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Abstract: Let X_1, ..., X_n be a random sample from an absolutely continuous distribution with non-negative support, and let Y_1, ..., Y_n be mutually independent lifetimes with proportional hazard rates. Let also X_(1) < ... < X_(n) and Y_(1) < ... < Y_(n) be their associated order statistics. It is shown that the pair (X_(1), X_(n)) is then more dependent than the pair (Y_(1), Y_(n)), in the sense of the right-tail increasing ordering of Avérous and Dortet-Bernadet [Canad. J. Statist., 2000]. Elementary consequences of this fact are highlighted.

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Louvain-la-Neuve, March 5, 2007

Dr Jan de Leeuw, Editor
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Dear Jan,

Attached please find, in PDF format, a revision of JMVA manuscript no 06-249 entitled "On the dependence between the extreme order statistics in the proportional hazards model," written in collaboration with Ali Dolati and Subhash C. Kochar.

Also enclosed is a point-by-point response sheet indicating how we dealt with the referee's minor comments in the revision.

We trust that the new version of the manuscript will be satisfactory and thank you for your interest in publishing this work in JMVA.

Best regards,

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Response to the referee's report on
JMVA manuscript no 06–249

The referee and Associate Editor recommended acceptance of this submission, subject to a minor revision. The reviewer simply asked for some condensation and pointed out two typographical errors.

Condensation

The referee asked for some reduction in the size of the manuscript but did not set a specific goal. His/her only suggestion was that the proof of the main result could possibly be made shorter if it weren't divided into subsections.

As the argument breaks up naturally into logical parts, we have a strong preference for keeping the division in subsections; in our view, it makes the exposition crisp and easy to follow. Very little space would actually be gained by eliminating the titles of the subsections anyway.

In fact, Section 3 of the revision includes one more short subsection because after our original submission, we made a simple observation that led to a significant extension of our main result. The additional argument is given in a new §3.1, whose content does not affect the rest of the proof.

In order to avoid making the paper any longer, we reduced the length of the introduction. The new version is thus exactly the same length as the original submission, but the main result is now much more valuable, because the restrictive condition on the proportional hazards has been eliminated.

We hope you agree with our course of action.

Typographical errors

1. At the request of the referee, the condition $(r_1 + \cdots + r_n)/n = r$ was replaced by $(\lambda_1 + \cdots + \lambda_n)/n = 1$ wherever appropriate in the manuscript.

2. The referee also asked to change "Sathe (1988)" for "Sathe (1980)" on p. 2, but upon verification, Sathe's paper was published in 1988. The reference to his paper is now consistent throughout the manuscript.

Furthermore, the text was thoroughly checked and a few additional typos were corrected.

We are grateful to the Associate Editor and referee(s) for their rapid and thoughtful treatment of our submission.

On the dependence between the extreme order statistics in the proportional hazards model

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Abstract

Let $X_1, \ldots, X_n$ be a random sample from an absolutely continuous distribution with non-negative support, and let $Y_1, \ldots, Y_n$ be mutually independent lifetimes with proportional hazard rates. Let also $X_{(1)} < \cdots < X_{(n)}$ and $Y_{(1)} < \cdots < Y_{(n)}$ be their associated order statistics. It is shown that the pair $(X_{(1)}, X_{(n)})$ is then more dependent than the pair $(Y_{(1)}, Y_{(n)})$, in the sense of the right-tail increasing ordering of Avérous and Dortet-Bernadet [*Canad. J. Statist.*, 2000]. Elementary consequences of this fact are highlighted.

Key words: Concordance ordering; Correlation; Dispersive ordering; Exponential distribution; Kendall's tau; Monotone regression dependence; Proportional Hazards; Right-tail increasingness; Spearman's rho.

1 Introduction

Let $X_1, \ldots, X_n$ be a random sample of $n \geq 2$ mutually independent lifetimes with survival function $\bar{F} = 1 - F$, and let $X_{(1)} < \cdots < X_{(n)}$ be the associated order statistics. As is well known, $X_{(n+1-k)}$ then characterizes the stochastic behavior of the so-called "k-out-of-n" system, which is designed to work so long as $k \in \{1, \ldots, n\}$ of its components are operational.

In practice, of course, systems are often made up of components whose lifetimes $Y_1, \ldots, Y_n$ are mutually independent but whose survival functions $\bar{F}_1, \ldots, \bar{F}_n$ are different. It is thus of general interest to study the impact of heterogeneity on the characteristics of a stochastic system.

This note focuses on the relative degree of dependence between the pairs $(X_{(1)}, X_{(n)})$ and $(Y_{(1)}, Y_{(n)})$ of extreme order statistics respectively associated with sets of homogeneous and heterogeneous sets of survival times. This work was motivated in part by a result of Sathe (1988), who showed that

$$\mathrm{corr}(Y_{(1)}, Y_{(n)}) \leq \mathrm{corr}(X_{(1)}, X_{(n)})$$

when $X_1, \ldots, X_n$ form a random sample from the exponential distribution with hazard rate $\lambda > 0$ while $Y_1, \ldots, Y_n$ are mutually independent exponentials with distinct hazard rates $\lambda_1, \ldots, \lambda_n > 0$ such that $(\lambda_1 + \cdots + \lambda_n)/n = \lambda$.

Although this observation is interesting, it merely compares the relative degree of *linear* association within the two pairs. It is now widely recognized, however, that margin-free measures of association are more appropriate than Pearson's correlation, because they are based on the unique underlying copula which governs the dependence between the components of a continuous random pair. For a discussion, see, e.g., Embrechts et al. (2002) and references therein.

Specifically, what is showed here is that the pair $(X_{(1)}, X_{(n)})$ is more dependent than the pair $(Y_{(1)}, Y_{(n)})$, according to the right-tail increasing ordering of Avérous and Dortet-Bernadet (2000). This implies in particular that

$$\kappa(Y_{(1)}, Y_{(n)}) \leq \kappa(X_{(1)}, X_{(n)}),$$

where $\kappa(S, T)$ represents any concordance measure between random variables S and T in the sense of Scarsini (1984), e.g., Spearman's rho or Kendall's tau.

This result is established under the assumption that $X_1, \ldots, X_n$ are absolutely continuous with common density $F' = f$ and hazard rate $r = f/\bar{F}$ while $Y_1, \ldots, Y_n$ have proportional hazard rates. In other words, it is assumed that there exist a hazard rate $r^\star$ and constants $\lambda_1, \ldots, \lambda_n \in (0, \infty)$ such that for each $k \in \{1, \ldots, n\}$, Y_k has hazard rate $r_k = \lambda_k r^\star$.

Section 2 recalls the notions required to state the result formally. The proof is then given in Section 3. A few concluding remarks are made in the Discussion.

2 Preliminaries

For $i = 1, 2$, let (S_i, T_i) be a pair of continuous random variables with joint cumulative function H_i and margins F_i, G_i. Let also

$$C_i(u, v) = H_i\{F_i^{-1}(u), G_i^{-1}(v)\}, \quad u, v \in (0, 1)$$

be the unique copula associated with H_i. In other words, C_i is the distribution of the pair $(U_i, V_i) \equiv (F_i(S_i), G_i(T_i))$ whose margins are uniform on the interval $(0, 1)$. See, e.g., Chapter 1 of Nelsen (1999) for details.

By analogy with the univariate notion of stochastic dominance, copula C_1 is said to be less dependent than copula C_2 in the *positive quadrant dependence ordering* (PQD), denoted $(S_1, T_1) \prec_{\mathrm{PQD}} (S_2, T_2)$, if and only if

$$C_1(u, v) \leq C_2(u, v), \quad u, v \in (0, 1).$$

This condition implies that $\kappa(S_1, T_1) \leq \kappa(S_2, T_2)$ for all concordance measures meeting the axioms of Scarsini (1984); see, e.g., Tchen (1980).

A stronger dependence ordering called *right-tail increasingness* (RTI) is defined below in terms of the conditional distributions

$$C^R_{i,u}(v) = \frac{v - C_i(u, v)}{1 - u} = \mathrm{P}(V_i \leq v | U_i > u)$$

and their (right continuous) inverses $(C^R_{i,u})^{-1}$, $i = 1, 2$.

Definition 1 *T_1 is said to be less right-tail increasing in S_1 than T_2 is in S_2, denoted $(T_1|S_1) \prec_{\mathrm{RTI}} (T_2|S_2)$, if and only if*

$$C^R_{2,u_2} \circ (C^R_{2,u_1})^{-1}(w) \leq C^R_{1,u_2} \circ (C^R_{1,u_1})^{-1}(w)$$

for all $0 < u_1 < u_2 < 1$ and $w \in (0, 1)$.

This notion is a restriction to copulas of the ordering proposed by Avérous and Dortet-Bernadet (2000). (Note that contradictory results may occur when their concept is used to compare random pairs other than through their associated copulas.) The present definition is also distinct from the RTI ordering of Hollander et al. (1990), which is not a dependence ordering in the usual sense of Kimeldorf and Sampson (1989).

The RTI ordering is stronger than PQD in the sense that

$$(T_1|S_1) \prec_{\text{RTI}} (T_2|S_2) \quad \Rightarrow \quad (S_1,T_1) \prec_{\text{PQD}} (S_2,T_2).$$

The classical dispersive ordering between univariate distributions, whose definition is recalled below, also plays a role in the sequel. See, e.g., Chapter 3B of Shaked and Shanthikumar (2007) for further information in this regard.

Definition 2 *A random variable X with distribution function F is said to be less dispersed than another variable Y with distribution G, written $X \prec_{\text{DISP}} Y$, if and only if*

$$F^{-1}(\beta) - F^{-1}(\alpha) \le G^{-1}(\beta) - G^{-1}(\alpha)$$

for all $0 < \alpha \le \beta < 1$, where F^{-1} and G^{-1} denote the right-continuous inverses of F and G, respectively. Equivalently, one must have $F\{F^{-1}(w) + c\} \ge G\{G^{-1}(w) + c\}$ for every $c > 0$ and $w \in (0,1)$.

3 Main result

This section gives a proof of the following result.

Proposition 1 *Let $X_1,\ldots,X_n$ and $Y_1,\ldots,Y_n$ be two sets of mutually independent lifetimes. Assume that for $k \in \{1,\ldots,n\}$, X_k has hazard rate r and Y_k has hazard rate $r_k = \lambda_k r^\star$, where $\lambda_1,\ldots,\lambda_n > 0$. The dependence in the pair $(X_{(1)}, X_{(n)})$ of extreme order statistics from the homogeneous set is then larger than the dependence in the pair $(Y_{(1)}, Y_{(n)})$ of extreme order statistics from the heterogeneous set, in the sense that*

$$(Y_{(n)}|Y_{(1)}) \prec_{\text{RTI}} (X_{(n)}|X_{(1)}) \quad \textit{and} \quad (Y_{(1)}, Y_{(n)}) \prec_{\text{PQD}} (X_{(1)}, X_{(n)}).$$

The argument leading to Proposition 1 can be decomposed in five easy steps, as detailed below.

3.1 Reduction to the case $r^\star = r$ and $\bar\lambda = 1$

There is clearly no loss of generality in assuming that $\bar\lambda = (\lambda_1+\cdots+\lambda_n)/n = 1$. For, one can always express the hazard rate of Y_k in the alternative form $r_k = \tilde\lambda_k \tilde r$ with $\tilde r = \bar\lambda\, r^\star$ and renormalized proportionality constant as $\tilde\lambda_k = \lambda_k/\bar\lambda$.

It is also sufficient to establish Proposition 1 in the case $r^\star = r$. Indeed, if $X^\star_{(1)} < \cdots < X^\star_{(n)}$ are the order statistics associated with a random sample

$X_1^\star, \dots, X_n^\star$ from a distribution with hazard rate $r^\star$, one has both

$$(X_{(n)}|X_{(1)}) \prec_{\text{RTI}} (X_{(n)}^\star|X_{(1)}^\star) \quad \text{and} \quad (X_{(n)}^\star|X_{(1)}^\star) \prec_{\text{RTI}} (X_{(n)}|X_{(1)}).$$

This comes from the fact that the copula associated with any pair of order statistics from a random sample does not depend on the parent distribution; see, e.g., Lemma 6 of Avérous et al. (2005).

3.2 Further reduction to the exponential case

Let the cumulative hazard rate associated with r be denoted by

$$R(t) = \int_0^t r(z)dz = -\log\{\bar{F}(t)\}, \quad t > 0$$

and for each $k \in \{1, \dots, n\}$, consider the transformation

$$X_k \mapsto X_k^* = R(X_k), \quad Y_k \mapsto Y_k^* = R(Y_k).$$

Let also $X_{(1)}^* < \cdots < X_{(n)}^*$ and $Y_{(1)}^* < \cdots < Y_{(n)}^*$ be the order statistics corresponding to the new sets of variables.

In view of their invariance by monotone increasing transformations of the margins, the copulas associated with the pairs $(X_{(1)}, X_{(n)})$ and $(X_{(1)}^*, X_{(n)}^*)$ coincide. Similarly, the pairs $(Y_{(1)}, Y_{(n)})$ and $(Y_{(1)}^*, Y_{(n)}^*)$ have the same copula.

Furthermore, the RTI dependence ordering is copula-based. Accordingly,

$$(Y_{(n)}|Y_{(1)}) \prec_{\text{RTI}} (X_{(n)}|X_{(1)}) \quad \Leftrightarrow \quad (Y_{(n)}^*|Y_{(1)}^*) \prec_{\text{RTI}} (X_{(n)}^*|X_{(1)}^*)$$

and hence one need only show the right-hand side to prove Proposition 3.

This is a convenient simplification which amounts to assuming a constant hazard rate or, equivalently, that all the variables involved are exponential. Indeed, given that $R^{-1}(t) = \bar{F}^{-1}(e^{-t})$, it is immediate that X_k^* is exponential with unit mean for each $k \in \{1, \dots, n\}$. Furthermore, the proportional hazards assumption on Y_k is equivalent to the statement that

$$\bar{F}_k(t) = \{\bar{F}(t)\}^{\lambda_k}, \quad t > 0. \tag{1}$$

Accordingly, $Y_k^* = R(Y_k)$ is exponential with mean $1/\lambda_k$ for each $k \in \{1, \dots, n\}$.

3.3 Translation into the LTD ordering

Left-tail decreasingness (LTD) is another dependence ordering due to Avérous and Dortet-Bernadet (2000). Following Colangelo et al. (2006), the most economical way of defining it is through the equivalence

$$(T_1|S_1) \prec_{\mathrm{RTI}} (T_2|S_2) \quad \Leftrightarrow \quad (-T_1|-S_1) \prec_{\mathrm{LTD}} (-T_2|-S_2). \tag{2}$$

A more explicit definition is given below, in terms of the conditional copulas

$$C^L_{i,u}(v) = \frac{1}{u}\, C_i(u,v) = \mathrm{P}(V_i \le v | U_i \le u)$$

and their (right continuous) inverses $(C^L_{i,u})^{-1}$, $i = 1,2$.

Definition 3 *T_1 is said to be less left-tail decreasing in S_1 than T_2 is in S_2, denoted $(T_1|S_1) \prec_{\mathrm{LTD}} (T_2|S_2)$, if and only if*

$$C^L_{2,u_2} \circ (C^L_{2,u_1})^{-1}(w) \le C^L_{1,u_2} \circ (C^L_{1,u_1})^{-1}(w)$$

for all $0 < u_1 < u_2 < 1$ and $w \in (0,1)$.

Using Definition 3 and the general fact that the copula D of $(-S,-T)$ is connected to the copula C of $(-S,T)$ through the relation $D(u,v) = u - C(u,1-v)$ for all $u,v \in (0,1)$, one can check easily that

$$(-T_1|-S_1) \prec_{\mathrm{LTD}} (-T_2|-S_2) \quad \Leftrightarrow \quad (T_2|-S_2) \prec_{\mathrm{LTD}} (T_1|-S_1). \tag{3}$$

In the light of relations (2) and (3), therefore, Proposition 1 is true so long as

$$(X^*_{(n)}|-X^*_{(1)}) \prec_{\mathrm{LTD}} (Y^*_{(n)}|-Y^*_{(1)}).$$

3.4 Determination of the copulas of $(-X^*_{(1)}, X^*_{(n)})$ and $(-Y^*_{(1)}, Y^*_{(n)})$

It is sufficient to find the copula C_λ of the pair $(-Y^*_{(1)}, Y^*_{(n)})$, because the pair $(-X^*_{(1)}, X^*_{(n)})$ corresponds to the special case where the components of the vector $\lambda = (\lambda_1, \ldots, \lambda_n)$ are all equal to one.

Remembering that $\lambda_1 + \cdots + \lambda_n = n$ by hypothesis, one can see that for arbitrary $s < 0$ and $t > 0$,

$$K(s) = e^{ns} \quad \text{and} \quad L_\lambda(t) = \prod_{k=1}^{n} \left(1 - e^{-\lambda_k t}\right)$$

are the marginal distribution functions of $-Y^*_{(1)}$ and $Y^*_{(n)}$, respectively. Whenever $s+t \geq 0$, one finds more generally that

$$\begin{aligned}\mathrm{P}(-Y^*_{(1)} \leq s, Y^*_{(n)} \leq t) &= \mathrm{P}(-s < Y^*_1 \leq t, \ldots, -s < Y^*_n \leq t) \\ &= \prod_{k=1}^{n} \left(e^{\lambda_k s} - e^{-\lambda_k t}\right) = K(s) L_\lambda(s+t). \end{aligned} \tag{4}$$

The copula C_λ of the pair $(-Y^*_{(1)}, Y^*_{(n)})$ may then be found by substituting

$$s = K^{-1}(u) = \log(u)/n, \quad t = L_\lambda^{-1}(v)$$

into equation (4). This yields

$$C_\lambda(u,v) = \begin{cases} u L_\lambda\{L_\lambda^{-1}(v) + \log(u)/n\} & \text{if } u \in A_\lambda(v), \\ 0 & \text{otherwise,} \end{cases}$$

where by definition, $A_\lambda(v) = \{u \in (0,1) : L_\lambda^{-1}(v) + \log(u)/n \geq 0\}$.

Although the expression for C is not algebraically closed in general, it turns out to be sufficiently explicit to establish Proposition 1. Note however that in the special case where $\lambda_1 = \cdots = \lambda_n = 1$, one gets

$$L_1(t) = (1-e^{-t})^n, \quad L_1^{-1}(v) = -\log(1-v^{1/n})$$

and hence

$$C_1(u,v) = \max\{0, (u^{1/n} + v^{1/n} - 1)^n\}, \quad u, v \in (0,1).$$

This copula turns out to be a member of Clayton's family, also known as the gamma frailty model in survival analysis; see, e.g., Oakes (1989). As already noted by Schmitz (2004), this copula characterizes the dependence between the extreme order statistics of a random sample of size $n \geq 2$ from any univariate continuous distribution.

3.5 Comparison of C_λ and C_1 via the LTD ordering

In order to prove Proposition 1, it remains to show that for arbitrary $0 < u_1 < u_2 < 1$ and $w \in (0,1)$,

$$C^L_{\lambda,u_2} \circ (C^L_{\lambda,u_1})^{-1}(w) \leq C^L_{1,u_2} \circ (C^L_{1,u_1})^{-1}(w), \tag{5}$$

where for arbitrary λ and $u \in (0,1)$,

$$C^L_{\lambda,u}(v) = \begin{cases} L_\lambda\{L_\lambda^{-1}(v) + \log(u)/n\} & \text{if } u \in A_\lambda(v), \\ 0 & \text{otherwise,} \end{cases}$$

and for all $w \in (0,1)$, $(C^L_{\lambda,u})^{-1}(w) = L_\lambda\{L_\lambda^{-1}(w) - \log(u)/n\}$.

For fixed $w \in (0,1)$, let $v_\lambda = (C^L_{\lambda,u_1})^{-1}(w)$ and observe that because $u_1 < u_2$,

$$L_\lambda^{-1}(v_\lambda) + \log(u_2)/n = L_\lambda^{-1}(w) + \log(u_2/u_1)/n > 0,$$

i.e., $u_2 \in A_\lambda(v_\lambda)$. Likewise, $u_2 \in A_1(v_1)$ with $v_1 = (C^L_{1,u_1})^{-1}(w)$.

Consequently, an equivalent expression for inequality (5) is given by

$$L_\lambda\{L_\lambda^{-1}(w) + \log(u_2/u_1)/n\} \le L_1\{L_1^{-1}(w) + \log(u_2/u_1)/n\}.$$

Writing $c = \log(u_2/u_1)/n > 0$ and allowing $u_1 \in (0,1)$ and $u_2 \in (u_1, 1)$ to vary freely in their domain, one can see that Proposition 1 holds if and only if

$$L_\lambda\{L_\lambda^{-1}(w) + c\} \le L_1\{L_1^{-1}(w) + c\} \tag{6}$$

for every $w \in (0,1)$ and $c > 0$, where L_1 and L_λ are the distribution functions of $X^*_{(n)}$ and $Y^*_{(n)}$ respectively. In view of Definition 2, however, condition (6) amounts to the statement that

$$X^*_{(n)} \prec_{\text{DISP}} Y^*_{(n)},$$

and this fact is already known from the work of Dykstra et al. (1997). Thus the proof is completes. □

4 Discussion

A few applications of, and complements to, Proposition 1 are briefly described below. For clarity, each topic is the object of a short subsection.

4.1 A lower bound on $\kappa(Y_{(1)}, Y_{(n)})$

It was mentioned in the Introduction that under the conditions of Proposition 1, the presence of heterogeneity in a set of observations from a proportional hazards model tends to decrease the degree of association between extreme order statistics as measured, e.g., by Spearman's rho or Kendall's tau.

In the light of the work of Schmitz (2004) and Avérous et al. (2005), it is also known that $\kappa(X_{(1)}, X_{(n)}) \geq 0$ for any concordance measure and any homogeneous sample of observations. One may wonder, therefore, whether the introduction of heterogeneity as per the terms of Proposition 1 reduces this dependence sufficiently to make it negative.

In fact, $\kappa(Y_{(1)}, Y_{(n)}) \geq 0$ also, as follows immediately from Theorem 3.4 of Boland et al. (1996). The latter states that $(Y_{(n)}|Y_{(1)})$ is in right-tail increasing dependence, i.e., that it is more right-tail increasing than any pair (S,T) of independent continuous random variables. Alternatively, it is easy to see from the above developments that $(Y_{(n)}| - Y_{(1)})$ is in negative dependence in the left-tail decreasing ordering. To this end, one must only show that

$$w \leq C^L_{\lambda,u_2} \circ (C^L_{\lambda,u_1})^{-1}(w)$$

for all $0 < u_1 < u_2 < 1$ and $w \in (0,1)$. But by the same arguments as before, the inequality reduces to

$$w \leq L_\lambda\{L_\lambda^{-1}(w) + \log(u_2/u_1)/n\},$$

which is immediate from the fact that $u_1 < u_2$. This is consistent with the result of Sathe (1988), who showed that $\mathrm{corr}(Y_{(1)}, Y_{(n)}) \geq 0$ in the special case of exponentials.

4.2 An upper bound on $\kappa(Y_{(1)}, Y_{(n)})$

Coming back to the introductory remarks that motivated this work, consider a set of mutually independent components whose lifetimes $Y_1, \ldots, Y_n$ follow a proportional hazards model of the form (1). To be explicit, assume that there exist a baseline survivor function $\bar{F}$ and scalars $\lambda_1, \ldots, \lambda_n > 0$ such that

$$\mathrm{P}(Y_k > t) = \{\bar{F}(t)\}^{\lambda_k}, \quad t > 0.$$

It may then be of interest to qualify the degree of association between the order statistics $Y_{(1)}$ and $Y_{(n)}$ which account for the reliability of the n-out-of-n (series) and 1-out-of-n (parallel) systems.

For concordance measures in the sense of Scarsini (1984), $\kappa(Y_{(1)}, Y_{(n)})$ does not depend on the baseline survivorship, and so the calculation would be simplified by assuming that $\bar{F}(t) = e^{-t}$, which implies that $Y_1, \ldots, Y_n$ are then exponential. Nevertheless, the calculation would remain exceedingly complex, in view of the heterogeneity.

Under the conditions of Proposition 1, an upper bound on $\kappa(Y_{(1)}, Y_{(n)})$ is given by $\kappa(X_{(1)}, X_{(n)})$, where $X_{(1)}$ and $X_{(n)}$ determine the reliability of the series and parallel systems in the homogeneous case. For Kendall's tau, in particular, one would get

$$\tau(Y_{(1)}, Y_{(n)}) \leq \frac{1}{2n-1}\,,$$

as per Theorem 5 of Schmitz (2004). Theorem 6 in the same paper could be used to give an upper bound on Spearman's correlation between $Y_{(1)}$ and $Y_{(n)}$.

4.3 Possible extensions

There are several ways in which Proposition 1 could be extended. An obvious option would be to investigate whether statements of the form

$$(Y_{(j)}|Y_{(k)}) \prec_{\mathrm{RTI}} (X_{(j)}|X_{(k)}) \quad \text{or} \quad (Y_{(j)}|Y_{(k)}) \prec_{\mathrm{PQD}} (X_{(j)}|X_{(k)})$$

could be established for other choices of $j, k \in \{1, \ldots, n\}$ with $j > k$. This problem seems difficult, however, given the intricate form of the dependence structure between order statistics from a heterogeneous set of observations.

An apparently simpler, yet unsolved, problem would consist of showing that

$$(Y_{(n)}|Y_{(1)}) \prec_{\mathrm{MRD}} (X_{(n)}|X_{(1)}) \tag{7}$$

using the *monotone regression dependence* (MRD) ordering. This concept, whose origin can be traced back to Yanagimoto and Okamoto (1969), is also known in the literature as the stochastically increasing (SI) ordering. Following Capéraà and Genest (1990), its definition is given below in terms of the conditional copulas

$$C_{i,u}(v) = \frac{\partial}{\partial u}\, C_i(u, v) = \mathrm{P}(V_i \leq v | U_i = u)$$

and their (right continuous) inverses $(C_{i,u})^{-1}$, $i = 1, 2$. For an equivalent alternative definition in terms of quantile functions, see Avérous et al. (2005).

Definition 4 *T_1 is said to be less monotone regression dependent in S_1 than T_2 is in S_2, denoted $(T_1|S_1) \prec_{\mathrm{MRD}} (T_2|S_2)$, if and only if*

$$C_{2,u_2} \circ (C_{2,u_1})^{-1}(w) \leq C_{1,u_2} \circ (C_{1,u_1})^{-1}(w)$$

for all $0 < u_1 < u_2 < 1$ and $w \in (0, 1)$.

If it turned out to be true, a statement such as (7) would represent a strengthening of Proposition 1, because of the following chain of implications established by Avérous and Dortet-Bernadet (2000):

$$(T_1|S_1) \prec_{\text{MRD}} (T_2|S_2) \;\Rightarrow\; \begin{array}{c} (T_1|S_1) \prec_{\text{LTD}} (T_2|S_2) \\ (T_1|S_1) \prec_{\text{RTI}} (T_2|S_2) \end{array} \;\Rightarrow\; (S_1,T_1) \prec_{\text{PQD}} (S_2,T_2).$$

In view of the copula-based definition of $\prec_{\text{MRD}}$, a proof of conjecture (7) could be limited to the exponential case. Although it remains elusive, the following connection seems well worth pointing out.

Proposition 2 *Let* $X_1^*, \ldots, X_n^*$ *and* $Y_1^*, \ldots, Y_n^*$ *be two sets of mutually independent exponential random variables. Assume that for* $k \in \{1, \ldots, n\}$, $\text{E}(X_k^*) = 1$ *and* $\text{E}(Y_k^*) = 1/\lambda_k > 0$. *If* $(\lambda_1 + \cdots + \lambda_n)/n = 1$, *then*

$$\begin{aligned} (Y_{(n)}^*|Y_{(1)}^*) \prec_{\text{MRD}} (X_{(n)}^*|X_{(1)}^*) \quad &\Leftrightarrow \quad (X_{(n)}^*|-X_{(1)}^*) \prec_{\text{MRD}} (Y_{(n)}^*|-Y_{(1)}^*) \\ &\Leftrightarrow \quad X_{(n)}^* - X_{(1)}^* \prec_{\text{DISP}} Y_{(n)}^* - Y_{(1)}^*. \end{aligned}$$

Proof. The first equivalence is a general property of the $\prec_{\text{MRD}}$ ordering which is easily verified from its definition. To establish the second equivalence, use (4) to express the copula of $(-Y_{(1)}^*, Y_{(n)}^*)$ in the alternative form

$$C_\lambda(u,v) = \begin{cases} \displaystyle\prod_{k=1}^n \left(e^{\lambda_k \log(u)/n} - e^{-\lambda_k L_\lambda^{-1}(v)}\right) & \text{if } u \in A_\lambda(v), \\ 0 & \text{otherwise.} \end{cases}$$

Upon differentiation and elementary algebra, it follows that

$$C_{\lambda,u}(v) = \frac{\partial}{\partial u}\, C_\lambda(u,v) = M_\lambda\{L_\lambda^{-1}(v) + \log(u)/n\}, \quad u \in A_\lambda(v)$$

where, as per David (1981, p. 26),

$$M_\lambda(t) = \text{P}(Y_{(n)}^* - Y_{(1)}^* \le t) = \left(\frac{1}{n}\sum_{k=1}^n \frac{\lambda_k}{1 - e^{-\lambda_k t}}\right) \times \prod_{k=1}^n (1 - e^{-\lambda_k t}).$$

Using the fact that $(C_{\lambda,u})^{-1}(w) = L_\lambda\{M_\lambda^{-1}(w) - \log(u)/n\}$ for all $w \in (0,1)$, one deduces that $(X_{(n)}^*|-X_{(1)}^*) \prec_{\text{MRD}} (Y_{(n)}^*|-Y_{(1)}^*)$ holds true if and only if

$$M_\lambda\{M_\lambda^{-1}(w) + \log(u_2/u_1)/n\} \le M_1\{M_1^{-1}(w) + \log(u_2/u_1)/n\}$$

for all $0 < u_1 < u_2 < 1$ and $w \in (0,1)$. Letting $c = \log(u_2/u_1)/n > 0$, one then sees at once that the latter statement is equivalent to the fact that $X^*_{(n)} - X^*_{(1)}$ is less dispersed than $Y^*_{(n)} - Y^*_{(1)}$ in the sense of Definition 2. □

An immediate consequence of this result is that the conjecture (7) is at least true in the case $n = 2$. For, Theorem 3.7 of Kochar and Korwar (1996) states that under the same set of conditions as Proposition 2, the normalized spacings are ordered by $\prec_{\mathrm{DISP}}$, viz.

$$(n-k+1)(X^*_{(k)} - X^*_{(k-1)}) \prec_{\mathrm{DISP}} (n-k+1)(Y^*_{(k)} - Y^*_{(k-1)}), \quad k \in \{2, \ldots, n\}.$$

When $n = 2$, this is precisely the desired result. Extensive numerical evidence collected by the authors leads them to believe that the relation

$$X^*_{(n)} - X^*_{(1)} \prec_{\mathrm{DISP}} Y^*_{(n)} - Y^*_{(1)}$$

is valid for any integer $n \geq 3$. Note that from Corollary 2.1 of Kochar and Rojo (1996), the weaker relation

$$X^*_{(n)} - X^*_{(1)} \prec_{\mathrm{ST}} Y^*_{(n)} - Y^*_{(1)}$$

is already known to hold, i.e., $M_\lambda(t) \leq M_1(t)$ for all $t > 0$.

In closing, it can be observed that because

$$M_\lambda\{M_\lambda^{-1}(w) + \log(u_2/u_1)/n\} \geq w, \quad w \in (0,1)$$

for all choices of $\lambda = (\lambda_1, \ldots, \lambda_n)$, both $(X^*_{(n)}|X^*_{(1)})$ and $(Y^*_{(n)}|Y^*_{(1)})$ are positive monotone regression dependent; in other words, their copula dominates the independence copula in the $\prec_{\mathrm{MRD}}$ ordering. In the homogeneous case, this was already known from Proposition 2 of Avérous et al. (2005); in the heterogeneous case, this reinforces the observation made in Subsection 4.1.

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