# Excess Wealth Order and Sample Spacings *


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**Abstract**

In this note, we further study the properties of excess wealth (or right spread) order and the location independent riskier order. It is proved that if $X$ is less variable than $Y$ according to excess wealth order, then $X_{n:\,n} - X_{k:\,n} \leq_{icx} Y_{n:\,n} - Y_{k:\,n}$ for $k = 0, 1, \cdots, n-1$, where $X_{0:\,n} = Y_{0:\,n} \equiv 0$. Similar results are obtained for location independent riskier order. An application in $k$-price business auction models is presented as well.



**Key Words** Auction; Right spread order; Increasing convex order; Location independent riskier order; Rent of winner; Sample range

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## 1 Introduction and motivation

The concept of variability is a basic one in statistics, probability and many other related areas, such as reliability theory, business, economics and actuarial science, among others.


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Most of the classical methods for variability comparisons are based only upon summary statistics such as variance and standard deviation which are usually quite noninformative though they are convenient to be dealt with. In the past two decades, several more refined stochastic orders which compare variabilities of random variables based on their entire distribution functions, have been introduced in the literature. *Shaked and Shanthikumar* (1994) and *Müller and Stoyan* (2002) present comprehensive discussions on most of those concepts and their properties. In this note, we further study some properties of a variability order, known as *excess wealth order* or *right spread order* as described below.

Let $X$ and $Y$ be two random variables with their distribution functions $F$ and $G$ and survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, respectively. Denote by $F^{-1}$ and $G^{-1}$ their corresponding right continuous inverses. A basic concept for comparing variability or spread between two probability distributions is that of dispersive ordering. $X$ is said to be *less dispersed* than $Y$, written as $X \leq_{disp} Y$ or $F \leq_{disp} G$, if $F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha)$ for all $0 < \alpha \leq \beta < 1$. *Muñoz-Perez* (1990) proved that

$$X \leq_{disp} Y \Longleftrightarrow (X - F^{-1}(p))^+ \leq_{st} (Y - G^{-1}(p))^+, \quad \text{for every } p \in (0,1), \tag{1}$$

where $(Z)^+ = \max\{Z, 0\}$ and the stochastic ordering $X \leq_{st} Y$ holds in the sense that $\bar{F}(x) \leq \bar{G}(x)$ for all $x$. Based on this observation, *Fernandez-Ponce et al.* (1998) proposed the *right spread ordering*, which is implied by dispersive ordering and hence is a weaker variability order.

**Definition 1** $X$ is said to be *less right spread out* than $Y$ ( $X \leq_{RS} Y$) if

$$\mathrm{E}[(X - F^{-1}(p))^+] \leq \mathrm{E}[(Y - G^{-1}(p))^+], \quad \text{for every } p \in (0,1), \tag{2}$$

provided the expectations exist. Or equivalently, if

$$\int_{F^{-1}(p)}^{\infty} \bar{F}(x)\,\mathrm{d}x \leq \int_{G^{-1}(p)}^{\infty} \bar{G}(x)\,\mathrm{d}x, \quad \text{for every } p \in (0,1). \tag{3}$$

This ordering was also independently studied in *Shaked and Shanthikumar* (1998) and was called as *excess wealth* ordering. Refer to *Kochar et al.* (2002) and *Li and Shaked* (2004) for its further properties.

Recall that $X$ is said to be smaller than $Y$ in the *increasing convex order* (denoted by $X \leq_{icx} Y$) if

$$\int_t^{+\infty} \bar{F}(x)\,\mathrm{d}x \leq \int_t^{+\infty} \bar{G}(x)\,\mathrm{d}x, \quad \text{for all } t.$$

Note that $X \leq_{st} Y \Longrightarrow X \leq_{icx} Y$, *Belzunce* (1999) developed the following useful characterization of the right spread order in terms of the increasing convex order,

$$X \leq_{RS} Y \Longleftrightarrow (X - F^{-1}(p))^+ \leq_{icx} (Y - G^{-1}(p))^+, \quad \text{for all } p \in (0,1).$$

The right hand side inequality is equivalent to

$$\int_t^{\infty} \bar{F}(x + F^{-1}(p))\,\mathrm{d}x \leq \int_t^{\infty} \bar{G}(x + G^{-1}(p))\,\mathrm{d}x, \quad \text{for all } p \in (0,1) \text{ and } t. \tag{4}$$

In order to compare two random assets in economics, *Jewitt* (1989) introduced the location independent riskier order as below, which is dual to the concept of *excess wealth order* and is also of interest in rank dependent expected utility frame work, see *Chateauneuf et al.*(2004).

**Definition 2** $X$ is said to be smaller than $Y$ in the *location independent riskier* order (denoted by $X \leq_{lir} Y$) if

$$\int_{-\infty}^{F^{-1}(p)} F(x)\,\mathrm{d}x \leq \int_{-\infty}^{G^{-1}(p)} G(x)\,\mathrm{d}x, \quad \text{for all } p \in (0,1). \tag{5}$$

It is of interest to study how a variability ordering between two probability distributions affects the relative positioning of the corresponding observations in random samples from the two distributions. Let $X_{1:\,n} \leq X_{2:\,n} \leq \cdots \leq X_{n:\,n}$ be the order statistics of a random sample $X_1, X_2, \ldots, X_n$ on $X$. Similarly, denote by $Y_{1:\,n} \leq Y_{2:\,n} \leq \cdots \leq Y_{n:\,n}$ the order statistics of a random sample $Y_1, Y_2, \ldots, Y_n$ on $Y$. Let $U_{i:\,n} \equiv X_{i:\,n} - X_{i-1:\,n}$ and $V_{i:\,n} \equiv Y_{i:\,n} - Y_{i-1:\,n}$ be the respective $i$-th sample spacings with $X_{0:\,n} = Y_{0:\,n} \equiv 0$, for $i = 1, \cdots, n$. *Bartoszewicz* (1986) proved that

$$X \leq_{disp} Y \Longrightarrow (U_{1:\,n}, \cdots, U_{n:\,n}) \leq_{st} (V_{1:\,n}, \cdots, V_{n:\,n}), \tag{6}$$

which implies

$$X_{j:\,n} - X_{i:\,n} \leq_{st} Y_{j:\,n} - Y_{i:\,n}, \quad \text{for } 1 \leq i < j \leq n. \tag{7}$$

This paper further investigates this problem and extends the implication in (7) when dispersive ordering between $X$ and $Y$ is replaced by either the excess wealth order or the location independent riskier order. It is shown in Section 2 that $X \leq_{ew} Y$ implies $X_{n:\,n} - X_{k:\,n} \leq_{icx} Y_{n:\,n} - Y_{k:\,n}$ for $1 \leq k \leq n-1$. A parallel result is proved in the case of the location independent riskier order. In particular, we will notice that the sample ranges are ordered according to the increasing convex order when the parent distributions are ordered according to the excess wealth order or the location independent riskier order. Some applications of these two results in the theory of reliability and business auction model are presented in Sections 2 and 3.

For the sake of convenience, throughout this note, the term *increasing* is used for *monotone nondecreasing* and *decreasing* is used for *monotone non-decreasing.* It is assumed in the sequel that all random variables involved are absolutely continuous and expectations exist when used.

## 2 Main results

The main results of this note are contained in the next two theorems.

**Theorem 3** If $X \leq_{RS} Y$, then,

$$X_{n:\,n} - X_{k:\,n} \leq_{icx} Y_{n:\,n} - Y_{k:\,n}, \quad \text{for } k = 0, 1, \cdots, n-1. \tag{8}$$

**Proof** The case $k = 0$ is proved in *Kochar et al.* (2002). Note that, for $1 \leq r < s \leq n$, the distribution $X_{s:\,n}$ given $X_{r:\,n} = x$ is the same as that of the $(s-r)$-th order statistic in a random sample of size $(n-r)$ from a distribution with pdf $f(y)/\bar{F}(x)$ for $y \geq x$. The survival function of $X_{n:\,n} - X_{k:n}$ is

$$\begin{aligned} \bar{H}_F(y) &= P[X_{n:\,n} - X_{k:\,n} \geq y] \\ &= \int_0^\infty P[X_{n:\,n} - X_{k:\,n} \geq y | X_{k:\,n} = x] \,\mathrm{d}F_{k:\,n}(x) \\ &= C(k,n) \int_0^\infty \left[ 1 - \left\{ \frac{\bar{F}(x) - \bar{F}(x+y)}{\bar{F}(x)} \right\}^{n-k} \right] F^{k-1}(x) \bar{F}^{n-k}(x) \,\mathrm{d}F(x) \\ &= C(k,n) \int_0^\infty \left[ \bar{F}^{n-k}(x) - \left\{ \bar{F}(x) - \bar{F}(x+y) \right\}^{n-k} \right] F^{k-1}(x) \,\mathrm{d}F(x) \\ &= C(k,n) \int_0^1 p^{k-1} \left[ (1-p)^{n-k} - \{ \bar{F}(y + F^{-1}(p)) - p \}^{n-k} \right] \,\mathrm{d}p, \end{aligned}$$

where $C(k,n)$ is a constant.

We have to prove that under the assumption of $X \leq_{RS} Y$,

$$\int_x^\infty \bar{H}_F(y) dy \leq \int_x^\infty \bar{H}_G(y) \,\mathrm{d}y. \tag{9}$$

By the characterization (4), $X \leq_{RS} Y$ is equivalent to

$$\int_t^\infty \left[ F\left(x + F^{-1}(p)\right) - G\left(x + G^{-1}(p)\right) \right] \,\mathrm{d}x \geq 0 \quad \text{for all } x \geq 0.$$

Since

$$h(y,p) = \sum_{i=0}^{n-k-1} \left[ F\left(y + F^{-1}(p)\right) - p \right]^i \left[ G\left(y + G^{-1}(p)\right) - p \right]^{n-k-1-i}$$

is nonnegative and increasing in $x$ for any fixed $p \in (0,1)$, by Lemma 7.1(a) of *Barlow and Proschan* (1981), it holds that for any $t$ and $p \in (0,1)$,

$$\int_t^\infty \left[ F\left(y + F^{-1}(p)\right) - G\left(y + G^{-1}(p)\right) \right] h(y,p) \,\mathrm{d}y \geq 0.$$

That is,

$$\int_t^\infty \left\{\left[F\left(y+F^{-1}(p)\right)-p\right]^{n-k}-\left[G\left(y+G^{-1}(p)\right)-p\right]^{n-k}\right\}\,\mathrm{d}y \geq 0.$$

Thus,

$$\int_0^1 p^{k-1}\int_t^\infty \left\{\left[F\left(y+F^{-1}(p)\right)-p\right]^{n-k}-\left[G\left(y+G^{-1}(p)\right)-p\right]^{n-k}\right\}\,\mathrm{d}y\,\mathrm{d}p \geq 0.$$

Interchanging the order of integration, this becomes, for all $t$,

$$\int_t^\infty \int_0^1 p^{k-1}\left\{\left[F\left(y+F^{-1}(p)\right)-p\right]^{n-k}-\left[G\left(y+G^{-1}(p)\right)-p\right]^{n-k}\right\}\,\mathrm{d}p\,\mathrm{d}y \geq 0,$$

which is equivalent to

$$\begin{aligned} &\int_t^\infty \int_0^1 \left\{(1-p)^{n-k}-\left[F\left(y+F^{-1}(p)\right)-p\right]^{n-k}\right\}p^{k-1}\,\mathrm{d}p\,\mathrm{d}y \\ \leq\ &\int_t^\infty \int_0^1 \left\{(1-p)^{n-k}-\left[G\left(y+G^{-1}(p)\right)-p\right]^{n-k}\right\}p^{k-1}\,\mathrm{d}p\,\mathrm{d}y, \end{aligned}$$

which in turn is equivalent to (9). This proves the desired result. ■

As observed in *Belzunce* (1999) and *Fagiuoli et al.* (1999),

$$X \leq_{lir} Y \quad \text{if and only if} \quad -X \leq_{RS} -Y.$$

By Theorem 3, if $X \leq_{lir} Y$, we have, for $1 \leq k \leq n$,

$$(-X)_{n:\,n}-(-X)_{k-1:\,n} \leq_{icx} (-Y)_{n:\,n}-(-Y)_{k-1:\,n}.$$

Since $(-X)_{n:\,n}-(-X)_{k:\,n} \stackrel{st}{=} X_{n-k+1:\,n}-X_{1:\,n}$, for all $1 \leq k \leq n$, the next result parallel to Theorem 3 follows immediately.

**Theorem 4** If $X \leq_{lir} Y$, then,

$$X_{k:\,n}-X_{1:\,n} \leq_{icx} Y_{k:\,n}-Y_{1:\,n}, \quad \text{for } k=2,\cdots,n. \tag{10}$$

Specifically, setting $k=n$ in (8) and (10), respectively, we immediately have the following corollary.

**Corollary 5** If $X \leq_{RS} Y$ or $X \leq_{lir} Y$, then, $X_{n:\,n}-X_{1:\,n} \leq_{icx} Y_{n:\,n}-Y_{1:\,n}$ for $n \geq 1$.

Recall that $X$ is NBUE (*new better than used in expectation*) if and only if $X \le_{ew} Y$, where $Y$ is exponential with the same mean as that of $X$ (see *Belzunce*, 1999). It follows directly from Theorem 3 that, if $X_1, \ldots, X_n$ is a random sample from a distribution which is NBUE, then,

$$\mathrm{E}[X_{n:\,n} - X_{k:\,n}] \le \left[1 + \cdots + \frac{1}{n-k}\right] \mathrm{E}[X], \quad \text{for any } 1 \le k \le n-1.$$

In the literature on applied probability and statistics, many authors have examined the effect of relative aging of two distributions on the variability of their sample observations. See for example, *Barlow and Proschan* (1981) and *Kochar and Wiens* (1987), among others. Recall that $X$ is said to be star-shaped with respect to $Y$ and denoted by $X \le_* Y$ if, $G^{-1}F(t)/t$ is increasing in $t$. In this case we also say that $X$ is more IFRA (increasing failure rate average) than $Y$. For the properties of star-shaped ordering, please refer to *Barlow and Proschan* (1981) and *Müller and Stoyan* (2002). *Bartoszewicz* (1998) claimed in Corollary 2 that

$$X \le_* Y \Longrightarrow \frac{\mathrm{E}[X_{n:\,n} - X_{1:\,n}]}{\mathrm{E}[X_1]} \le \frac{\mathrm{E}[Y_{n:\,n} - Y_{1:\,n}]}{\mathrm{E}[Y_1]}. \tag{11}$$

Further, *Li and Zuo* (2004) successfully relaxed the star-shaped ordering assumption in (11) to the NBUE (new better than used in expectation) ordering $X \le_{nbue} Y$ (*Kochar and Wiens* , 1987). Note that star ordering implies NBUE ordering. Taking into account the fact that

$$X \le_{nbue} Y \Longleftrightarrow \frac{X}{\mathrm{E}[X]} \le_{RS} \frac{Y}{\mathrm{E}[Y]},$$

(*Kochar et al.* 2002), the next corollary follows directly from Theorem 3 and thus presents a more general version of the moment inequality in (11).

**Corollary 6** $\quad X \le_{nbue} Y \Longrightarrow \dfrac{X_{n:\,n} - X_{k:\,n}}{\mathrm{E}[X]} \le_{icx} \dfrac{Y_{n:\,n} - Y_{k:\,n}}{\mathrm{E}[Y]}$, for all $1 \le k \le n-1$.

Another useful notion of variability is the dilation order, which was studied in *Belzunce et al.* (1997). $X$ is said to be smaller than $Y$ in the *dilation order* (denoted by $X \le_{dil} Y$), if $\mathrm{E}\phi(X - \mathrm{E}X) \le \mathrm{E}\phi(Y - \mathrm{E}Y)$ for all convex functions $\phi$. Readers can refer to *Fagiuoli et al.* 1999) for other details. In particular, they have shown that $X \le_{ew} Y \Longrightarrow X \le_{dil} Y$, from which one can prove that if the supports of $X$ and $Y$ are bounded from below and $\ell_X$, $\ell_Y$, their left end points of the supports satisfy $\ell_X \le \ell_Y$, then,

$$X \le_{ew} Y \Longrightarrow X \le_{icx} Y. \tag{12}$$

On the other hand, *Fagiuoli et al.* (1999) proved that

$$X \leq_{lir} Y \Longrightarrow X \leq_{dil} Y. \tag{13}$$

In view of (12) and (13), one may wonder whether Theorems 3 and 4 are still valid with the excess wealth order and the location independent riskier order replaced by the increasing convex order and the dilation order, respectively. The following example gives a negative answer.

**Example 7** (*Shaked and Shanthikumar*, 1994) For a fixed $\varepsilon \in (0, \frac{1}{3})$ and a number $M \geq 2$, let $X$ assign probability masses $\varepsilon$, $\frac{2}{3}$ and $\frac{1}{3} - \varepsilon$ at the points 0, 1 and $M$, respectively. Let $Y$ assign probability masses $\frac{1}{3}+\varepsilon$, $\frac{1}{3}$ and $\frac{1}{3}-\varepsilon$ at the points 0, 2 and $M$, respectively. It is obvious that $\mathrm{E}[X] = \mathrm{E}[Y]$. In Example 3.9 (*Shaked and Shanthikumar*, 1998), it is stated that $X \leq_{cx} Y$ but $X \not\leq_{RS} Y$ for larger number $M$. Direct evaluation also reveals that $X \leq_{dil} Y$.

It can be easily evaluated that

$$\int_0^\infty F^2(x)\bar{F}(x)\,\mathrm{d}x = \varepsilon^2(1-\varepsilon) + \left(\frac{2}{3}+\varepsilon\right)^2\left(\frac{1}{3}-\varepsilon\right)(M-1),$$

$$\int_0^\infty G^2(x)\bar{G}(x)\,\mathrm{d}x = 2\varepsilon\left(\frac{1}{3}+\varepsilon\right)^2 + \left(\frac{2}{3}+\varepsilon\right)^2\left(\frac{1}{3}-\varepsilon\right)(M-2).$$

Then,

$$\mathrm{E}(Y_{3:3}-Y_{2:3}) - \mathrm{E}(X_{3:3}-X_{2:3}) = -\frac{4}{9} + \frac{2}{3}\varepsilon + 4\varepsilon^2 + 12\varepsilon^3,$$

which is negative for smaller $\varepsilon \geq 0$. So, neither the convex order nor the increasing convex order imply (8).

On the other hand,

$$\int_0^\infty F(x)\bar{F}^2(x)\,\mathrm{d}x = \varepsilon(1-\varepsilon)^2 + \left(\frac{2}{3}+\varepsilon\right)\left(\frac{1}{3}-\varepsilon\right)^2(M-1),$$

$$\int_0^\infty G(x)\bar{G}^2(x)\,\mathrm{d}x = 2\left(\frac{1}{3}+\varepsilon\right)\left(\frac{2}{3}-\varepsilon\right)^2 + \left(\frac{2}{3}+\varepsilon\right)\left(\frac{1}{3}-\varepsilon\right)^2(M-2).$$

Then, as $\varepsilon \to \frac{1}{3}$,

$$\mathrm{E}(Y_{2:3}-Y_{1:3}) - \mathrm{E}(X_{2:3}-X_{1:3}) = \frac{2}{3} - \varepsilon - 6\varepsilon^2 + 6\varepsilon^3 \to -\frac{1}{9}.$$

As a result, for some $\varepsilon$ near $\frac{1}{3}$, the above difference may be negative. That is, the dilation order $X \leq_{dil} Y$ is not a sufficient condition for (10) to hold. ■

# 3 An application in auction theory

In an auction with a setup in which a seller and a number of buyers gather to the auction of some good, all bidders respectively submit their own bids for the good, which are known only to themselves. The most favorable one will be awarded the good at a price that is some function of the submitted bids. Let $X_1, \cdots, X_n$ be the sample of bidders' valuations. If prices are bid in an ascending sequence by individual bidders until only one bidder remains, the highest one, and the price paid by the winner is the $(n-k+1)$-th largest price reached in the sequence, this is called a $k$-price buyer's auction. The *rent of the winner*, which is the difference between the largest price reached and the $k$-th largest price reached from bidders, can be characterized as the sample spacing $X_{n:\,n} - X_{k:\,n}$. If the prices are bid in an descending sequence by individual bidders until only one bidder remains, the lowest bidder, who is awarded the good at a price corresponding to the $k$-th largest price reached in the sequence, this is called a $k$-price reverse auction. The *rent of the winner* is then the sample spacing $X_{k:\,n} - X_{1:\,n}$, the difference between the $k$-th smallest price and the smallest one from bidders.

Typically, the first price auction with $k = 1$ and the second price auction with $k = 2$ are very popular in practical situations. *Monderer and Tennenholtz* (2000) suggested that such auctions may play an important role in the new economics evolving in the internet and are widely used as a selling mechanism for relatively cheap items like TV sets or computer products. *Paul and Gutierrez* (Theorem 4, 2004) proved that, with the assumption of $\mathrm{E}[X] = \mathrm{E}[Y]$,

$$X \le_* Y \Longrightarrow \mathrm{E}[X_{k:\,n} - X_{k-1:\,n}] \le \mathrm{E}[Y_{k:\,n} - Y_{k-1:\,n}], \quad k = 1, 2, \cdots, n,$$

and hence

$$\mathrm{E}[X_{k:\,n} - X_{1:\,n}] \le \mathrm{E}[Y_{k:\,n} - Y_{1:\,n}], \quad \mathrm{E}[X_{n:\,n} - X_{k:\,n}] \le \mathrm{E}[Y_{n:\,n} - Y_{k:\,n}].$$

This claims that an increase of the bid in the sense of star-shaped order will result in an increase of the expected winner's rent in both the $k$-price buyer's auction and the $k$-price reverse auction. Recently, *Li* (2005) further showed that

$$X \le_{RS} Y \Longrightarrow \mathrm{E}[X_{n:\,n} - X_{n-1:\,n}] \le \mathrm{E}[Y_{n:\,n} - Y_{n-1:\,n}],$$

which states that in the second price buyer's auction an increase of the bid in the sense of the right spread order results in an increase in the expected winner's rent.

According to Theorems 3 and 4 in Section 2, we draw the stronger conclusions that in a $k$-price buyer's auction, an increase of the bid in the sense of excess wealth order will result in an increase of the winner's rent in the sense of increasing convex order. And in a $k$-price reverse auction, an increase of the bid in the sense of the location independent riskier order will result in an increase of the winner's rent in the sense of increasing convex order as well.


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