COMMUN. STATIST.-THEORY METH., 19(1), 299-306 (1990)

SOME PARTIAL ORDERING RESULTS ON RECORD VALUES

Subhash C. Kochar

Department of Statistics and Actuarial Science
University of Iowa, Iowa City, IA 52242

Key Words and Phrases: Nonhomogeneous Poisson process; IFR; DFR; dispersive ordering; likelihood ratio ordering; stochastic ordering; minimal repair process.

ABSTRACT

It has been shown that if the nth record value R_n of a sequence of i.i.d. r.vs. has an increasing failure rate (IFR) distribution then so does R_{n+1}. On the other hand R_{n-1} has decreasing failure rate (DFR) distribution if R_n is DFR. Some other partial ordering results for the record values have also been obtained.

1. INTRODUCTION

Let $\{X_i, i \geq 1\}$ be a sequence of independent and identically distributed random variables from a distribution with common distribution function F(x). We assume that F is continuous with $F(0) = 0$. X_n is called a (upper) record value of this sequence if $X_n > X_i$ for $i = 1,2,\ldots,n-1$. By convention X_1 is a record value. The serial numbers at which record values occur are given by the random variables $\{L_n, n \geq 1\}$ defined recursively by $L_1 = 1$, $L_n = \min\{k : k > L_{n-1}, X_k > X_{L_{n-1}}\}$, $n \geq 2$. $\{L_n, n \geq 1\}$ is called the sequence of (upper) record times and $\{X_{L_n}, n \geq 1\}$ the sequence of (upper) record values corresponding to $\{X_n, n \geq 1\}$.

For the convenience of notation we shall denote X_{L_n} by R_n so that $\{R_n, n \geq 1\}$ is the sequence of record values.

There is an intimate connection between record values and occurrence times of a nonhomogeneous Poisson process. The sequence $\{R_n, n \geq 1\}$ has the same distribution as the occurrence times of a nonhomogeneous Poisson process with mean value function $M(t) = -\ln \bar{F}(t)$, where $\bar{F}(t) = 1-F(t)$. The converse is also true. If the mean value function $M(t)$ of a nonhomogeneous Poisson process is continuous and tends to ∞ as $t \longrightarrow \infty$, then the sequence of occurrence times of this process can be considered as the record values of a sequence of independent and identically distributed random variables each having distribution function $F(t) = 1 - \exp(M(t))$, $t \geq 0$. Thus the study of record values is essentially the study of the occurrence times of a nonhomogeneous Poisson process.

Ascher and Feingold (1984, page 51) provide an excellent survey and discussion of the role of the nonhomogeneous Poisson processes in the modeling of minimal repair systems and reliability growth. For a decent review of the above, refer to Gupta and Kirmani (1988) and to Glick (1978) for additional results and references.

Gupta and Kirmani (1988) have proved many interesting results for the occurrence and interoccurrence times of a nonhomogeneous Poisson process. In particular, they have proved that the record values from an increasing failure rate (IFR) distribution have IFR distributions. This result has been generalized in this note. It is shown in Section 2 that

R_{n+1} is IFR if R_n is IFR and R_{n-1} is DFR if R_n is DFR. In the third section, some additional results on record values have been obtained.

2. IFR AND DFR PROPERTIES OF RECORD VALUES

We first prove a lemma

Lemma 2.1: For $x > 0$

$$\varphi(x) = \left[\sum_{j=0}^{n-1} \frac{x^{j+1}}{j!}\right]\left[\sum_{j=0}^{n} \frac{x^j}{j!}\right]^{-1} \tag{2.1.}$$

is nondecreasing in x.

Proof: Differentiating $\varphi(x)$, we have

$$\varphi'(x) = \left[\sum_{j=0}^{n} \frac{x^j}{j!}\right]^{-2} \left[\left[\sum_{j=0}^{n} \frac{x^j}{j!}\right]\left[\sum_{j=0}^{n-1} (j+1)\frac{x^j}{j!}\right] - \left[\sum_{j=0}^{n-1} \frac{x^{j+1}}{j!}\right]\left[\sum_{j=0}^{n} \frac{x^{j-1}}{j!}\right]\right]$$

The second factor is simplified to

$$\left\{\sum_{j=0}^{n} \frac{x^j}{j!}\right\}\left\{\sum_{j=1}^{n-1} \frac{x^j}{(j-1)!} + \sum_{j=0}^{n-1} \frac{x^j}{j!}\right\} - \left\{\sum_{j=0}^{n-1} \frac{x^{j+1}}{j!}\right\}\left\{\sum_{j=1}^{n} \frac{x^{j-1}}{(j-1)!}\right\}$$

$$= \left[\sum_{j=0}^{n-2} \frac{x^j}{j!}\right]\left[\frac{x^{n+1}}{n!}\right] - \frac{x^n}{(n-1)!}\sum_{j=0}^{n-1}\frac{x^j}{j!} + \left[\sum_{j=0}^{n} \frac{x^j}{j!}\right]\left[\sum_{j=0}^{n-1} \frac{x^j}{j!}\right]$$

$$\equiv \sum_{k=0}^{2n-1} c_k x^k \quad \text{say}$$

Obviously for $k \le n$, $c_k \ge 0$. For $n+1 \le k \le 2n-1$

$$c_k = \sum_{j=k-n}^{n-1} \frac{1}{j!(k-j)!} \frac{2n-k}{n!(k-n)!} \ge 0,$$

since for $k-n \le j \le n-1$, the terms $a_j = [j!(k-j)!]^{-1}$ are first increasing and then decreasing taking their minima at $j = k-n$ for $k-n \le j \le n-1$. As a result each term in the summation is greater than the first.

Therefore, $\varphi'(x) > 0$ and hence $\varphi(x)$ is increasing.

Theorem 2.1: (a) If R_n is IFR, then R_{n+1} is also IFR

(b) If R_n is DFR, then R_{n-1} is also DFR

Proof: Let

$$M(t) = -\log \bar{F}(t). \tag{2.2}$$

The density function f_n and the distribution function F_n of R_n are, respectively,

$$f_n(t) = \frac{[M(t)]^{n-1}}{\Gamma(n)} \cdot f(t) \quad \text{and} \quad F_{(n)}(t) = 1 - \bar{F}(t) \sum_{k=0}^{n-1} \frac{[M(t)]^k}{k!} \tag{2.3}$$

(see Glick (1974) and Nagaraja (1982)). The failure rate of R_n is

$$r_{(n)}(t) = \frac{[M(t)]^{n-1}}{\Gamma(n) \sum_{j=0}^{n-1} \frac{[M(t)]^j}{j!}} \frac{f(t)}{\bar{F}(t)} \tag{2.4}$$

Hence

$$\frac{r_{(n+1)}(t)}{r_{(n)}(t)} = \frac{\sum_{j=0}^{n-1} \frac{[M(t)]^{j+1}}{j!}}{n \sum_{j=0}^{n} \frac{[M(t)]^{j}}{j!}} = \frac{1}{n}\,\phi(M(t)).$$

Since $M(t)$ is nondecreasing, it follows from the above lemma, that this ratio of failure rates $\frac{r_{(n+1)}(t)}{r_{(n)}(t)}$ is nondecreasing in t. Therefore, if $r_{(n)}(t)$ is increasing in t, then $r_{(n+1)}(t)$ is also increasing in t. This proves (a). The proof of (b) follows similarly.

Since $r_{(1)}(t) = r(t)$, the failure rate of F, this theorem implies that the record values from an IFR distribution are IFR, as already proved by Gupta and Kirmani (1988).

<u>Theorem 2.2</u>: Let F be strictly increasing. Then not all record values can have DFR distribution

<u>Proof</u>: Let $x_2 > x_1$. Then

$$\frac{r_{(n)}(x_2)}{r_{(n)}(x_1)} = \left[\frac{M(x_2)}{M(x_1)}\right]^{n-1} \frac{\sum_{j=0}^{n-1} \frac{[M(x_1)]^j}{j!}}{\sum_{j=0}^{n-1} \frac{[M(x_2)]^j}{j!}} \; \frac{f(x_2)}{f(x_1)} \; \frac{\bar{F}(x_1)}{\bar{F}(x_2)} \longrightarrow \infty \text{ as } n \longrightarrow \infty$$

since the second factor goes to $e^{M(x_1)-M(x_2)}$ and $M(x_2) > M(x_1)$. Hence there exist some n such that R_n is not DFR.

Takahasi (1988) has obtained similar results for order statistics.

3. SOME MONOTONICITY RESULTS

Gupta and Kirmani (1988) have obtained some general monotonicity results for record values and their increments. Results of this section supplement theirs.

Theorem 3.1: For $n > m$, R_m is likelihood ratio ordered with respect to R_n ($R_m \overset{L.R.}{<} R_n$) in the sense that $f_{(n)}(t)/f_{(m)}(t)$ is nondecreasing in t.

Proof: From (2.5)

$$\frac{f_{(n)}(t)}{f_{(m)}(t)} = \frac{\Gamma(m)}{\Gamma(n)} [M(t)]^{n-m}$$

The required result follows since $M(t)$ is nondecreasing in t. This observation leads to the following interesting corollary.

Corollary 3.1: For $n > m$

(a) The failure rate of R_n is smaller than that of R_m.

(b) If R_m is DFR, then R_n is more dispersed than R_m ($R_m \overset{disp}{<} R_n$) in the sense that $F^{-1}_{(n)}F_{(m)}(t) - t$ is nondecreasing in t and as a consequence $\text{var}(R_m) \le \text{var}(R_n)$.

Proof: (a) follows from the relationships between the various partial orderings (see, Ross (1983)).

(b) follows from Theorem 2.1(b) of Bagai and Kochar (1986).

It follows from the above corollary that if we are sampling from a DFR distribution, the variance of any record value cannot be less than the variance of the parent distribution.

Below, we obtain some stochastic ordering results between the interoccurrence times of a nonhomogeneous Poisson process.

Theorem 3.2: Let $G \overset{disp}{<} F$ and let $R_n(S_n)$ be the nth record value from $F(G)$. Then

$$R_{n+k} - R_k \overset{st}{\geq} S_{n+k} - S_k, \quad n,k = 1,2,...$$

Proof: It is easy to observe using (2.3) that

$$F^{-1}G(S_n) \overset{dist}{=} R_n \tag{3.1}$$

Since $\Delta(x) = F^{-1}G(x) - x$ is nondecreasing and $S_{n+k} \geq S_n$, it follows that

$$F^{-1}G(S_{n+k}) - S_{n+k} \geq F^{-1}G(S_n) - S_n$$

or

$$F^{-1}G(S_{n+k}) - F^{-1}G(S_n) \geq S_{n+k} - S_n$$

Hence,

$$P[F^{-1}G(S_{n+k}) - F^{-1}G(S_n) > x] \geq P[S_{n+k} - S_n > x], \quad x > 0$$

The required results follows from this and (3.1).

As a consequence of this theorem and Theorem 2.1(b) of Bagai and Kochar (1986), we obtain the following corollary, which has also been proved in Gupta and Kirmani (1988).

Corollary 3.2: Let $r_F(r_G)$ be the failure rate of $F(G)$. Then if $r_F \leq r_G$

and if either F or G is DFR, then

$$R_{n+k} - R_n \overset{st}{\geq} S_{n+k} - S_n \quad \text{for } n,k = 1,2,\ldots$$

ACKNOWLEDGEMENTS

I am grateful to Professors Gupta and Kirmani for providing me a preprint of their paper. Thanks are due to Professor James Lynch and the referee for carefully reading the paper and making important suggestions on an earlier draft.

REFERENCES

Ascher, H.E. and Feingold, H. (1984). *Repairable Systems Reliability*, Marcel Dekker, New York.

Bagai, I, and Kochar, S.C. (1986). On tail–ordering and comparison of failure rates. *Commun. Statist. Theor. Meth.*, 15, 1377–1388.

Glick, N. (1978). Breaking records and breaking boards. *Amer. Math. Monthly*, 85, 2–26.

Gupta, R.C. and Kirmani, S.N.U.A. (1988). Closure and monotonicity properties of nonhomogeneous Poisson processes and record values. To appear in *Probability in the Engineering and Information Sciences*, 2, 475–484.

Nagaraja, H.N. (1982). Record values and extreme value distributions. *J. Appl. Prob.*, 19, 233–239.

Takahasi, K. (1988). A note on hazard rates of order statistics. *Commun. Statist. Theor. Meth.*, 17, 4133–4136.

Ross, S.M. (1983). *Stochastic Processes.* Wiley, New York.

Received April 1989; Revised November 1989.

Recommended by James Lynch, University of South Carolina, Columbia, SC.

Refereed Anonymously.