$.*

Let $G(\alpha,\lambda)$ and $W(\alpha,\lambda)$ denote gamma and Weibull random variables with shape parameter $\alpha$ and scale parameter $\lambda$. For these scale models, Sun and Zhang (2005) and Khaledi and Kochar (2006), respectively, extended the above result from component-wise stochastic ordering to multivariate stochastic ordering. They proved the following theorem.

**Theorem 1.2.** *Let $(X_1, \ldots, X_n)$ and $(Y_1, \ldots,Y_n)$ be two independent random vectors with*

(a) *$X_i \sim W(\alpha,\lambda_i)$ and $Y_i \sim W(\alpha,\mu_i)$, $i= 1,\ldots, n$ or*
(b) *$X_i \sim G(\alpha,\lambda_i)$ and $Y_i \sim G(\alpha,\mu_i)$, $i= 1,\ldots,n$.*

*Then for $0 < \alpha \le 1$,*

$$\boldsymbol{\lambda} \overset{m}{\succcurlyeq} \boldsymbol{\mu} \Rightarrow (X_{1:n},\ldots,X_{n:n}) \overset{st}{\succcurlyeq} (Y_{1:n},\ldots,Y_{n:n}). \tag{1.1}$$

Hu (1995) also considered the general scale problem and proved the following result.

**Theorem 1.3.** *Let $X_1,\ldots,X_n$; $Y_1,\ldots,Y_n$ be independent nonnegative random variables with $X_i \sim F(\lambda_i x)$, $Y_i \sim F(\mu_i x)$, where $\lambda_i > 0$, $\mu_i > 0$, $i=1,\ldots,n$ are such that*

$$(\lambda_1,\ldots,\lambda_n) \overset{m}{\succcurlyeq} (\mu_1,\ldots,\mu_n).$$

*Assume that the failure rate of $F$, $r(x)$ is decreasing and $xr(x)$ is increasing. Then on some probability space $(\Omega,\mathcal{F},\mathcal{P})$, there exist random variables $X'_1,\ldots,X'_n$; $Y'_1,\ldots,Y'_n$ such that*

$$(X_1,\ldots,X_n) \overset{st}{=} (X'_1,\ldots,X'_n),$$

$$(Y_1,\ldots,Y_n) \overset{st}{=} (Y'_1,\ldots,Y'_n),$$

$$X_{k:n} \geq Y_{k:n} \quad a.s. \text{ for } k=1,\ldots,n.$$

*This implies in particular that*

$$(X_{1:n},\ldots,X_{n:n}) \overset{st}{\succcurlyeq} (Y_{1:n},\ldots,X_{n:n}).$$

Hu (1994) applied Theorem 1.3 to several important life distributions (e.g., Weibull, gamma, half-normal distributions).

Next, in this section we introduce generalized gamma and power-generalized Weibull distributions to show that Hu's result can be applied to these general families of distributions which covers gamma and Weibull distributions as special cases.

### 1.1. *Generalized gamma distribution*

A random variable $X$ is said to have a generalized gamma distribution, denoted by $X \sim GG(p,q)$, if it admits the following density function:

$$g_{p,q}(x) = \frac{p}{\Gamma\left(\frac{q}{p}\right)} x^{q-1} e^{-x^p} x > 0,$$

where $p,q$ $(>0)$ are the shapes parameters. It was introduced by Stacy (1962) and includes the wildly used exponential $(p=1,\ q=1)$, Weibull $(p=q)$, and gamma $(p=1)$ distributions as special cases. It is a flexible family of distributions, having an increasing failure rate when $q \geq 1, p \geq 1$, a bathtub failure rate when $q<1, p>1$, an upside down bathtub (or unimodal) failure rate when $q>1, p<1$.

It is easy to show that for $p \leq 1, q \leq 1$, the hazard rate of generalized gamma distribution is decreasing. It is shown in Lemma A.3 of the Appendix that $xr(x)$ is an increasing function of $x$ for all $p,q>0$. Thus the conditions of Theorem 1.3 are satisfied when the baseline distribution is generalized gamma distribution with $p \leq 1, q \leq 1$ and thus extending the result of Theorem 1.1 in Sun and Zhang (2005) from standard gamma distribution to generalized gamma distribution.

### 1.2. *Power-generalized Weibull distribution*

A random variable $X$ is said to have power-generalized Weibull distribution, denoted by $X \sim PGW(\nu,\gamma)$, if its density function is

$$f(t,\nu,\gamma) = \frac{\nu}{\gamma} t^{\nu-1}(1+t^{\nu})^{1/\gamma-1} e^{1-(1+t^{\nu})^{1/\gamma}}, \quad t>0, \ \nu,\gamma>0, \tag{1.2}$$

and its survival function is

$$\overline{F}(t,\nu,\gamma) = e^{1-(1+t^{\nu})^{1/\gamma}}, \quad t>0. \tag{1.3}$$

It has a decreasing failure rate when $\nu \leq \gamma, \nu \leq 1$, an increasing failure rate when $\nu \geq \gamma, \nu \geq 1$, a bathtub failure rate when $0<\gamma<\nu<1$ and an upside down bathtub (or unimodal) failure rate when $\gamma>\nu>1$. It includes Weibull and exponential distributions as special cases. For more details on this family and its applications in probability and statistics, the reader is referred to Bagdonavicius and Nikulin (2002). Let $r(x)$ be the hazard rate function of power-generalized Weibull distribution. It is known that for $\nu \leq \gamma, 0<\nu \leq 1$, $r(x)$ is a decreasing function (cf. Bagdonavicius and Nikulin, 2002).

For all values of $\nu$ and $\gamma$, the function

$$xr(x) = \frac{\nu}{\gamma} x^{\nu}(1+x^{\nu})^{1/\gamma-1}$$

is increasing in $x$. That is, the conditions of Theorem 1.3 are satisfied by the power-generalized Weibull distribution with $\nu \leq \gamma, 0<\nu \leq 1$, thus extending the result of Theorem 2.2 in Khaledi and Kochar (2006) from standard Weibull distribution to power-generalized Weibull distribution.

In Section 2, we prove that Theorem 1.3 can be strengthened from stochastic ordering to the hazard rate ordering for series systems. In Section 3, we consider the notion of $p$-larger order (Definition 3.1), which is weaker than majorization, to prove stochastic ordering between two parallel systems consisting of components whose lifetimes distributions belong to the scale model. We also show in the same section that Theorem 1.3 can be extended from stochastic order to reverse hazard rate order when we compare to parallel systems. We observe that the results obtained in these two sections can be applied to generalized gamma and power generalized Weibull distributions.

## 2. Series systems

Next theorem shows that under some additional conditions the results of Theorem 1.3 can be extended from the usual stochastic order to the hazard rate order when we compare two series systems.

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**Theorem 2.1.** *Let $X_{\lambda_1},\ldots,X_{\lambda_n}$ be independent nonnegative random variables with $X_{\lambda_i}\sim F(\lambda_i x)$, $i=1,\ldots,n$, where $\lambda_i>0$, $i=1,\ldots,n$ and F is an absolutely continuous distribution. Let r be the hazard rate functions of F. If $x^2r'(x)$ is decreasing and $(\lambda_1,\ldots,\lambda_n)\overset{m}{\succeq}(\mu_1,\ldots,\mu_n)$, then*

(i) $X^{\lambda}_{1:n}\geq_{hr}X^{\mu}_{1:n}$, *and*
(ii) *if $r(x)$ is decreasing then $X^{\lambda}_{1:n}\geq_{disp}X^{\mu}_{1:n}$.*

**Proof.**

(i) We have to show that the hazard rate of $X_{(1)}$, $r_{X^{\lambda}_{1:n}}(x)$ is Schur-concave in $(\lambda_1,\ldots,\lambda_n)$. The partial derivative of $r_{X^{\lambda}_{1:n}}(x)$ with respect to $\lambda_i$ is

$$\frac{\partial r_{X^{\lambda}_{1:n}}(x)}{\partial\lambda_i}=x\lambda_i r'(x\lambda_i)+r(x\lambda_i),\quad i=1,\ldots,n.$$

Now, using Theorem 3.A.4 in Marshall and Olkin (1979), to prove the required result we have to show that

$$(\lambda_i-\lambda_j)(x\lambda_i r'(x\lambda_i)+r(x\lambda_i)-x\lambda_j r'(x\lambda_j)-r(x\lambda_j))\leq 0.$$

That is, $ur'(u)+r(u)$ is decreasing in $u$ which in turn is equivalent to the assumption that $u^2r'(u)$ is decreasing in $u$. This proves (i).

(ii) Using the assumption that $r(x)$ is decreasing in $x$ and part (i), the required result follows from Theorem 2.1 in Bagai and Kochar (1986) and Theorem 1 in Bartoszewicz (1985). □

**Remark 2.1.** The inequalities in (i) and (ii) of Theorem 2.1 are reversed if $x^2r'(x)$ is increasing.

Let the baseline distribution be $PGW(\nu,\gamma)$. The function $x^2r'(x)$ for this distribution can be simplified as

$$\begin{aligned}x^2r'(x)&=\frac{\nu}{\gamma}x^{\nu}(1+x^{\nu})^{1/\gamma-2}\left[(\nu-1)(1+x^{\nu})+x^{\nu}\nu\left(\frac{1}{\gamma}-1\right)\right]\\&=\frac{\nu}{\gamma}\left(x^{\nu}(1+x^{\nu})^{1/\gamma-1}(\nu-1)+\nu\left(\frac{1}{\gamma}-1\right)x^{\nu}(1+x^{\nu})^{1/\gamma-1}-\nu\left(\frac{1}{\gamma}-1\right)x^{\nu}(1+x^{\nu})^{1/\gamma-2}\right)\\&=\frac{\nu}{\gamma}\left[\left(\frac{\nu}{\gamma}-1\right)x^{\nu}(1+x^{\nu})^{1/\gamma-1}-\nu\left(\frac{1}{\gamma}-1\right)x^{\nu}(1+x^{\nu})^{1/\gamma-2}\right].\end{aligned}\tag{2.1}$$

To prove $x^2r'(x)$ is decreasing (increasing), it is enough to show that the function

$$g(x)=\frac{\nu}{\gamma}\left\{\left(\frac{\nu}{\gamma}-1\right)x(1+x)^{1/\gamma-1}-\nu\left(\frac{1}{\gamma}-1\right)x(1+x)^{1/\gamma-2}\right\}$$

is decreasing (increasing). The derivative of $g(x)$ is

$$g'(x)=(1+x)^{1/\gamma-3}\left\{x^2\left(\frac{\nu}{\gamma}-1\right)\frac{1}{\gamma}+x\left(\left(\frac{\nu}{\gamma}-1\right)\left(\frac{1}{\gamma}+1\right)-\nu\left(\frac{1}{\gamma}-1\right)^2\right)+\nu-1\right\}.\tag{2.2}$$

Let

$$\phi(x)=x^2\left(\frac{\nu}{\gamma}-1\right)\frac{1}{\gamma}+x\left(\left(\frac{\nu}{\gamma}-1\right)\left(\frac{1}{\gamma}+1\right)-\nu\left(\frac{1}{\gamma}-1\right)^2\right)+\nu-1,\tag{2.3}$$

then

$$\lim_{x\to 0}\phi(x)=\nu-1\quad\text{and}\quad\lim_{x\to\infty}\phi(x)=\begin{cases}\infty, & \nu>\gamma,\\ -\infty, & \nu<\gamma.\end{cases}\tag{2.4}$$

For $\nu\leq\gamma$ and $\nu<1$, the function $\phi(x)$ is negative, from which it follows that $x^2r'(x)$ is decreasing. That is, the result of Theorem 2.1(i) can be applied to $PGW(\nu,\gamma)$ with restriction $0<\nu\leq\gamma$ and $\nu<1$. Now, let $\nu>\gamma$ and $\nu>1$. At

$$t=-\frac{\left(\left(\frac{\nu}{\gamma}-1\right)\left(\frac{1}{\gamma}+1\right)-\nu\left(\frac{1}{\gamma}-1\right)^2\right)}{2\left(\frac{\nu}{\gamma}-1\right)},$$

$\phi'(x)$ is zero and $\phi(x)$ attains its minimum. On the other hand under the assumptions that $\nu>\gamma$ and $\nu>1$, $\phi(t)>0$. Combining these observations we have that $x^2r'(x)$ is increasing, from which the result of Remark 2.1 can be applied to $PGW(\nu,\gamma)$ with restriction $\nu>\gamma$ and $\nu>1$.

In Lemma A.4 of the Appendix, we show that the condition of Theorem 2.1(i) holds when the baseline distribution in the scale model is the $GG(p,q)$ with parameters $p,q<1$. We know that for $p,q\leq 1$, the hazard rate function $r(x)$ is decreasing.

That is, Theorem 2.1(ii) can be applied to this case as well. We also show that the result of Remark 2.1 can be applied to $GG(p,q)$ with parameters $p,q>1$.

With the help of the following counterexample we show that the above observation may not be true for other order statistics.

**Example 2.1.** Let $(X_1,X_2,X_3)$ be a set of independent random variables corresponding to a scale model with baseline $GG(0.2,0.5)$ and scale parameters $(\lambda_1,\lambda_2,\lambda_3)=(0.1,2,5)$ and $(X_1^*,X_2^*,X_3^*)$ be another set of independent random variables corresponding to a scale model with baseline $GG(0.2,0.5)$ and scale parameters $(\mu_1,\mu_2,\mu_3)=(0.1,3,4)$. It is easily seen that

$$(\lambda_1,\lambda_2,\lambda_3)\overset{m}{\succeq}(\mu_1,\mu_2,\mu_3).$$

However,

$$r_{X_{3:3}}(1.4)\approx 0.35 > r_{X^*_{3:3}}(1.4)\approx 0.30.$$

## 3. Parallel systems

Bon and Paltanea (1999) have considered a pre-order on $\mathbb{R}^{+^n}$ called $p$-larger order.

**Definition 3.1.** A vector $\mathbf{x}$ in $\mathbb{R}^{+^n}$ is said to be *p-larger* than another vector $\mathbf{y}$ also in $\mathbb{R}^{+^n}$ (written $\mathbf{x}\overset{p}{\succeq}\mathbf{y}$) if for $j=1,\ldots,n$,

$$\prod_{i=1}^{j} x_{(i)} \le \prod_{i=1}^{j} y_{(i)}.$$

It is known that when $\mathbf{x},\mathbf{y}\in\mathbb{R}^{+^n}$, then

$$\mathbf{x}\overset{m}{\succeq}\mathbf{y}\Longrightarrow\mathbf{x}\overset{p}{\succeq}\mathbf{y}. \tag{3.1}$$

The converse is, however, not true (cf. Khaledi and Kochar, 2002).

Khaledi and Kochar (2006) proved the following result for the proportional hazard rate (PHR) model.

**Theorem 3.1.** *Let $X_1,\ldots,X_n$ be independent random variables with $X_i$ having survival function $\overline{F}^{\lambda_i}(x)$, $i=1,\ldots,n$. Let $Y_1,\ldots,Y_n$ be another set of independent random variables with $Y_i$ having survival function $\overline{F}^{\mu_i}(x)$, $i=1,\ldots,n$. Then*

$$\boldsymbol{\lambda}\overset{p}{\succeq}\boldsymbol{\mu}\Longrightarrow X_{n:n}\ge_{st} Y_{n:n}.$$

In the next theorem we prove that under a mild condition a similar result holds for the scale model. The following lemma is used to prove the result.

**Lemma 3.1** (*Khaledi and Kochar, 2002*). *The function $\psi:\mathbb{R}^{+^n}\to IR$ satisfies*

$$\mathbf{x}\overset{p}{\succeq}\mathbf{y}\Longrightarrow\psi(\mathbf{x})\ge\psi(\mathbf{y}) \tag{3.2}$$

*if and only if,*

(i) $\psi(e^{a_1},\ldots,e^{a_n})$ *is Schur-convex in* $(a_1,\ldots,a_n)$,
(ii) $\psi(e^{a_1},\ldots,e^{a_n})$ *is decreasing in* $a_i$, *for* $i=1,\ldots,n$,

*where* $a_i=\log x_i$, *for* $i=1,\ldots,n$.

**Theorem 3.2.** *Let $X_1,\ldots,X_n$ be a set of independent nonnegative random variables with $X_i\sim F(\lambda_i x)$, $i=1,\ldots,n$, where $F$ is an absolutely continuous distribution function with density function $f$. Let $Y_1,\ldots,Y_n$ be another set of independent nonnegative random variables with $Y_i\sim F(\mu_i x)$, $i=1,\ldots,n$. If $x\tilde{r}(x)$ is decreasing in $x$, then*

$$(\lambda_1,\ldots,\lambda_n)\overset{p}{\succeq}(\mu_1,\ldots,\mu_n)\Longrightarrow X_{n:n}\ge_{st} Y_{n:n}. \tag{3.3}$$

**Proof.** The survival function of $X_{n:n}$ can be written as

$$\overline{F}_{X_{n:n}}(t)=1-\prod_{i=1}^{n} F(e^{a_i}t), \tag{3.4}$$

where $a_i=\log\lambda_i$, $i=1,\ldots,n$. Using Lemma 3.1, it is enough to show that the function $\overline{F}_{X_{n:n}}(t)$ given in (3.4) is Schur-convex and decreasing in $a_i$'s. To prove its Schur-convexity, it follows from Theorem 3.A.4 in Marshall and Olkin (1979) that we have to show that for $i\neq j$,

$$(a_i-a_j)\left(\frac{\partial\overline{F}_{X_{n:n}}}{\partial a_i}-\frac{\partial\overline{F}_{X_{n:n}}}{\partial a_j}\right)\ge 0,$$



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that is, for $i \neq j$,

$$(a_i - a_j) \prod_{i=1}^{n} F(e^{a_i} t) \left[ t e^{a_j} \frac{f(t e^{a_j})}{F(t e^{a_j})} - t e^{a_i} \frac{f(t e^{a_i})}{F(t e^{a_i})} \right] \geq 0. \tag{3.5}$$

The assumption $x\tilde{r}(x)$ is decreasing implies that the function $e^{a_i t}\tilde{r}(e^{a_i} t)$ is decreasing in $a_i$, $i=1,\ldots,n$, from which it follows that (3.5) holds. The partial derivative of $\overline{F}_{X_{n:n}}(t)$ with respect to $a_i$ is negative, which in turn implies that the survival function of $X^{\lambda}_{n:n}$ is decreasing in $a_i$ for $i=1,\ldots,n$. This completes the proof of the required result. □

**Remark 3.1.** Khaledi and Kochar (2000) proved (3.3) when the baseline distribution in the scale model is exponential.

The above theorem immediately leads to the following corollary.

**Corollary 3.1.** *Let $X_1,\ldots,X_n$ be a set of independent nonnegative random variables with $X_i \sim F(\lambda_i x)$, $i=1,\ldots,n$, where $F$ is an absolutely continuous distribution function with density function $f$. Let $Y_1,\ldots,Y_n$ be i.i.d. random variables with common c.d.f. $F(\tilde{\lambda} x)$, where $\tilde{\lambda}$ is the geometric mean of the $\lambda_i$' s. If $x\tilde{r}(x)$ is decreasing in $x$, then $X_{n:n} \geq_{st} Y_{n:n}$.*

The above corollary gives a lower bound on the survival function of a parallel system with nonidentical components in terms of the one with i.i.d. components when the common scale parameter is the geometric mean of the scale parameters. The new bound is better than the one that follows from Hu (1995) which is in terms of the arithmetic mean of the scale parameters since $\overline{F}_{X_{n:n}}(x)$ is a nonincreasing function of $\tilde{\lambda}$ and the fact that the geometric mean of the $\lambda_i$' s is smaller than their arithmetic mean.

Under the conditions of Theorems 1.3 and 3.2 the improvements on the bounds are relatively more if the $\lambda_i$'s are more dispersed in the sense of majorization. This fact follows from the fact that the geometric mean is Schur concave whereas the arithmetic mean is Schur constant and the survival function of a parallel system of i.i.d. components with baseline distribution $F(x)$ and common parameter $\tilde{\lambda}$ is decreasing in $\tilde{\lambda}$.

We show in Lemmas 4.5 and 4.6 in the Appendix that the conditions of Theorem 3.2 are satisfied when the baseline distributions in the scale model are generalized gamma distribution and power-generalized Weibull distribution with arbitrary parameters.

In Fig. 1, we plot the survival function of a parallel system consisting of three components with generalized gamma distributions with scale parameters $\boldsymbol{\lambda}_1 = (0.01, 1.8, 5.99)$ and shape parameters $q=0.2$ and $p=0.5$ along with the lower bounds as given by Corollary 3.1 based on the geometric mean and the arithmetic mean of the parameters. The vector of parameters in Fig. 2 is $\boldsymbol{\lambda}_2 = (0.5, 3.5, 3.8)$. Note that $\boldsymbol{\lambda}_1 \overset{m}{\succcurlyeq} \boldsymbol{\lambda}_2$. As discussed above, the improvements on the bounds are relatively more if the $\lambda_i$' s are more dispersed in the sense of majorization.

Next theorem extends Theorem 1.3 from the usual stochastic order to the reverse hazard rate order when we compare two parallel systems, a result which is similar to Theorem 2.1.

**Theorem 3.3.** *Let $X_{\lambda_1},\ldots,X_{\lambda_n}$ be independent nonnegative random variables with $X_{\lambda_i} \sim F(\lambda_i x)$, $i=1,\ldots,n$, where $\lambda_i > 0$, $i=1,\ldots,n$ and $F$ is an absolutely continuous distribution. Let $f$ and $\tilde{r}$ be the density and the reverse hazard rate functions of $F$, respectively. If $x^2\tilde{r}'(x)$ is increasing, then*

$$(\lambda_1,\ldots,\lambda_n) \overset{m}{\succcurlyeq} (\mu_1,\ldots,\mu_n) \Rightarrow X^{\lambda}_{n:n} \geq_{rh} X^{\mu}_{n:n}. \tag{3.6}$$

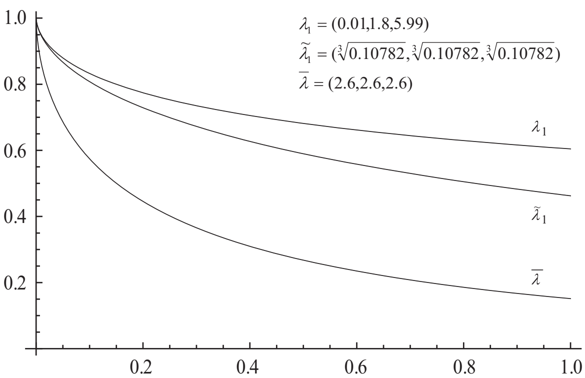


**Fig. 1.** Graphs of survival functions of $X_{3:3}$.

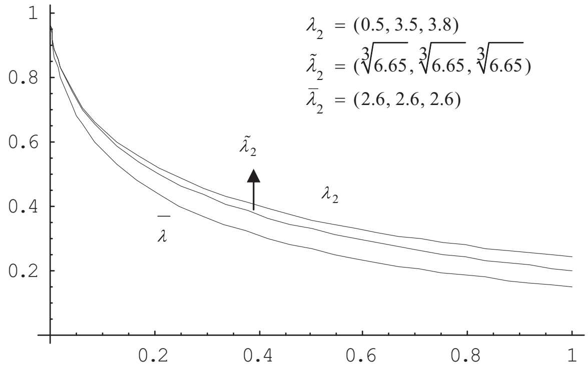


**Fig. 2.** Graphs of survival functions of $X_{3:3}$.

**Proof.** We have to show that $\tilde{r}_{X^{\lambda}_{n:n}}(x)$ is Schur-convex in $(\lambda_1,\ldots,\lambda_n)$. The partial derivative of $\tilde{r}_{X^{\lambda}_{n:n}}(x)$ with respect to $\lambda_i$ is

$$\frac{\partial \tilde{r}_{X^{\lambda}_{n:n}}(x)}{\partial \lambda_i} = x\lambda_i \tilde{r}'(x\lambda_i)+\tilde{r}(x\lambda_i), \quad i=1,\ldots,n.$$

Now, using Theorem 3.A.4 in Marshall and Olkin (1979), to prove the required result we have to show that

$$(\lambda_i-\lambda_j)(x\lambda_i\tilde{r}'(x\lambda_i)+\tilde{r}(x\lambda_i)-x\lambda_j\tilde{r}'(x\lambda_j)-\tilde{r}(x\lambda_j)) \geq 0.$$

That is, $u\tilde{r}'(u)+\tilde{r}(u)$ is increasing in $u$ which in turn is equivalent to the assumption that $u^2\tilde{r}'(u)$ is increasing in $u$. This proves the required result. □

**Remark 3.2.** The inequality in (3.6) is reversed if $x^2\tilde{r}'(x)$ is decreasing.

It is proved in Lemma A.8 in the Appendix that the conditions of Theorem 3.3 are satisfied by generalized gamma distribution with the parameters $p<1$ and $q>0$. That is, the reverse hazard rate of a parallel system consisting of independent components with $GG(p,q)$ lifetimes is Schur convex in the vector of scales parameters, when either $p<1$ and $q<1$ (that is $F$ is $DFR$) or $p<1$ and $q>1$ (that is $F$ has an upside down bathtub failure rate).

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## Appendix A

To prove the required results in this section the following observations are required. First, we need the following crucial relations between $r(x)$, the hazard rate of generalized gamma random variable and its derivative, $r'(x)$. By definition,

$$r(x) = \frac{x^{q-1}e^{-x^p}}{\int_x^\infty t^{q-1}e^{-t^p}\,dt} \tag{A.1}$$

and its derivative with respect to $x$ is

$$r'(x) = \frac{x^{q-2}e^{-x^p}(q-1-px^P)\int_x^\infty t^{q-1}e^{-t^p}\,dt+(x^{q-1}e^{-x^p})^2}{(\int_x^\infty t^{q-1}e^{-t^p}\,dt)^2}.$$

Combining these, we have

$$x\frac{r'(x)}{r(x)} = q-1-px^p+xr(x) \tag{A.2}$$

and

$$\left(x\frac{r'(x)}{r(x)}\right)' = r(x)\left(1-p^2\frac{x^{p-1}}{r(x)}+x\frac{r'(x)}{r(x)}\right), \tag{A.3}$$

where $(xr'(x)/r(x))'$ denote the derivative of $xr'(x)/r(x)$ with respect to $x$. Using (A.1)–(A.3), after some simplification and manipulations we obtain the following lemma that is used to prove Lemma A.2.

**Lemma A.1.** *Let $r(x)$ be the hazard rate of a generalized gamma random variable with parameters $p,q>0$. Then*

$$\text{(a)}\quad \lim_{x\longrightarrow 0}\frac{x^{p-1}}{r(x)}=\begin{cases}0, & q<p\\ \infty, & q>p\end{cases}\quad \text{and}\quad \lim_{x\longrightarrow\infty}\frac{x^{p-1}}{r(x)}=\frac{1}{p}, \tag{A.4}$$

$$\text{(b)}\quad \lim_{x\longrightarrow 0}x\frac{r'(x)}{r(x)}=q-1\quad \text{and}\quad \lim_{x\longrightarrow\infty}x\frac{r'(x)}{r(x)}=p-1. \tag{A.5}$$

**Lemma A.2.** *Let $U(x)=x^{p-1}/r(x)$ and $g(x)=xr'(x)/r(x)$, then for all $x>0$ and $p>q$ we have*

(i) $g(x)>q-1$,
(ii) $U(x)<1/p$ *and*
(iii) $g(x)<p-1$ *for all* $x>0$

*where $r(x)$ and $xr'(x)/r(x)$, respectively, are defined as in* (A.1) *and* (A.2).

**Proof.**

(i) Suppose $\min_{x>0}g(x)\le q-1$. In this case, it follows from Lemma A.1(b) that $g(x)$ crosses the level $q-1$ at least once. That is, there exist a point $v>0$ such that $vr'(v)/r(v)=q-1$. Now, it follows from (A.2) that $U(v)=1/p$. Using these observations in (A.3), we obtain that $(v^*r'(v^*)/r(v^*))'=r(v^*)(q-p)<0$. That is,

$$\text{for } x<v \text{ we have } x\frac{r'(x)}{r(x)}\ge q-1. \tag{A.6}$$

Now, first we show that if the point $v$ exists, it is unique. It follows from Lemma A.1(b), $\lim_{x\longrightarrow\infty}xr'(x)/r(x)=p-1$ and assumption $p>q$ that the function $g(x)$ has to cross the level $q-1$ at least at another point, say $v^*$ such that $(v^*r'(v^*)/r(v^*))'>0$, but this is impossible, since the derivative of $g(x)$ at crossing point $v$ and $v^*$ is negative. Thus, $v$ is unique and $v$ is the only point at which the function $g(x)$ crosses the level $q-1$ and for $x>v$ we have $xr'(x)/r(x)\le q-1$. But, this contradict $\lim_{x\longrightarrow\infty}xr'(x)/r(x)=p-1$. That is, such a point $v$ does not exist. This proves the required result of (i).

(ii) Suppose that $\max_{x>0}U(x)>1/p$. From Lemma A.1(a), $U(\infty)=1/p$ and $U(0)$=0. Therefore, $U(x)$ crosses level $1/p$, from which it follows that $g(x)$ has to cross the level $q-1$. But, from (i), this is not the case. This completes the proof of (ii).

(iii) Suppose that $\max_{x>0}xr'(x)/r(x)\ge p-1$ Then, it follows from Lemma A.1(b) that the function $g(x)$ must cross the level $p-1$ at least once. Let $\mu$ be one of the crossing points. That is $\mu r'(\mu)/r(\mu)=p-1$. Now using this in (A.3), it follows that

$$\left(x\frac{r'(x)}{r(x)}\right)'_{x=\mu}=r(\mu)\left[p-p^2\frac{\mu^{p-1}}{r(\mu)}\right].$$

Now, it follows from (ii) that $(xr'(x)/r(x))'/_{x=\mu}>0$. Using similar kind of arguments as used in part (i), it is easy to show that the crossing point $\mu$ is unique. Now, it follows from Lemma A.1(b) that

$$\text{for } x<\mu,\ x\frac{r'(x)}{r(x)}<p-1\quad\text{and}\quad \text{for } x>\mu,\ x\frac{r'(x)}{r(x)}>p-1. \tag{A.7}$$

On the other hand,

$$U'(x)=\frac{x^{p-2}}{r(x)}\left[p-1-x\frac{r'(x)}{r(x)}\right].$$

Combining these observations we obtain for $x<\mu$, $U'(x)>0$ and for $x>\mu$, $U'(x)<0$, but these observations contradict Lemma A.1(a). That is, the crossing point $\mu$ does not exist, from which the required result follows. □

**Remark A.1.** The inequalities in (i), (ii) and (iii) of Lemma A.2 are reversed for $p<q$.

**Lemma A.3.** *Let $X\sim GG(p,q)$, $p,q>0$, then $xr(x)$ is an increasing function.*

**Proof.** The derivative of $xr(x)$ is

$$(xr(x))'=r(x)\left(1+\frac{xr'(x)}{r(x)}\right).$$

Now, the proof of the required result simply follows from Lemma A.2(i) and (iii) and Remark A.1. □

**Lemma A.4.** *Let $X\sim GG(p,q)$, $p,q<1$, with hazard rate $r(x)$, then $x^2r'(x)$ is a deceasing function.*

**Proof.** We prove the result for the case when $q<p$. The proof for the case when $q>p$ is similar and hence is omitted. We write

$$x^2 r'(x) = (xr(x))\left(x\frac{r'(x)}{r(x)}\right)$$

derivative of this function with respect to $x$ is

$$r(x)\left(1+x\frac{r'(x)}{r(x)}\right)\left[x\frac{r'(x)}{r(x)}+xr(x)-p^2\frac{x^p}{1+x\frac{r'(x)}{r(x)}}\right]. \tag{A.8}$$

It follows from Lemma A.2 that $q<1+xr'(x)/r(x)<p$. Now, to prove the required result we show that

$$L(x)=\left[x\frac{r'(x)}{r(x)}+xr(x)-p^2\frac{x^p}{1+x\frac{r'(x)}{r(x)}}\right]\leq 0. \tag{A.9}$$

Using the above relations, Lopital's formula and some manipulations, we obtain that

$$\lim_{x\longrightarrow 0} L(x)=q-1 \tag{A.10}$$

and

$$\lim_{x\longrightarrow\infty} L(x)=p-1. \tag{A.11}$$

Suppose $\max_{x>0} L(x)>0$. Then, it follows from (A.10) and (A.11) that $L(x)$ has to cross the level 0 at least twice. The derivative of $L(x)$ with respect to $x$, after some simplifications, is

$$L'(x)=r(x)\left(1+x\frac{r'(x)}{r(x)}\right)\left[1-\frac{p^2x^{p-1}}{r(x)\left(1+x\frac{r'(x)}{r(x)}\right)}\right]+(r(x)+r'(x)x)-p^3\frac{x^{p-1}}{1+x\frac{r'(x)}{r(x)}}+\frac{p^2x^pr(x)}{1+x\frac{r'(x)}{r(x)}}\left[1-\frac{p^2x^{p-1}}{r(x)\left(1+x\frac{r'(x)}{r(x)}\right)}\right]. \tag{A.12}$$

At any point $v$ that $L(v)=0$, we have $vr'(v)/r(v)+vr(v)=p^2v^p/(1+vr'(v)/r(v))$. Using this in (A.12),

$$L'(v)=\left(1+v\frac{r'(v)}{r(v)}\right)\left(-2\frac{r'(v)}{r(v)}+\frac{1-p}{1+v\frac{r'(v)}{r(v)}}\times\frac{1}{v}\times\left[vr(v)+v\frac{r'(v)}{r(v)}\right]\right)>0.$$

But, this observation contradict both relations (A.10) and (A.11). That is, the function $x^2r'(x)$ is a decreasing function. □

The following interesting remark can be proved using the similar arguments used to prove Lemma A.4.

**Remark A.2.** Let $X\sim GG(p,q)$, $p,q>1$, with hazard rate $r(x)$, then $x^2r'(x)$ is an increasing function.

**Lemma A.5.** *Let $X\sim GG(p,q)$, $p,q>0$, with reverse hazard rate $\tilde{r}(x)$, then $x\tilde{r}(x)$ is a decreasing function.*

**Proof.** By definition,

$$x\tilde{r}(x)=\frac{x^qe^{-x^p}}{\int_0^x t^{q-1}e^{-t^p}\,dt} \tag{A.13}$$

and its derivative

$$(x\tilde{r}(x))'=\frac{x^{q-1}e^{-x^p}[q-px^p]\int_0^x t^{q-1}e^{-t^p}\,dt-x(x^{q-1}e^{-x^p})^2}{\left(\int_0^x t^{q-1}e^{-t^p}\,dt\right)^2}=\tilde{r}(x)[q-px^p-x\tilde{r}(x)]. \tag{A.14}$$

Let $\psi(x)=px^p+x\tilde{r}(x)$. That is, we have to show that $\psi(x)\geq q$ for all $x>0$. It is easy to see that

$$\lim_{x\longrightarrow\infty}\psi(x)=\infty \tag{A.15}$$

and

$$\lim_{x\longrightarrow 0}\psi(x)=q. \tag{A.16}$$

Now, suppose $\min_{x>0}\psi(x)<q$. Then $\psi(x)$ has to cross the level $q$ at least once, since $\psi(0)=q$ and $\psi(\infty)=\infty$. That is, there exist a point $v>0$ such that $\psi(v)=q$. Then, using this in

$$\psi'(x)=p^2x^{p-1}+\tilde{r}(x)(q-px^p-x\tilde{r}(x))$$

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we obtain that

$$\psi'(v) = p^2 v^{p-1} > 0.$$

Suppose that $v$ is the first point that $\psi(v) = q$. That is,

$$\text{for } x < v \text{ we have } \psi(x) \le q. \tag{A.17}$$

Then for $x < v$, $\psi(x)$ is increasing and $\psi(x) > \psi(0) = q$. That is a contradiction with $\psi(x) < q$. Then for $x < v$, $\psi(x) > q$, but this is impossible, since the derivative of $\psi(x)$ at crossing point $v$ is positive. That is, such a point $v$ does not exist. This proves that $\psi(x) > q$ and then according to (A.14), the function $x\tilde{r}(x)$ is decreasing and the required result is proved. □

The following lemma for power-generalized Weibull distribution can be proved using similar kind of arguments used above.

**Lemma A.6.** *Let $X \sim PGW(v,\gamma)$, $v,\gamma > 0$, with reverse hazard rate $\tilde{r}(x)$, then $x\tilde{r}(x)$ is a decreasing function.*

**Lemma A.7.** *Let $\tilde{r}(x)$ be the hazard rate of a generalized gamma random variable with shape parameters $p,q > 0$. Then*

$$\text{(a)} \quad \lim_{x\longrightarrow 0} x\tilde{r}(x) = q \quad \textit{and} \quad \lim_{x\longrightarrow \infty} x\tilde{r}(x) = 0, \tag{A.18}$$

$$\text{(b)} \quad \lim_{x\longrightarrow 0} x\frac{\tilde{r}'(x)}{\tilde{r}(x)} = -1 \quad \textit{and} \quad \lim_{x\longrightarrow \infty} x\frac{\tilde{r}'(x)}{\tilde{r}(x)} = -\infty. \tag{A.19}$$

**Proof.** The reverse hazard rate of $GG(p,q)$ is

$$\tilde{r}(x) = \frac{x^{q-1}e^{-x^p}}{\int_0^x t^{q-1}e^{-t^p}\,dt}, \tag{A.20}$$

its derivative with respect to $x$ is

$$\tilde{r}'(x) = \frac{x^{q-2}e^{-x^p}(q-1-px^p)\int_0^x t^{q-1}e^{-t^p}\,dt - (x^{q-1}e^{-x^p})^2}{\left(\int_0^x t^{q-1}e^{-t^p}\,dt\right)^2}.$$

Combining these observation, we have

$$x\frac{\tilde{r}'(x)}{\tilde{r}(x)} = q-1-px^p - x\tilde{r}(x) \tag{A.21}$$

and

$$\left(x\frac{\tilde{r}'(x)}{\tilde{r}(x)}\right)' = \tilde{r}(x)\left(-1-p^2\frac{x^{p-1}}{\tilde{r}(x)} - x\frac{\tilde{r}'(x)}{\tilde{r}(x)}\right), \tag{A.22}$$

where $(xr'(x)/r(x))'$ denote the derivative of $x\tilde{r}'(x)/\tilde{r}(x)$ with respect to $x$. Now the required results follow from similar kind of arguments used to prove Lemma A.1. □

**Lemma A.8.** *Let $X \sim GG(p,q)$, $p < 1$, with reverse hazard rate $\tilde{r}(x)$, then $x^2\tilde{r}'(x)$ is an increasing function.*

**Proof.** Derivative of negative function

$$x^2\tilde{r}'(x) = (x\tilde{r}(x))\left(x\frac{\tilde{r}'(x)}{\tilde{r}(x)}\right)$$

with respect to $x$ is

$$\tilde{r}(x)\left(1 + x\frac{\tilde{r}'(x)}{\tilde{r}(x)}\right)\left[x\frac{\tilde{r}'(x)}{\tilde{r}(x)} - x\tilde{r}(x) - p^2\frac{x^p}{1 + x\frac{\tilde{r}'(x)}{\tilde{r}(x)}}\right]. \tag{A.23}$$

In Lemma A.5 we proved that $x\tilde{r}(x)$ is decreasing from which it follows that $x\tilde{r}'(x)/\tilde{r}(x) < -1$. Now we prove that

$$L(x) = \left[x\frac{\tilde{r}'(x)}{\tilde{r}(x)} - x\tilde{r}(x) - p^2\frac{x^p}{1 + x\frac{\tilde{r}'(x)}{\tilde{r}(x)}}\right] \le 0, \tag{A.24}$$

which completes the proof of required result. Using (A.21) in (A.24), for $p > q$, we obtain that

$$\lim_{x\longrightarrow 0} L(x) = p-1 \le 0. \tag{A.25}$$

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On the other hand, part(a) of Lemma A.7 implies that

$$\lim_{x \longrightarrow \infty} L(x) = -\infty. \tag{A.26}$$

Suppose $\max_{x>0} L(x) > 0$. Then, it follows from (A.25) and (A.26) that $L(x)$ has to cross the level 0 at least twice. The derivative of $L(x)$ with respect to $x$ is

$$L'(x) = -p^2 \frac{x^{p-1}}{1+x\frac{\tilde{r}'(x)}{\tilde{r}(x)}}\left(1+x\frac{\tilde{r}'(x)}{\tilde{r}(x)}\right) - 2\tilde{r}(x)\left(1+x\frac{\tilde{r}'(x)}{\tilde{r}(x)}\right) - p^3 \frac{x^{p-1}}{1+x\frac{\tilde{r}'(x)}{\tilde{r}(x)}}\left(1+x\frac{\tilde{r}'(x)}{\tilde{r}(x)}\right) - p^4 \frac{x^{2p-1}}{1+x\frac{\tilde{r}'(x)}{\tilde{r}(x)}}\left(1+x\frac{\tilde{r}'(x)}{\tilde{r}(x)}\right)$$
$$-p^2 \frac{x^{p-1}\tilde{r}(x)}{1+x\frac{\tilde{r}'(x)}{\tilde{r}(x)}}\left(1+x\frac{\tilde{r}'(x)}{\tilde{r}(x)}\right). \tag{A.27}$$

At any point $v$, where $L(v) = 0$, we have $v\tilde{r}'(v)/\tilde{r}(v) + v\tilde{r}(v) = p^2 v^p/(1+v\tilde{r}'(v)/\tilde{r}(v))$. Using this observation in (A.27),

$$L'(v) = -2\left(1+v\frac{\tilde{r}'(v)}{\tilde{r}(v)}\right)\frac{\tilde{r}'(v)}{\tilde{r}(v)} + (1-p)\left(v\frac{\tilde{r}'(v)}{\tilde{r}(v)} - \tilde{r}(v)\right) < 0.$$

But, this observation contradict both relations (A.25) and (A.26). That is, the function $x^2\tilde{r}'(x)$ is an increasing function. □

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